Indecomposability in Partial Differential Fields

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- K is equipped with $\Delta = \{\delta_1, \ldots, \delta_m\}$, a *commuting* set of derivations. We will be working inside of a *large* differentially closed field.
- G will be a *differential algebraic group*
- There are several ways of thinking about this:
  Kolchin Gives an intrinsic definition of differential algebraic groups.
  Model Theory The *definable* groups in differentially closed fields.
  Subgroups Differential algebraic subgroups of algebraic groups.

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An example

For simplicity, let $\Delta = \{\delta_1, \delta_2\}$.

$$\begin{pmatrix} 1 & u_1 & u \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\delta_i(u_i) = 0$.

The basic question

Vague Question: When is the subgroup generated by a uniform family of constructible subsets of G actually a differential algebraic subgroup?

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The commutator is not a differential algebraic group. The commutator consists of all matrices of this form with $u_i = 0$ and $u \in \mathbb{Q}[C_{\delta_1} \cup C_{\delta_2}]$.

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Kolchin Polynomials

- $\omega_{\eta/K}(s) = td_K(K(\theta\eta)_{\theta\in\Theta(s)})$
- This is a polynomial for large enough values of s.
- This polynomial is a birational invariant, but not a differential birational invariant.
- The degree of the polynomial is called the *differential type*. Notation: $\tau(\eta)$.
- The leading coefficient is called the *typical differential dimension*.

The idea of this work is to apply model theoretic techniques to differential algebraic groups, replacing model theoretic notions (Morley rank, Morley degree, Lascar rank) by differential algebraic ones.

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- The *strong identity component* $G_0 \lhd G$ is the smallest definable subgroup H of G such that $\tau(G/H) < \tau(G)$.
- G is *strongly connected* if $G = G_0$.
- G is *almost simple* if for any normal proper definable subgroup H of G we have $\tau(H) < \tau(G)$.
- Cassidy and Singer showed that every differential algebraic group has a composition series in which the successive quotients are almost simple.
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Zilber Indecomposability

Let G be a group of finite Morley rank.
A definable subset of $A \subseteq G$ is *indecomposable* if for every definable subgroup H of G, A is contained in a single coset or intersects infinitely many cosets.

Theorem

Let G be a group of finite Morley rank. Let $\{X_i : i \in I\}$ be a family of indecomposable subsets of G, each containing the identity. Then the subgroup generated by the family is definable and connected.

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X is Indecomposable $\leftrightarrow$ for any $H \leq G$, X/H is either large or $|X/H| = 1$.

Here large means "of the same differential type as G"

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Application 1

"We also do not have an example of a noncommutative almost simple linear differential algebraic group whose commutator subgroup is not closed in the Kolchin topology." (Cassidy, Singer 2008)

Theorem

(Cassidy, Singer 2008) Let G be an almost simple linear differential algebraic group of Δ-type 1. The G is either commutative or perfect.

Theorem

Let G be a strongly connected (nonabelian) differential algebraic group. Then the commutator of G is a strongly connected differential algebraic subgroup with $\tau([G, G]) = \tau(G)$.

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(Sketch of the) Proof:
Apply the indecomposability theorem to the family $g^{-1}g^G$ as $g \in G$ varies.
We need to know that g^G is indecomposable.
It is enough to prove the result for all $N \lhd G$.
G acts on g^G/N,

$$h * g^{h_1}N \mapsto hg^{h_1}Nh^{-1} = hg^{h_1}h^{-1}hNh^{-1} = g^{hh_1}N$$

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The kernel must be a subgroup of G of Δ-type n and of the same dimension as G.
But G is strongly connected, thus $|g^G/N| = 1$.

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What are non-commutative almost simple differential algebraic groups?

Assume G is almost simple, non-commutative. By using Pillay's version of a "Chevalley-Barsotti" type structure theorem, plus the indecomposability theorem, one can show that such a G must be linear.

$G/Z(G)$ is simple, so by Cassidy's theorem, the quotient is isomorphic to an algebraic group restricted to a definable field. So,

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Conjecture

Every non-commutative almost simple differential algebraic group is an algebraic group (perhaps restricted to some definable subfield).

Second approach: concentrate on typical differential dimension. The first interesting case is when the typical dimension is 3.

Theorem

A non-commutative almost simple differential algebraic group of typical differential dimension 3 is either (definably isomorphic to) $SL_2(F)$ or $PSL_2(F)$ for some definable subfield F.

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Thanks for listening

Thanks to D. Marker for several useful conversations about this work.

Thanks to P. Cassidy for directing me to her work with M. Singer.

Thanks to the Kolchin seminar for useful feedback, especially Phyllis Cassidy and William Sit.

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