Generic Differential Galois Extensions

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Comercial

Consider $\int f$, where

$$f = \frac{(3xe^x + 3)\sqrt[3]{(\log x + e^x)^2} - xe^x - 1}{24x\sqrt[3]{(\log x + e^x)^2}(\log x + e^x - \sqrt[3]{\log x + e^x})}.$$

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Brian Miller's Algorithm:

$$\int \frac{(3xe^x + 3)\sqrt[3]{(\log x + e^x)^2} - xe^x - 1}{24x\sqrt[3]{(\log x + e^x)^2}(\log x + e^x - \sqrt[3]{\log x + e^x})} = \frac{1}{8}\log\left(\sqrt[3]{(\log x + e^x)^2} - 1\right) + \frac{1}{24}\log\left(\log x + e^x\right).$$

Liouville's Theorem

Theorem (Liouville's Theorem)

Let K be a differential field of characteristic zero with constant field C and let $f \in K$. If the equation $g' = f$ has a solution $g \in L$ where L is an elementary extension of K having the same constant field C, then there exist $v, u_1, u_2, \ldots, u_n \in E$ and constants $c_1, \ldots, c_n \in C$ such that

$$f = v' + \sum_{i=1}^{n} c_i \frac{u_i'}{u_i}$$

therefore, $\int f = v + \sum_{i=1}^{n} c_i \log u_i$.

The Structure of Picard-Vessiot Extensions

The structure of Picard-Vessiot G-extensions can be described in terms of G-torsors.

Let G be a linear algebraic group defined over a field k. A *k-homogeneous space* for G is a k-affine variety together with a morphism $G \times V \mapsto V$ of k-varieties inducing a transitive action of $G(\overline{k})$ on $V(\overline{k})$, where $\overline{k}$ denotes the algebraic closure of k. If moreover the action is faithful, V is called a *principal k-homogeneous space* for G or a *G-torsor*. The group G itself is called the trivial G-torsor.

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Theorem

The set of G-torsors (up to G-isomorphism) maps bijectively to the first Galois cohomology set $H^1(k, G)$.

Structure Theorem (Kolchin)

Theorem

Let k be a differential field with field of constants $\mathcal{C}$ and let $E \supset k$ be a Picard-Vessiot extension with group G. Then E is the function field $k(V)$ of some k-irreducible G-torsor where the action of the Galois group on E is the same as the action resulting from $G(\mathcal{C})$ acting on V. Moreover, $E = k(v)$, for some E-point $v \in V$.

Generic Extensions

Definition.

A Picard-Vessiot extension $\mathcal{E} \supset \mathcal{F}$ with group G is called *generic* when the following condition holds:
For any differential field F with field of constants $\mathcal{C}$ there is a PVE $E \supset F$ with differential Galois group $H \leq G$ if and only if there are $f_i \in F$ such that the matrix $\mathcal{A}(f_1, \ldots, f_k)$ is well defined and the equation $X' = X\,\mathcal{A}(f_1, \ldots, f_k)$ gives rise to the extension $E \supset F$.

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Definition.

Let $\mathbf{s} = (s_1, \ldots, s_m)$ be indeterminates over a field K, and let G be a finite group. A monic polynomial $P(\mathbf{s}, X) \in K(\mathbf{s})[X]$ is called a *generic G-polynomial* over K if the following conditions are satisfied:

1. The splitting field of $P(\mathbf{s}, X)$ over $K(\mathbf{s})$ is a G-extension, that is, a Galois extension with Galois group isomorphic to G.

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1. The splitting field of $P(\mathbf{s}, X)$ over $K(\mathbf{s})$ is a G-extension, that is, a Galois extension with Galois group isomorphic to G.
2. Every G-extension of a field L containing K is the splitting field (over L) of the polynomial $P(\mathbf{a}, X)$ for some $\mathbf{a} = (a_1, \ldots, a_n) \in L^n$. The polynomial $P(\mathbf{a}, X)$ is called a *specialization* of $P(\mathbf{s}, X)$.

Goldman's Equation

Definition.
Let G be a linear algebraic group over $\mathcal{C}$ and assume that a faithful representation in $\mathrm{GL}_n(\mathcal{C})$ is given. Let

$$L(t,y) = Q_0(t_1,\dots,t_r)y^{(n)}+\cdots+Q_n(t_1,\dots,t_r)y \in \mathcal{C}\{t_1,\dots,t_r,y\}$$

and write $(\pi_1,\dots,\pi_n)$ for a fundamental system of zeros of $L(t,y)$ such that $\mathcal{C}\langle t_1,\dots,t_r,\pi_1,\dots,\pi_n\rangle$ is a PVE of $\mathcal{C}\langle t_1,\dots,t_r\rangle$ with group G. Then $L(t,y)=0$ will be called a *generic equation with group G* if:

1. $t_1, \ldots, t_r$ are differentially independent over $\mathcal{C}$, and $\mathcal{C}\langle t_1, \ldots, t_r\rangle \subset \mathcal{C}\langle \pi_1, \ldots, \pi_n\rangle$.

