# Automorphisms of Hurwitz Series

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This talk is dedicated to the memory of Jerald Kovacic - colleague, friend and source of inspiration

- All rings are associative, commutative and unitary.
- $A$ and $B$ will denote rings (of any characteristic).
- Lemma - If $d : A \longrightarrow A$ is a derivation on $A$, and if $f : A \longrightarrow B$ and $g : B \longrightarrow A$ are ring homomorphisms with $f \circ g = id_B$, then $f \circ d \circ g : B \longrightarrow B$ is a derivation on $B$.
- $\mathrm{Der}A$ will denote the set of derivations on $A$.
- For any $m, n \in \mathbf{N}, \delta_n^m$ will denote the Kronecker delta, i.e., $\delta_n^m = 1$ if $m = n$ and $\delta_n^m = 0$ if $m \neq n$.

**The ring of Hurwitz series over $A$:**

- The ring of Hurwitz series over $A$ is denoted by $HA$
- Elements of $HA$ are sequences in $A$, i.e., $a : \mathbf{N} \to A$, written $(a_n)$
- Addition: $(a_n) + (b_n) = (a_n + b_n)$
- Zero: $0 = (0, 0, 0, \ldots)$

**Hurwitz multiplication:**

- $(a_n) * (b_n) = (\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k})$
- So $(a_0, a_1, a_2, \ldots) * (b_0, b_1, b_2, \ldots) = (a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + 2a_1 b_1 + a_2 b_0, \ldots)$
- Example: $(1, 1, 1, 1, \ldots) * (1, 2, 4, 8, \ldots) = (1, 3, 9, 27, \ldots)$
- Identity: $1 = 1_{HA} = (1, 0, 0, 0, \ldots)$

**Derivation on $HA$:**

- $\partial : HA \to HA : (a_n) \mapsto (a_{n+1})$ - "shift operator"
- $\partial((a_0, a_1, a_2, a_3, \ldots)) = (a_1, a_2, a_3, \ldots)$
- $\partial$ is a derivation on $HA$
- There is a differential ring homomorphism

$$\psi : (A[[t]], \frac{d}{dt}) \to (HA, \partial) : \sum_{n=0}^{\infty} a_n t^n \mapsto (n! a_n),$$

and if $\mathbf{Q} \subseteq A$, then $HA \cong A[[t]]$.

**Some natural ring homomorphisms involving Hurwitz series:**

- $\varepsilon : HA \to A : (a_n) \mapsto a_0$
- $\delta : HA \to HHA : (a_n) \mapsto ((b_m)_n)$ with $((b_m)_n) = a_{m+n}$; i.e., $\delta((a_n)) = (a_{m+n})$
- $\lambda : A \to HA : a \mapsto (a, 0, 0, 0, \ldots)$
- If $d$ is a derivation on $A$ then

  $$\widetilde{d} : A \to HA : a \mapsto (a, d(a), d^2(a), \ldots)$$

  is a ring homomorphism, called the *Hurwitz homomorphism* of $d$. Note that $\widetilde{0} = \lambda$.

It is well-known that if $d$ is a derivation on $A$ and $\mathbf{Q} \subseteq A$, then there is a differential ring homomorphism

$$T : (A, d) \to (A[[t]], d/dt) : a \mapsto \sum_{n=0}^{\infty} \frac{d^n(a)}{n!} t^n,$$

called the *Taylor homomorphism* of $d$.

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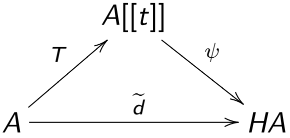
$$T : (A, d) \to (A[[t]], d/dt) : a \mapsto \sum_{n=0}^{\infty} \frac{d^n(a)}{n!} t^n,$$

called the *Taylor homomorphism* of $d$. When $\mathbf{Q} \subset A$, $\psi : A[[t]] \to HA$ is an isomorphism, and $T$ and $\widetilde{d}$ are related by the commutative diagram




However, the Taylor homomorphism $T$ is defined only in case $\mathbf{Q} \subseteq A$, while the Hurwitz homomorphism $\widetilde{d}$ is defined for any differential ring $A$ of any characteristic.

**The order of a Hurwitz series:**

- We define the *order* of $0 \neq h \in HA$, denoted by $\mathrm{ord}(h)$, to be the minimum $i \in \mathbf{N}$ such that $h(i) \neq 0$ and when $h = 0$, $\mathrm{ord}(h) := \infty$.

**Divided power structure on $HA$:**

- The divided powers $x^{[m]}$ in $HA$ are given by $x^{[m]}(n) = \delta_n^m$, so e.g., $x^{[2]} = (0, 0, 1, 0, \ldots, 0, \ldots)$
- $x^{[m]} * x^{[n]} = \binom{m+n}{n} x^{[m+n]}$
- $(a_0, a_1, a_2, \ldots) = \sum a_n x^{[n]}$

## Comorphisms on $A$

- A *comorphism* $\alpha$ on a ring $A$ is a ring homomorphism $\alpha : A \to HA$ such that the diagrams

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & HA \\ & \searrow^{\mathrm{id}_A} & \downarrow \varepsilon \\ & & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{\alpha} & HA \\ \alpha \downarrow & & \downarrow \delta \\ HA & \xrightarrow{H\alpha} & HHA \end{array}$$

  commute.

- Examples of comorphisms on $A$ include $\lambda$ and $\widetilde{d}$, where $d$ is a derivation on $A$.

- The set of all comorphisms on $A$ will be denoted by $\mathrm{Comor}A$.

