Constructive approaches to Kovacic's reduced forms of linear differential systems

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Introduction (1)

$$[A] \ : \quad Y' = A(x)Y, \qquad A \in \mathcal{M}_n(k) \text{ (take } k = C(x)\text{)}.$$

Gauge transformation : $P \in GL_n(C(x))$,

$$Y = PZ \longrightarrow Z' = BZ, B = P^{-1}(-P' + AP)$$

Two such systems are called *equivalent* (or isomorphic) over k.

Our goals :

- among all systems equivalent to $[A]$, define a "reduced form"
- find how to characterize and compute such a "reduced form"
- apply this to integrability in hamiltonian systems

Introduction (2)

Reduction of linear differential systems : a very simple example

Y'=AY with

```
> A:=array( 1 .. 2, 1 .. 2, [( 1, 1 ) = (2*x-1)/((x-1)*x^2), ( 1, 2
) = (x-1)^2/x, ( 2, 2 ) = -(2*x-1)/((x-1)*x^2), ( 2, 1 ) = -(x^2
-3*x+1)/(x^3*(x-1)^2) ] );
```

$$A := \begin{bmatrix} \frac{2x-1}{(x-1)x^2} & \frac{(x-1)^2}{x} \\ -\frac{x^2-3x+1}{x^3(x-1)^2} & -\frac{2x-1}{(x-1)x^2} \end{bmatrix} \qquad (1)$$

```
> v:=Mratsolde(A,x);  ## Rational Solutions, from Barkatou
```

$$v := \begin{bmatrix} (x-1)\,c_1 & \frac{c_1}{x} \end{bmatrix} \qquad (2)$$

```
> P:=augment(v,vector([0,1/v[1]]));
```

$$P := \begin{bmatrix} (x-1)\,c_1 & 0 \\ \frac{c_1}{x} & \frac{1}{(x-1)\,c_1} \end{bmatrix} \qquad (3)$$

Letting Y=P.Z, we have Z'=BZ with

```
> B:=inverse_gauge_change(A,P);
```

$$B := \begin{bmatrix} 0 & \frac{1}{c_1^2\,x} \\ 0 & 0 \end{bmatrix} \qquad (4)$$

And the system cannot be simplified further without adding a new transcendental function : ln(x)

Introduction (3)

$$[A] : \quad Y' = A(x)Y, \qquad A \in \mathcal{M}_n(k) \text{ (take } k = C(x)).$$

Differential Galois group G with Lie algebra $\mathfrak{g}$.

Definition

We say that $[A]$ is in `reduced form` if $A \in \mathfrak{g}(k)$.

In this talk, we explain

- Why this notion means `"simplest form obtained without introducing new transcendents"`
- Why this notion is useful, somehow *concrete* and computable (mostly)
- How to apply it to integrability (cf also G. Duval's talk yesterday)

Result

Given a reduced (integrable) m-th variational equation, we reduce the $(m+1)$-th variational.

I. Reduced Forms of Linear Differential Systems

Ingredient #1 : Differential Galois group

$$Y' = A(x)Y. \tag{1}$$

$k = \mathbb{C}(x)$ coefficient field
Picard-Vessiot Ext. : $K = k(U_1)$, U_1 fundamental solution matrix
Differential Galois Group : $G := \mathrm{Aut}_\partial(K/k)$

$$\mathrm{Aut}_\partial(K/k) := \{\sigma \in \mathrm{Aut}(K) \,:\, \sigma|_k \equiv \mathrm{id}_k \text{ and } \sigma \circ \partial \equiv \partial \circ \sigma\}$$

G is a group of matrices (linear algebraic group).

G stabilizes the *ideal of differential relations* between solutions

$\mathfrak{g}$, Lie algebra of G : tangent space of G at Id.

