Hrushovski's Algorithm for Computing Galois Groups of Linear Differential Equations

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Basic Terminology: Galois Groups

Notation: $\delta = \frac{d}{dt}$, $\quad Y = (y_1, y_2, \cdots, y_n)^t$.

Consider matrix equations:

$$\delta(Y) = AY, \quad A \in \mathrm{Mat}_n(\bar{\mathbb{Q}}(t)).$$

K: Picard-Vessiot extension field $/\bar{\mathbb{Q}}(t)$.

Galois group

$$G = \{\ \bar{\mathbb{Q}}(t)\text{-diff. automorphisms of } K\}$$

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$\mathcal{F}$: a fundamental matrix, $X = (x_{i,j})$.

Set

$$\mathfrak{M} = \left\{ P(X) \in \bar{\mathbb{Q}}(t)[X] \mid P(\mathcal{F}) = 0 \right\}.$$

Remark: $\mathfrak{M}$ consists of all "alg. relations" of $\mathcal{F}$.

Group action of $\mathrm{GL}_n(\bar{\mathbb{Q}})$ on $\bar{\mathbb{Q}}(t)[X]$:

$$g \cdot P = P(Xg), \quad \forall\, g \in \mathrm{GL}_n(\bar{\mathbb{Q}}), P \in \bar{\mathbb{Q}}(t)[X].$$

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Key Point of Hrushovski's Algorithm

Theorem: For any alg. group $N \subseteq \mathrm{GL}_n(\bar{\mathbb{Q}})$, there are alg. groups H s.t.

$$H^t \trianglelefteq N^\circ \leq N \leq H$$

where

- ⋆ N°: identity component of N;
- ⋆ H^t: alg. subgr. generated by unipotent elements of H.

Moreover there is H bounded by an integer d only depending on n.

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> H is said to be bounded by d if there are $f_1, \cdots, f_m \in \bar{\mathbb{Q}}[X]$ with $\deg(f_i) \leq d$ s.t.
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> $$H = \mathrm{Zero}(f_1, \cdots, f_m) \cap \mathrm{GL}_n(\bar{\mathbb{Q}}).$$

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A rough estimate,

$$d = O\left(n^{n^{n^{n^{n^{n^{3}}}}}}\right).$$

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Remark: Let H be a proto-Galois group and $\chi_1, \cdots, \chi_\ell$ be a basis of character group $X(H^\circ)$ of H° and

$$\phi = (\chi_1, \cdots, \chi_\ell)^t : H^\circ \to \left(\bar{\mathbb{Q}}^*\right)^\ell.$$

Then $G^\circ = \phi^{-1}(S)$, $S = \phi(G^\circ)$: a subtorus of $\left(\bar{\mathbb{Q}}^*\right)^\ell$.

Hrushovski's Algorithm

- Compute a proto-Galois group H and therefore H°.
- Construct a morphism $\phi : H^\circ \to \left(\bar{\mathbb{Q}}^*\right)^m$ for some m s.t. $\ker(\phi) = H^t$, and compute a subtorus S of $\phi(H^\circ)$ s.t.

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From these, we obtain G°.
- Compute a finite Galois extension $L/\bar{\mathbb{Q}}(t)$ and $P_1, \cdots, P_s \in L[X]$ s.t.

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For any $\rho \in \mathbb{Z}_{\geq 0}$, set

$$\mathfrak{M}_\rho = \left\{ P \in \bar{\mathbb{Q}}(t)[X] \mid \deg(P) \leq \rho \ \& \ P(\mathcal{F}) = 0 \right\}.$$

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Remark:

- $\mathfrak{M}_\rho$ is a $\bar{\mathbb{Q}}(t)$-vector space of finite dimension.
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Lemma: Fixing ρ, we can compute a basis of $\mathfrak{M}_\rho$ and then obtain G_ρ.

Theorem (Feng2013): Let d be the bound of proto-Galois groups. Then G_d is a proto-Galois group.

Remark: The above results enable us to compute a proto-Galois group.