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2. For every specialization $(t_1, \ldots, t_r, \pi_1, \ldots, \pi_n) \to (\bar{t}_1, \ldots, \bar{t}_r, \bar{\pi}_1, \ldots, \bar{\pi}_n)$ over $\mathcal{C}$ such that $\mathcal{C}\langle \bar{t}_1, \ldots, \bar{t}_r, \bar{\pi}_1, \ldots, \bar{\pi}_n \rangle$ is a PVE of $\mathcal{C}\langle \bar{t}_1, \ldots, \bar{t}_r \rangle$ and the field of constants of the latter is $\mathcal{C}$, the differential Galois group of this extension is a subgroup of G.

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3. If $(\omega_1, \ldots, \omega_n)$ is a fundamental system of zeros of $L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y \in F\{y\}$, where F is any differential field with field of constants $\mathcal{C}$, and $F\langle \omega_1, \ldots, \omega_n\rangle$ is a PVE of F with differential Galois group $H \leq G$, then there exists a specialization $(t_1, \ldots, t_r) \to (\bar{t}_1, \ldots, \bar{t}_r)$ over F with $\bar{t}_i \in F$ such that $Q_o(\bar{t}_1, \ldots, \bar{t}_r) \neq 0$ and

$$a_i = Q_i(\bar{t}_1, \ldots, \bar{t}_r) Q_o^{-1}(\bar{t}_1, \ldots, \bar{t}_r).$$

Bhandari-Sankaran

(3′) If F is a differential field with field of constants $\mathcal{C}$ and E is a PVE of F with differential Galois group $H \leq G$, then there exists a linear differential equation

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0, \qquad a_i \in F$$

such that that $Q_o(\bar{t}_1, \ldots, \bar{t}_r) \neq 0$,
$a_i = Q_i(\bar{t}_1, \ldots, \bar{t}_r) Q_o^{-1}(\bar{t}_1, \ldots, \bar{t}_r)$, $i = 1, \ldots, n$, for suitable $\bar{t}_i \in F$ and $E = F\langle \omega_1, \ldots, \omega_n \rangle$ for a fundamental system of zeros of $L(y)$.

Connected Case (Trivial Torso)

Let $\dim(G)=n$ and suppose that Y_{ij}, $1 \leq i, j \leq n$ are differentially independent indeterminates over $\mathcal{C}$, and put $\mathcal{F} = \mathcal{C}\langle Y_{ij} \rangle$. Given a faithful representation of G in GL_m, the Lie algebra $\mathcal{G}$ maps to a Lie subalgebra of gl_m and a basis $\{D_1, \ldots, D_n\}$ of the former can be identified with a linearly independent set $\{A_1, \ldots, A_n\}$ of $m \times m$ matrices. Let $\mathcal{A}(Y_{ij}) = \sum_{i=1}^{n} Y_i A_i \in \mathcal{G}(\mathcal{F})$ and X a generic point of G, then $X' = A(Y_{ij})X$ gives a derivation on $\mathcal{F}(G)$

Theorem

The extension $\mathcal{F}(G) \supset \mathcal{F}$ is a generic Picard-Vessiot extension for G relative to the trivial G-torsor and, furthermore, it descends to subgroups of G as follows:
Let F be a differential field with field of constants $\mathcal{C}$.

1. *If $E \supset F$ is a Picard-Vessiot extension with connected differential Galois group $G' \leq G$ such that $E = F(G')$, then there is a specializaton $Y_i \to f_i \in F$ such that the equation $X' = \mathcal{A}(f_1, \ldots, f_n)X$ gives rise to this extension.*
2. *For every specialization $Y_i \to f_i \in F$, the differential equation $X' = \mathcal{A}(f_1, \ldots, f_n)X$ gives rise to a Picard-Vessiot extension $E \supset F$ with differential Galois group $G' \leq G$.*