$\mathfrak{g}$ measures the transcendence of K over k.

$$\dim_{\mathbb{C}} \mathfrak{g} = \mathrm{trdeg}(K/k).$$

Examples of linear algebraic groups and Lie algebras

G linear algebraic group, $\mathfrak{g}$ its Lie algebra.
$N \in \mathfrak{g} \iff Id + \epsilon N \in G(C[\epsilon]),\ \epsilon^2 = 0.$

- $M \in \mathrm{SL_n(C)}$: $\det(M) = 1 \longrightarrow N \in \mathfrak{sl}_n(\mathrm{C})$: $Tr(N) = 0.$
- $M \in \mathrm{Sp_{2n}(C)}$: $M^T \cdot J \cdot M = J \longrightarrow N \in \mathfrak{sp}_n(\mathrm{C})$: $N^T J + JN = 0.$
  $J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$
- $\mathbb{G}_a = \{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \in C \} \longrightarrow \mathfrak{g}_a = Span_C \{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \}.$
- $\mathbb{G}_m = \{ \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}, a \in C^* \} \longrightarrow \mathfrak{g}_m = Span_C \{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \}$
- $M \in \mathrm{SO(3)}$: $M^T M = Id$ & $\det(M) = 1 \longrightarrow N \in \mathfrak{so}(3)$: $N^T + N = 0$
  Generators : $N_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $N_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, $N_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Ingredient #2 : Lie Algebra associated to $A \in \mathcal{M}_n(k)$

Decompose the matrix $A \in \mathcal{M}_n(C(x))$ as

$$A := \sum_{i=1}^{r} a_i(x) M_i, \qquad M_i \in \mathcal{M}_n(C)$$

where the $a_i \in C(x)$ are linearly independent over C.
The C-vector space generated by the M_i is unique.

$\longrightarrow$ **Magnus, Wei-Norman 63/64**

Definition

The Lie algebra $Lie(A)$ generated by $M_1, \ldots, M_r$ will be called the *Lie algebra associated to* $A(x)$

i.e generated by the M_i and their iterated Lie brackets

Example

$$A = \begin{pmatrix} 0 & x & 0 \\ -x & 0 & x^2 \\ 0 & -x^2 & 0 \end{pmatrix} \quad M_i = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad Lie(A) = \mathfrak{so}(3)$$

Lie Algebra associated to a Matrix (2)

$A := \sum_{i=1}^{r} a_i(x) M_i,\ Lie(A) := Lie(M_1, \ldots, M_r).$

Theorem (Kovacic,Kolchin)

$$\mathtt{Lie}(\mathtt{Y}' = \mathtt{AY}) \subset \mathtt{Lie}(\mathtt{A})$$

Example

$$A = \begin{pmatrix} 0 & x & 0 \\ -x & 0 & x^2 \\ 0 & -x^2 & 0 \end{pmatrix} \quad M_i = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$Lie(A) = \mathfrak{so}(3)$ so $Lie(Y' = AY) \subset \mathfrak{so}(3)$.

Definition

A system $Y' = A(x)Y$ is in *reduced form* if the Lie algebra associated to $A(x)$ is the Lie algebra $Lie(Y' = AY)$ of the differential Galois group.

Existence : Kovacic's theorem (non constructive)

$A := \sum_{i=1}^{r} a_i(x) M_i$, $Lie(A) := Lie(M_1, \ldots, M_r)$.
$G = Gal(Y' = AY)$, $\mathfrak{g} = Lie(G) = Lie(Y' = AY)$.

Theorem (Kovacic,Kolchin)

$$\mathtt{Lie}(\mathtt{Y}' = \mathtt{AY}) \subset \mathtt{Lie}(\mathtt{A})$$

Remark : $k = C(x)$ is a C_1-field.

Theorem (Kovacic,Cassidy, Kolchin ?)

Let $\mathfrak{h} = Lie(A)$, $H = exp(\mathfrak{h})$ *connected group.*
Then $G \subset H$ *and* $\exists P \in H(\overline{k})$ *such that,*
letting $Y = PF$ *and* $\tilde{A} = P^{-1}(AP - P')$, *we have* $\mathfrak{g} = Lie(\tilde{A})$, *i.e*

$\tilde{A}$ is a reduced form of A

So a *reduced form* always exists ! how to find it ? ? →back to example

Aparicio-Compoint-Weil criterion for reduced form

$$Y' = AY, \quad V = Sol(Y' = AY) \text{ C-space.}$$
$$A := \sum_{i=1}^{r} a_i(x) M_i, \quad Lie(A) := Lie(M_1, \ldots, M_r).$$
$$G = Gal(Y' = AY), \quad \mathfrak{g} = Lie(G) = Lie(Y' = AY).$$

Tensor Construction $Const$: iteration of $\otimes, \oplus, \star, Sym, \Lambda$.
May construct system $Y' = \mathfrak{const}(A)Y$ whose solution space is $Const(V)$.

