Indecomposability in Partial Differential Fields

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- ▶ K is equipped with $\Delta = \{\delta_1, \ldots, \delta_m\}$, a commuting set of derivations. We will be working inside of a *large* differentially closed field.
- G will be a differential algebraic group
- There are several ways of thinking about this:
 - Kolchin Gives an intrinsic definition of differential algebraic groups.
 - Model Theory The *definable* groups in differentially closed fields.
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An example

For simplicity, let $\Delta = \{\delta_1, \delta_2\}$.

$$\left(\begin{array}{cccc}
1 & u_1 & u \\
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\end{array}\right)$$

where $\delta_i(u_i) = 0$.

The basic question

Vague Question: When is the subgroup generated by a uniform family of constructible subsets of *G* actually a differential algebraic subgroup?

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The commutator is not a differential algebraic group. The commutator consists of all matrices of this form with $u_i = 0$ and $u \in \mathbb{Q}[C_{\delta_1} \cup C_{\delta_2}]$.

This is not a constructible subset in the Kolchin topology. Model theorists: This is not a definable subgroup.

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- $\qquad \qquad \omega_{\eta/K}(s) = td_K(K(\theta\eta)_{\theta \in \Theta(s)})$
- ▶ This is a polynomial for large enough values of s.
- This polynomial is a birational invariant, but not a differential birational invariant.
- ▶ The degree of the polynomial is called the *differential type*. Notation: $\tau(\eta)$.
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- ▶ The strong identity component $G_0 \triangleleft G$ is the smallest definable subgroup H of G such that $\tau(G/H) < \tau(G)$.
- ▶ *G* is strongly connected if $G = G_0$.
- ▶ *G* is almost simple if for any normal proper definable subgroup *H* of *G* we have $\tau(H) < \tau(G)$.
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A definable subset of $A \subseteq G$ is *indecomposable* if for every definable subgroup H of G, A is contained in a single coset or intersects infinitely many cosets.

Theorem

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Differential Indecomposability

X is Indecomposable \leftrightarrow for any $H \leq G$, X/H is either large or |X/H| = 1.

Here large means "of the same differential type as G"

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Let G be a differential algebraic group. Let X_i for $i \in I$ be a family of indecomposable definable subsets of G, each containing the identity. Then $\langle \bigcup_i X_i \rangle$ is a strongly connected differential algebraic subgroup of G.

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"We also do not have an example of a noncommutative almost simple linear differential algebraic group whose commutator subgroup is not closed in the Kolchin topology." (Cassidy, Singer 2008)

Theorem

(Cassidy, Singer 2008) Let G be an almost simple linear differential algebraic group of Δ -type 1. The G is either commutative or perfect.

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Apply the indecomposability theorem to the family $g^{-1}g^G$ as $g \in G$ varies.

We need to know that g^G is indecomposable

It is enough to prove the result for all $N \lhd G$.

G acts on g^G/N .



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The kernel must be a subgroup of G of Δ -type n and of the same dimension as G.

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Assume G is almost simple, non-commutative. By using Pillay's version of a "Chevalley-Barsotti" type structure theorem, plus the indecomposability theorem, one can show that such a G must be linear.

G/Z(G) is simple, so by Cassidy's theorem, the quotient is isomorphic to an algebraic group restricted to a definable field. So

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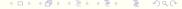
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Conjecture

Every non-commutative almost simple differential algebraic group is an algebraic group (perhaps restricted to some definable subfield).

Second approach: concentrate on typical differential dimension. The first interesting case is when the typical dimension is 3.

Theorem

A non-commutative almost simple differential algebraic group of typical differential dimension 3 is either (definably isomorphic to) $SL_2(F)$ or $PSL_2(F)$ for some definable subfield F.

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Thanks for listening

Thanks to D. Marker for several useful conversations about this work.

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