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Definition.
Suppose that Y_{ij} are differentially independent indeterminates over $\mathcal{C}$ and put $\mathcal{F} = \mathcal{C}\langle Y_{ij}\rangle$. We say that a Picard-Vessiot G-extension $\mathcal{E} \supset \mathcal{K}$ is *generic for G relative to split G-torsors* if there is a faithful differential H-action on $\mathcal{F}$, with $\mathcal{K} = \mathcal{F}^H$, such that

1. $\mathcal{E} = \mathcal{K}(\mathcal{W} \times G^0)$ for some $\mathcal{K}$-irreducible H-torsor $\mathcal{W}$, and

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1. $\mathcal{E} = \mathcal{K}(\mathcal{W} \times G^0)$ for some $\mathcal{K}$-irreducible H-torsor $\mathcal{W}$, and
2. for every faithful representation of G in a GL_m, the G^0-extension $\mathcal{F}(G^0) \supset \mathcal{F}$ has an H-equivariant equation $X' = \mathcal{A}(Y_{ij})X$, such that, given a Picard-Vessiot G-extension of the form $k(W \times G^0) \supset k$, where k is a differential field with field of constants $\mathcal{C}$ and W a k-irreducible H-torsor, there is an H-equivariant specialization $Y_i \to f_i$ with $f_i \in k(W)$, such that the G^0-extension $k(W)(G^0) \supset k(W)$ has equation $X' = \mathcal{A}(f_1, \ldots, f_n)X$.

Proposition.

Let $H, G' \leq \mathrm{GL}_m$ be algebraic groups over $\mathcal{C}$, with H finite and G' not necessarily connected. Le F be a differential field with field of constants $\mathcal{C}$ on which H acts faithfully as a group of differential automorphisms with $\mathcal{C} \subset F^H$. Let W be an F^H-irreducible H-torsor such that $F = F^H(W)$.
Let $A \in \mathrm{gl}_m(F)$ and assume that

1. *A is H-equivariant.*
2. *The Picard-Vessiot extension E of F corresponding to the equation $X' = AX$ has Galois group G'.*

Then there is a conjugation action of H on G' such that E is the function field of an F^H-irreducible $H \ltimes G'$-torsor $W \times V$, and a Picard-Vessiot extension of F^H with Galois group $H \ltimes G'$. Furthermore the action of the Galois group corresponds to the action of $H \ltimes G'$ on E induced by the action of $H \ltimes G'$ on $W \times V$.

Non-Trivial Torsors - Joint work with Arne Ledet

Let K be a differential field with field with algebraically closed field of constants $\mathcal{C}$.
Let $G \subseteq \mathrm{GL}_n(\mathcal{C})$ be a connected linear algebraic group, and let $H = K[X, 1/\det(X)]$ be the coordinate ring over K, where X is a generic point of G.

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Let $G \subseteq \mathrm{GL}_n(\mathcal{C})$ be a connected linear algebraic group, and let $H = K[X, 1/\det(X)]$ be the coordinate ring over K, where X is a generic point of G.
A crossed homomorphism $e \colon \mathrm{Gal}(K) \to G(\bar{K})$, gives an e-twisted Galois action on $\bar{H} = \bar{K} \otimes_K H$ by

$$^{\sigma}z = e_{\sigma}(\sigma z),$$

and a corresponding coordinate ring for a torsor $T = \bar{H}^{\mathrm{Gal}(K)}$.

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The G-action on H (and $\bar{H}$) is given by

$$^{g}X = Xg, \quad g \in G.$$

By Speiser's Theorem there exists $P \in \mathrm{GL}_n(\bar{K})$ with $e_\sigma = P\sigma P^{-1}$, and with $Y = XP$ we have

$$^{\sigma}Y = XP\sigma P^{-1}\sigma P = XP = Y,$$

from which it follows that we can realize T explicitly inside $\bar{H}$ as $T = K[Y, 1/\det(Y)]$.

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We then have a G-action on T given by

$$^{g}Y = g^{-1}Y, \quad g \in G.$$

Define a derivation on T by

$$Y' = YB$$

for some $B \in M_n(K)$. The fact that the derivation is expressed by multiplication from the right guarantees that the G-action on T is differential.

It then extends to $\bar{H}$, where

$$X' = XA = X(PBP^{-1} - P'P^{-1}),$$

and hence, if we let $\mathfrak{g}$ denote the Lie algebra $\mathrm{Lie}(G)$, we see that

$$A \in \mathfrak{g}(\bar{K}),$$

and

$$B = P^{-1}AP + P^{-1}P' \in [P^{-1}P' + P^{-1}\mathfrak{g}(\bar{K})P] \cap M_n(K).$$

Generic Torsors and Extensions?

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