$\mathfrak{const}$ vector-space morphism, $Const$ group morphism.

Theorem (Aparicio-Compoint-JA. W.)

Assume G reductive + unimodular. System $[A]$ is in reduced form $\Longleftrightarrow$ any `rational` *solution of any $[\mathfrak{const}(A)]$ has* `constant coefficients` .

Note : similar criterion if G not reductive (semi-invariants) + algorithm.

A-C-W criterion for reduced form (proof sketch)

Theorem (Aparicio-Compoint-JA. W.)

Assume G reductive + unimodular. System $[A]$ is in reduced form $\Longleftrightarrow$ any `rational` *solution of any $[\mathfrak{const}(A)]$ has* `constant coefficients` *.*

$$A = \sum a_i(x) N_i, \text{ reduced form :}$$
$$\mathfrak{g} = Span_C(N_1, \ldots, N_r) = \{N \in \mathcal{M}_n(C) | \forall j, D_N(P_j) = 0\}$$

$$D_{N_i}(P) = 0$$
$$\uparrow\downarrow$$
$$\mathfrak{sym}^m(N_i) \cdot v_P = 0$$
$$\uparrow\downarrow$$
$$\sum_i a_i(x) \mathfrak{sym}^m(N_i) \cdot v_P = 0$$
$$\uparrow\downarrow$$
$$\mathfrak{sym}^m(A).v_P = 0 = v_P'$$

so v_P solution in C^d (not just $C(x)$) of $Y' = \mathfrak{sym}^m(A).Y$.

and all invariants are polynomials in the P_i

Back to the log example

Reduction of linear differential systems : a very simple example

Y'=AY with

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And the system cannot be simplified further without adding a new transcendental function : ln(x)

Another $SO(3)$ example

$$A = \begin{bmatrix} \frac{2x^2-2x+1}{x(-1+x^2)} & \frac{5x-3x^3+2x^5-1+x^2-x^4}{(x-1)x} & -\frac{2x^4-3x^3+x+2}{x^2(x-1)^2} \\ -\frac{(2x-1)x}{(x+1)(-1+x^2)} & -\frac{x^5-x^4-x^3+x^2+4x-1}{-1+x^2} & \frac{x^4-2x^3+2x^2+1}{(x+1)(x-1)^2 x} \\ -\frac{x^2(x-1)}{x+1} & -(x-1)x\left(1-x^2+x^4\right) & \frac{x^5-2x^4+x^3+2x-1}{(x-1)x} \end{bmatrix}$$

May show (Aparicio & Compoint & J-A W. 2011) that the gauge change $Y = PZ$ with

$$P = \begin{bmatrix} \frac{(x-1)x}{-1+x^2} & 1 & 0 \\ 0 & -1+x^2 & -x^{-1} \\ 0 & 0 & \frac{1}{(x-1)x} \end{bmatrix}$$

gives $B = P[A]$ with $B = \begin{bmatrix} 0 & x & 1 \\ -x & 0 & x^2 \\ -1 & -x^2 & 0 \end{bmatrix}$

Matrix B depends only on 3 coefficients : "sparse".

II. Symplectic linear differential systems and Integrability

Complete Integrability of Hamiltonian Systems.

A *Hamiltonian system* over domain U of $\mathbb{R}^{2n}$:

$$(S) : \begin{cases} \dot{q}_i &= \quad \frac{\partial H}{\partial p_i}(q,p) \\ \dot{p}_i &= - \frac{\partial H}{\partial q_i}(q,p) \end{cases} \qquad i = 1, \ldots, n$$

where $H : U \longrightarrow \mathbb{R}$ is the *Hamiltonian* function.