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Bessel Equation:

$$\delta \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{4t^2} - 1 & -\frac{1}{t} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Fundamental matrix:

$$\mathcal{F} = \begin{pmatrix} sin(t)/\sqrt{t} & cos(t)/\sqrt{t} \\ \left(sin(t)/\sqrt{t}\right)' & \left(cos(t)/\sqrt{t}\right)' \end{pmatrix}$$

$$\mathfrak{M}_1 = \mathrm{Span}_{\bar{\mathbb{Q}}(t)} \left\{ \tfrac{1}{2t} x_{11} - x_{12} + x_{21},\ x_{11} + \tfrac{1}{2t} x_{12} + x_{22} \right\}$$

$$G_1 = \mathrm{stab}(\mathfrak{M}_1) = \left\{ \begin{pmatrix} c_{11} & -c_{12} \\ c_{12} & c_{11} \end{pmatrix} \middle| c_{11}^2 + c_{12}^2 \neq 0 \right\}$$

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Constructing morphism ϕ

Assume $H \subseteq \mathrm{GL}_n(\bar{\mathbb{Q}})$ is a connected alg. subgroup.

Theorem(Feng2013): There are generators of $X(H)$ that are represented by polynomials with degree $\leq \kappa$, an integer only depending on n. Moreover

$$\kappa = O\left(n^{n^{n^{n^3}}}\right).$$

Using the bound κ, compute a set of generators of $X(G_d^\circ)$, say $\chi_1, \cdots, \chi_\ell$ and set

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Computing Subtorus S

Lemma: There is $\alpha \in \mathrm{Zero}(\mathfrak{M}_d) \cap \mathrm{GL}_n(\overline{\mathbb{Q}(t)})$ s.t. $\alpha^{-1}\mathcal{F} \in G_d^\circ(\bar{K})$ and

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How to compute α?

- pick any $\tilde{\alpha} \in \mathrm{Zero}(\mathfrak{M}_d) \cap \mathrm{GL}_n(\overline{\mathbb{Q}(t)})$, then $\tilde{\alpha}^{-1}\mathcal{F} \in G_d\left(\bar{K}\right)$;
- Decompose $G_d = G_d^\circ \cup C_1 G_d^\circ \cup \cdots \cup C_m G_d^\circ$ with $C_i \in \mathrm{GL}_n(\bar{\mathbb{Q}})$ and find C_i s.t. $C_i\tilde{\alpha}^{-1}\mathcal{F} \in G_d^\circ\left(\bar{K}\right)$;
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Using the method in Compoint and Singer (1999), one can determine the above Galois group.

Example (continued)

Bessel Equation and a fundamental matrix:

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G_1 is a torus and

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is a basis of $X(G_1)$.

Let $\phi = \left(\frac{1}{2}\left(x_{11} - ix_{12} + ix_{21} + x_{22}\right), \ \frac{1}{2}\left(x_{11} + ix_{12} - ix_{21} + x_{22}\right)\right)^t$.

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$$\left\{ \frac{1}{2}\left(x_{11} - ix_{12} + ix_{21} + x_{22}\right), \ \frac{1}{2}\left(x_{11} + ix_{12} - ix_{21} + x_{22}\right) \right\}$$

is a basis of $X(G_1)$.

Let $\phi = \left(\frac{1}{2}\left(x_{11} - ix_{12} + ix_{21} + x_{22}\right), \ \frac{1}{2}\left(x_{11} + ix_{12} - ix_{21} + x_{22}\right)\right)^t$.

Pick $\alpha = \begin{pmatrix} 0 & 1 \\ 1 & -1/2t \end{pmatrix} \in \mathrm{Zero}(\mathfrak{M}_1) \cap \mathrm{GL}_2\left(\overline{\mathbb{Q}(t)}\right)$

Set $h_1 = \chi_1\left(\alpha^{-1}\mathcal{F}\right),\ h_2 = \chi_2\left(\alpha^{-1}\mathcal{F}\right)$. Then

$$\left.\begin{aligned} h_1' &= (i - 1/2t)\, h_1 \\ h_2' &= (-i - 1/2t)\, h_2 \end{aligned}\right\} \Longrightarrow h_1 h_2 = \frac{1}{t}.$$