- **Theorem.** There is a one-to-one correspondence:

$$\mathrm{Der}A \rightleftarrows \mathrm{Comor}A.$$

  - Given a derivation $d$ on $A$, we get a ring homomorphism $\widetilde{d} : A \longrightarrow HA : a \mapsto (d^{(n)}(a))$, the Hurwitz homomorphism.
  - Given a ring homomorphism $f : A \longrightarrow HA$ with $\varepsilon \circ f = id_A$ and $\delta \circ f = Hf \circ f$ , we get a derivation $\varepsilon \circ \partial \circ f$ on $A$.

## Hurwitz automorphisms

A ring endomorphism $\sigma$ of $HA$ is called a *Hurwitz endomorphism* if for all $n \in \mathbf{N}$ and $h \in HA$, $\sigma$ satisfies the following conditions,

$$(\varepsilon \circ \partial \circ \sigma \circ \lambda)^n = \varepsilon \circ \partial^n \circ \sigma \circ \lambda \tag{1}$$

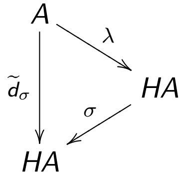
$$\sigma(x^{[n]}) = x^{[n]} \tag{2}$$

$$\mathrm{ord}(h) \leq \mathrm{ord}(\sigma(h)) \tag{3}$$

If $\sigma$ is bijective, then we call $\sigma$ a *Hurwitz automorphism* of $HA$. The set of all Hurwitz automorphisms of $HA$ will be denoted by $\mathrm{Haut}A$.

**Lemma.** Let $\sigma \in \mathrm{Haut}A$, $a \in A$, $k \in \mathbf{N}$, $h \in HA$ and define $d_\sigma$ by $d_\sigma := \varepsilon \circ \partial \circ \sigma \circ \lambda$. Then

1. $d_\sigma$ is a derivation on $A$ and $\sigma \circ \lambda = \widetilde{d_\sigma}$, i.e., the diagram




commutes, and

2. $\sigma(ax^{[k]})(n) = \begin{cases} 0, & \text{if } n < k; \\ \binom{n}{k} d_\sigma^{n-k}(a), & \text{if } n \geq k. \end{cases}$

**Theorem.** Let $\sigma \in \mathrm{Haut} A$ and $h \in HA$. Then for each $n \in \mathbf{N}$,

$$\sigma(h)(n) = \sum_{k=0}^{n} \binom{n}{k} d_\sigma^{n-k}(h(k)),$$

where $d_\sigma$ is the derivation given by $d_\sigma := \varepsilon \circ \partial \circ \sigma \circ \lambda$.

- For any $d \in \mathrm{Der} A$, $h \in HA$, and $n \in \mathbf{N}$, define

$$\sigma_d : HA \to HA$$

by

$$\sigma_d(h)(n) = \sum_{k=0}^{n} \binom{n}{k} d^{n-k}(h(k)).$$

- Note that for the zero derivation $0_A$ on $A$, $\sigma_{0_A} = \mathrm{id}_{HA}$.

**Theorem.** For any $g, h \in HA$ and $d \in \mathrm{Der} A$:

- $\sigma_d(g + h) = \sigma_d(g) + \sigma_d(h)$
- $\sigma_d(g * h) = \sigma_d(g) * \sigma_d(h)$

so that $\sigma_d$ is a Hurwitz endomorphism of $HA$.

**Lemma.** If $d_1, d_2 \in \mathrm{Der} A$ with $d_1 \circ d_2 = d_2 \circ d_1$, then $\sigma_{d_1} \circ \sigma_{d_2} = \sigma_{d_1 + d_2} = \sigma_{d_2} \circ \sigma_{d_1}$.

**Theorem.** For any $d \in \mathrm{Der} A$, $\sigma_d$ is a Hurwitz automorphism of $HA$ and $\sigma_d^{-1} = \sigma_{-d}$.

**Theorem.** Let $\Phi : \mathrm{Der}A \to \mathrm{Haut}A$ and $\Psi : \mathrm{Haut}A \to \mathrm{Der}A$ be defined by $\Phi(d) = \sigma_d$ and $\Psi(\sigma) = d_\sigma$ where $d_\sigma := \varepsilon \circ \partial \circ \sigma \circ \lambda$. Then $\Phi \circ \Psi = \mathrm{id}_{\mathrm{Haut}A}$ and $\Psi \circ \Phi = \mathrm{id}_{\mathrm{Der}A}$. Thus $\mathrm{Der}A$ and $\mathrm{Haut}A$ are isomorphic sets.

**Corollary.** For any ring $A$, $\mathrm{Der}A \cong \mathrm{Comor}A \cong \mathrm{Haut}A$.

**Seidenberg automorphisms over $A$**

If $\mathbf{Q} \subseteq A$, a *Seidenberg automorphism* over $A$ is an automorphism $E$ of $A[[t]]$ leaving $t$ fixed and reducing to the identity modulo $t$.

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Such an $E$ restricted to $A$ gives a derivation on $A$, and conversely every derivation on $A$ extends uniquely to a Seidenberg automorphism over $A$.

Further, if $\mathbf{Q} \subseteq A$ then

$$\psi : A[[t]] \to HA$$

is an isomorphism, and if $E$ is a Seidenberg automorphism over $A$ and $d$ is the derivation on $A$ from $E$, then the diagram

$$\begin{array}{ccc} A[[t]] & \xrightarrow{\psi} & HA \\ {\scriptstyle E}\downarrow & & \downarrow{\scriptstyle \sigma_d} \\ A[[t]] & \xrightarrow{\psi} & HA \end{array}$$

commutes. Thus, a Hurwitz automorphism is the analog of a Seidenberg automorphism for any ring $A$.