Condensed form : $\mathrm{x}'(t) = J\,\nabla H(\mathrm{x}(t))$ where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

First Integral : function $I(p,q)$ that remains constant along solutions.

Liouville (complete) integrability : "sufficiently many good first integrals".

Complete Integrability (Hamiltonian) (2).

$$(S): \qquad \dot{\mathrm{x}}(t) = J\,\nabla H(\mathrm{x}(t)) = X_H(\mathrm{x}(t))$$

Poisson Bracket :

$$\{G_1, G_2\} = \sum_{i=1}^{n} \left(\frac{\partial G_1}{\partial p_i}\frac{\partial G_2}{\partial q_i} - \frac{\partial G_1}{\partial q_i}\frac{\partial G_2}{\partial p_i} \right) = < \nabla G_1(\mathrm{x}), J\nabla G_2(\mathrm{x}) >$$

First integral G of (S) defined by $\{H, G\} = 0$

Definition

(S) is *completely integrable* (Liouville) in a class $\mathcal{F}$ of functions if it admits n first integrals $G_1 = H, G_2, \ldots, G_n \in \mathcal{F}$ such that

the G_i are functionally independent, and

in involution : $\{G_i, G_j\} = 0$

Variational Equation

For each particular solution $\phi(t, \boldsymbol{z})$ of

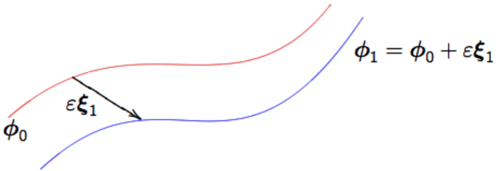
$$(S) : \left\{ \frac{d\boldsymbol{q}}{dt} = \frac{\partial H}{\partial \boldsymbol{p}}(\boldsymbol{x}), \quad \frac{d\boldsymbol{p}}{dt} = -\frac{\partial H}{\partial \boldsymbol{q}}(\boldsymbol{x}) \right.$$

Def : VE^k_ϕ is the differential system satisfied by $\boldsymbol{\xi}_k := \partial^k \phi / \partial \boldsymbol{z}^k$.

Taylor expansion along ϕ_0

For $k = 1$: the variational system is linear :

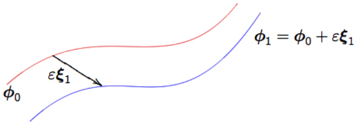
$$\boxed{\dot{\boldsymbol{\xi}}_1 = A_1 \boldsymbol{\xi}_1,} \qquad A_1 := \mathrm{Jac}_\phi(X) \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C}(\phi)). \qquad (\mathrm{VE}^1_\phi)$$



(S) is Hamiltonian , (VE^1_ϕ) is : $\dot{\boldsymbol{\xi}}_1 = J\,\mathrm{Hess}_\phi(H)\,\boldsymbol{\xi}_1$.

Hamiltonian systems and linearization (2)

$$(S) : \left\{ \frac{d\boldsymbol{q}}{dt} = \frac{\partial H}{\partial \boldsymbol{p}}(\boldsymbol{x}), \quad \frac{d\boldsymbol{p}}{dt} = -\frac{\partial H}{\partial \boldsymbol{q}}(\boldsymbol{x}) \right.$$



Theorem (Morales & Ramis)

(S) is Liouville integrable only if Lie$((\mathrm{VE}^1_\phi))$ is abelian.

Our tools to apply this theorem : Aparicio & Weil, J. Symb. Comp. 2011

- Compute a reduced form of $((\mathrm{VE}^1_\phi))$: if not abelian then stop
- If abelian, this gives a simpler form : use that Gauge transformation to simplify higher VEs.