It implies that

$$S = \mathrm{Gal}\left(\overline{\mathbb{Q}(t)}(h_1, h_2) \big/ \overline{\mathbb{Q}(t)}\right) = \left\{ \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix} \middle| \, c \in \bar{\mathbb{Q}}^* \right\}.$$

Then

$$G^\circ = \phi^{-1}(S) = \left\{ \begin{pmatrix} c_{11} & -c_{12} \\ c_{12} & c_{11} \end{pmatrix} \middle| \, c_{11}^2 + c_{12}^2 = 1 \right\}.$$

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Computing G

- Compute a finite Galois extension $L/\bar{\mathbb{Q}}(t)$ and $P_1, \cdots, P_s \in L[X]$ s.t.

$$\mathcal{F}G^\circ = \mathrm{Zero}(P_1, \cdots, P_s) \cap \mathcal{F}\mathrm{GL}_n(\bar{\mathbb{Q}}).$$

Assume that $f_1, \cdots, f_s$ satisfy $G^\circ = \mathrm{Zero}(f_1, \cdots, f_s) \cap \mathrm{GL}_n(\bar{\mathbb{Q}})$.

Lemma: There is $\beta \in \mathrm{Zero}(\mathfrak{M}_d) \cap \mathrm{GL}_n(\overline{\mathbb{Q}(t)})$ s.t.

$$\mathcal{F}G^\circ = \mathrm{Zero}(f_1(\beta^{-1}X), \cdots, f_s(\beta^{-1}X)) \cap \mathcal{F}\mathrm{GL}_n(\bar{\mathbb{Q}}).$$

Remark: Such β can be computed by a small modification of the argument used to find α.

Let L be the Galois closure of $\bar{\mathbb{Q}}(t)(\beta^{-1})/\bar{\mathbb{Q}}(t)$ and $P_i = f_i(\beta^{-1}X)$.

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$$G = \bigcup_{\sigma \in \mathrm{Gal}(L/\bar{\mathbb{Q}}(t))} \left\{ g \in \mathrm{GL}_n(\bar{\mathbb{Q}}) \mid \forall\, i, \sigma(P_i)(\mathcal{F}g) = 0 \right\} ?$$

Idea: Expand $\mathcal{F}$ as a formal power series at $t = c$ for some $c \in \bar{\mathbb{Q}}$. By Bertrand&Beukers(1985), one can compute $\ell \in \mathbb{Z}$ s.t. $\forall\, i, \sigma, g$

$$\sigma(P_i)(\mathcal{F}g) = 0 \text{ or } \mathrm{ord}_{t=c}\left(\sigma(P_i)(\mathcal{F}g)\right) \le \ell.$$

Let $\mathbf{c} = (c_{i,j})$ with $c_{i,j}$ indeterminates and

$$\mathcal{S}_\sigma = \left\{ \mathrm{coeff}(\sigma(P_i)(\mathcal{F}\mathbf{c}), t, j) \mid i = 1, \cdots, s, j = 0, \cdots, \ell \right\}.$$

Then $\mathcal{S}_\sigma$ is a system in $c_{i,j}$ and

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Example (continued)

$$G^\circ = \left\{ \begin{pmatrix} c_{11} & -c_{12} \\ c_{12} & c_{11} \end{pmatrix} \middle| \, c_{11}^2 + c_{12}^2 = 1 \right\} = \mathrm{Zero}(P_1, P_2, P_3) \cap \mathrm{GL}_2(\bar{\mathbb{Q}}).$$

where $P_1 = x_{11} - x_{22}, P_2 = x_{12} + x_{21}, P_3 = x_{11}^2 + x_{12}^2 - 1.$

Pick $\beta = \begin{pmatrix} \frac{i(1-t)}{2t} & \frac{t+1}{2t} \\ \frac{2t(t+1)+i(t-1)}{4t^2} & \frac{2it(t-1)-(t+1)}{4t^2} \end{pmatrix} \in \mathrm{Zero}(\mathfrak{M}_1) \cap \mathrm{GL}_2\left(\overline{\mathbb{Q}(t)}\right).$

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