Higher Variational Equations

$$(S): \ \dot{\boldsymbol{x}} = X_H(\boldsymbol{x}).$$

Flow : $\phi(t, \boldsymbol{z}) = \phi_0(t) + \xi_1(\boldsymbol{z} - \phi_0) + \frac{1}{2}\xi_2(\boldsymbol{z} - \phi_0)^2 + \ldots$

Variational equations of order m for $\xi_m := \frac{\partial^m \phi(t, \boldsymbol{z})}{\partial \boldsymbol{z}^m}$:

$$\begin{aligned}
(\mathrm{VE}^1_{\phi_0}) &: \quad \dot{\xi}_1 = (d_{\phi_0} X_H)\xi_1 \\
(\mathrm{VE}^2_{\phi_0}) &: \quad \dot{\xi}_2 = d^2_{\phi_0} X_H(\xi_1, \xi_1) + (d_{\phi_0} X_H)\xi_2 \\
(\mathrm{VE}^3_{\phi_0}) &: \quad \dot{\xi}_3 = d^3_{\phi_0} X_H(\xi_1, \xi_1, \xi_1) + 3d^2_{\phi_0} X_H(\xi_1, \xi_2) + (d_{\phi_0} X_H)\xi_3
\end{aligned}$$

$\left(\mathrm{VE}^m_{\phi_0}\right)$ may be seen as linear differential systems $\left(\mathrm{LVE}^m_{\phi_0}\right)$,

$$(\mathrm{LVE}^3_{\phi_0}) \quad : \quad \begin{pmatrix} (\mathrm{Sym}^3 \xi_1)^{\cdot} \\ (\xi_1 \bullet \xi_2)^{\cdot} \\ \dot{\xi}_3 \end{pmatrix} = \begin{pmatrix} \mathrm{sym}^3 A & 0 & 0 \\ B_{2,3} & \mathrm{sym}^2 A & 0 \\ B_{1,3} & B_{1,2} & A \end{pmatrix} \begin{pmatrix} \mathrm{Sym}^3 \xi_1 \\ \xi_1 \bullet \xi_2 \\ \xi_3 \end{pmatrix}$$

For $n = 4$ we have Size = 4+10+20=34 (!)

Morales-Ramis-Simo

Theorem (Morales & Ramis & Simo)

(S) is Liouville integrable only if all *Lie*$(\mathrm{VE}^{k}_{\phi_0})$ *are abelian.*

Our approach (see also G. Duval yesterday) : compute reduced forms iteratively.

Aparicio & Weil 2010, 2012 (in progress)

Partial reduction for $(\mathrm{LVE}_\phi^{m+1})$

Variational Equation of order $m+1$:

$$(VE_{m+1}) : Y' = A_{m+1} Y \quad \text{with} \quad A_{m+1} = \begin{pmatrix} sym^{m+1}(A_1) & 0 \\ B & A_m \end{pmatrix}$$

Q_m reduces VE_m $m \geq 1$ $\rightarrow$ P_{m+1} (partially) reduces VE_{m+1} with :

$$P_{m+1} := \begin{pmatrix} Sym^{m+1}(Q_1) & 0 \\ 0 & Q_m \end{pmatrix}$$

So we reduce for free diagonal part of $(VE_{m+1}) : Y' = P_{m+1}[A_{m+1}]Y$

$$A_{m+1} = \begin{pmatrix} sym^{m+1}(A_{1,red}) & 0 \\ B & A_{1,red} \end{pmatrix}$$

Partial reduction for $(\mathrm{LVE}_\phi^{m+1})$ (continued)

We know :

- Q_1 reduces $A_1 \rightsquigarrow \quad Q_1[A_1] \quad \in Lie(Y' = A_1 Y)$
- Q_m reduces $A_m \rightsquigarrow \quad Q_m[A_m] \in Lie(Y' = A_m Y)$

Then :

- $Sym^{m+1} Q_1[sym^{m+1} A_1] \quad \in Lie(Y' = sym^{m+1} A_1 Y)$
- $P_{m+1} \; := \begin{bmatrix} \mathrm{Sym}^{m+1}\mathrm{Q}_1 & 0 \\ 0 & \mathrm{Q}_m \end{bmatrix}$ partially reduces A_{m+1}

we have

$$\mathrm{P}_{m+1}[A_{m+1}] := \begin{bmatrix} \mathrm{Sym}^{m+1}\mathrm{Q}_1[\mathrm{sym}^{m+1}\mathrm{A}_1] & 0 \\ \mathrm{B} & \mathrm{Q}_m[\mathrm{A}_m] \end{bmatrix}$$

where diagonal blocks are reduced.

We now want to reduce block B

Reduction of the lower triangular part : block B

Consider

$$\mathtt{A} := P_{m+1}[A_{m+1}] = \mathtt{A}_{\mathtt{diag}} + \mathtt{A}_{\mathtt{sub}} = \begin{bmatrix} \mathtt{Sym}^{\mathtt{m+1}}\mathtt{Q}_1[\mathtt{sym}^{\mathtt{m+1}}\mathtt{A}_1] & 0 \\ \mathtt{B} & \mathtt{Q}_{\mathtt{m}}[\mathtt{A}_{\mathtt{m}}] \end{bmatrix}$$

Let $\mathfrak{h}_{\mathtt{diag}} := Lie(A)_{\mathtt{sub}}$ and $\mathfrak{h}_{\mathtt{sub}} := Lie(A)_{\mathtt{sub}}$. We have

$$\mathtt{A}_{\mathtt{sub}} := \beta_1 B_1 + \ldots + \beta_d B_d \text{ where } \left\{ \begin{array}{ll} \beta_i \in \boldsymbol{k} & \\ (B_i)_{i=1}^{d} \quad \text{basis of} & \mathfrak{h}_{\mathtt{sub}} \end{array} \right\}$$

Lemma

Let $[A_{diag}\, , \mathtt{B}_1] = \gamma_1 \mathtt{B}_1 + \gamma_2 B_2 + \ldots + \gamma_d B_d$ *with* $\gamma_i \in \boldsymbol{k}$.
If $y' = \gamma_1 y + \beta_1$ *admits a solution* $g_1 \in \boldsymbol{k}$, *then*

$$\mathtt{P} := \exp(\mathtt{g}_1 \mathtt{B}_1) \text{ satisfies } \mathtt{P}[\mathtt{A}] = \mathtt{A}_{\mathtt{diag}} + \tilde{\beta}_2 \mathtt{B}_2 + \ldots + \tilde{\beta}_d B_d$$

i.e. $P[A]$ *does NOT have any term involving* $\mathtt{B}_1$.

Reduction of the lower triangular part : block B (continued)

$$\mathtt{A} := P_{m+1}[A_{m+1}] = \mathtt{A}_{\mathtt{diag}} + \mathtt{A}_{\mathtt{sub}} = \begin{bmatrix} \mathtt{Sym}^{\mathtt{m+1}}\mathtt{Q}_1[\mathtt{sym}^{\mathtt{m+1}}\mathtt{A}_1] & 0 \\ \mathtt{B} & \mathtt{Q}_{\mathtt{m}}[\mathtt{A}_{\mathtt{m}}] \end{bmatrix}$$

Lemma :

Let $[A_{diag}\,,\,\mathtt{B}_1] = \gamma_1\mathtt{B}_1 + \gamma_2 B_2 + \ldots + \gamma_d B_d$ with $\gamma_i \in \boldsymbol{k}$.
If $y' = \gamma_1 y + \beta_1$ admits a solution $g_1 \in \boldsymbol{k}$, then

$$\mathtt{P} := \exp(\mathtt{g}_1\mathtt{B}_1) \text{ satisfies } \mathtt{P}[\mathtt{A}] = \mathtt{A}_{\mathtt{diag}} + \tilde{\beta}_2\mathtt{B}_2 + \ldots + \tilde{\beta}_d B_d$$

i.e. $P[A]$ does NOT have any term involving $\mathtt{B}_1$.

Let $\mathfrak{h}_{\mathtt{diag}} := Lie(A)_{\mathtt{sub}} = span(A_1, \ldots, A_r)$. Maps $[A_i, \bullet] : \mathfrak{h}_{\mathtt{sub}} \to \mathfrak{h}_{\mathtt{sub}}$ commute so may choose B_i such that $[A_i, \bullet]$ are in simultaneous triangular form.

$\longrightarrow$ Conjecture : gives effective Morales-Ramis-Simo theorem .