Algebraic differential equations from covering maps

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The exponential function $\exp : \mathbb{C} \to \mathbb{C}^\times$ has a many-valued analytic inverse $\log : \mathbb{C}^\times \to \mathbb{C}^\times$ where log is well-defined only up to the adding an element of $2\pi i\mathbb{Z}$.

Treating exp and log as functions on functions does not help: If Δ is some connected Riemann surface and $f : \Delta \to \mathbb{C}^\times$ is analytic, then we deduce a "function" $\log(f) : \Delta \to \mathbb{C}$.

However, because $\log(f)$ is well-defined up to an additive constant, $\partial \log(f) := \frac{d}{dz}(\log(f))$ is a well defined function. That is, for $M = \mathscr{M}(U)$ the differential field of meromorphic functions we have a well-defined differential-analytic function $\partial \log : \mathbb{G}_m(M) \to \mathbb{G}_a(M)$.

Of course, one computes that $\partial \log(f) = \frac{f'}{f}$ is, in fact, differential algebraic.

The logarithmic derivative

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Kolchin's differential logarithm

If M is a differential field with field of constants C and G is an algebraic group over C, then

- we have a map of groups $\nabla : G(M) \to TG(M)$ given in coordinates by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n; \partial(x_1), \ldots, \partial(x_n))$,
- the tangent bundle splits as $TG = G \ltimes T_e G$ (where $T_e G$ is the tangent space to G at the identity) via $(g, v) \mapsto (g, d(g^{-1} \cdot)v)$, and
- the map $\partial \log_G : G(M) \to T_e G(M)$ given by sending g to the $T_e G$-component of $\nabla(g)$ via the splitting is differential algebraic and its fibres are torsors for $G(C)$.

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The usual $\partial \log$ is $\partial \log_{\mathbb{G}_m}$.

We are given:

- complex algebraic groups $K < G$,
- a complex submanifold $U \subseteq (G/K)(\mathbb{C})$,
- a discrete, Zariski dense subgroup $\Gamma < G(\mathbb{C})$ for which $\Gamma \curvearrowright U$,
- an algebraic variety X, and
- an analytic covering map $\pi : U \to X(\mathbb{C})$ expressing $X(\mathbb{C}) = \Gamma \backslash U$.

For example, we may take $G = \mathrm{PGL}_2$, $U = \mathfrak{h} = \{z \in \mathbb{C} \ : \ \mathrm{Re}(z) > 0\}$, $\Gamma = \Gamma_0(N)$ a congruence group in $\mathrm{PSL}_2(\mathbb{Z})$, $X = Y_0(N)$ a modular curve and $\pi = j_N : \mathfrak{h} \to Y_0(N)(\mathbb{C})$ the associated covering map.

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Generalized differential logarithm

As with the logarithm, the inverse function $\pi^{-1} : X \to (G/K)$ is locally analytic, but is only well-defined up to the action of Γ and in the same way if Δ is some connected Riemann surface and $f : \Delta \to X(\mathbb{C})$ is analytic, then we deduce a mutlivalued function $\pi^{-1}(f)$. Put another way, if $M = \mathscr{M}(\Delta)$ is the differential field of meromorphic functions on Δ, we have a multivalued analytic function $\pi^{-1} : X(M) \to (G/K)(M)$ well-defined up to the action of Γ.

If we had a differential algebraic map η defined on (G/K) so that $\eta(x) = \eta(y) \iff (\exists \gamma \in G(\mathbb{C}))[\gamma \cdot x = y]$, then we would have a well-defined differential analytic function χ defined by $\chi := \eta \circ (\pi^{-1})$.

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Proposition

If M is a differential field of characteristic zero with algebraically closed field of constants C, then the differential rational map $S : K \dashrightarrow K$ defined by $S(x) := (\frac{x''}{x'})' - \frac{1}{2}(\frac{x''}{x'})^2$ enjoys the property that $S(x) = S(y)$ if and only if there is some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(C)$ with $y = \frac{ax+b}{cx+d}$.

Another way of putting it, the map $S : \mathbb{P}^1 \to \mathbb{P}^1$ expresses $\mathbb{P}^1$ as the quotient $\mathrm{GL}_2(C) \backslash \mathbb{P}^1 = \mathrm{GL}_2(C) \backslash \mathrm{GL}_2 / K$ where K is the group of upper triangular matrices.

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Theorem (Poizat)

The theory of differentially closed fields of characteristic zero eliminates imaginaries. That is, if M is a differentially closed field of characteristic zero, Y is some differentially constructible set over M, and $E \subseteq Y \times Y$ is a differentially constructible equivalence relation, then there is a differentially constructible function η with domain Y having the property that $\eta(x) = \eta(y) \Longleftrightarrow xEy$.

Taking $Y = (G/K)$ and $xEy :\Longleftrightarrow (\exists g \in G(C))[g \cdot x = y]$, we obtain the existence of generalized Schwartzians.

Corollary

If $K < G$ are complex algebraic groups, then there is a differentially constructible function η on (G/K) having the property that for any differential field M with field of constants $\mathbb{C}$ and any two points $x, y \in (G/K)(M)$ one has $\eta(x) = \eta(y) \Longleftrightarrow (\exists \gamma \in G(\mathbb{C}))[\gamma \cdot x = y]$.

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Poizat's theorem is itself a consequence of Weil's theorem that the quotient of a constructible set by a constructible equivalence relation may be realized as a constructible set.

In general, for an algebraic variety Y over $\mathbb{C}$ and a natural number n, there is a truncated arc space $\mathscr{A}_n X \to X$ which represents $X(\mathbb{C}[\epsilon]/(\epsilon^{n+1}))$. For any differential field M with field of constants $\mathbb{C}$, we have a map $\nabla : X(M) \to \mathscr{A}_n X(M)$ corresponding to the map of rings $M \to M[\epsilon]/(\epsilon^{n+1})$ given by $x \mapsto \sum_{j=0}^{n} \frac{\partial^j(x)}{j!}\epsilon^j$.

There is a natural action $G \curvearrowright \mathscr{A}_n(G/K)$. By Weil's theorem on constructible quotients we obtain a constructible quotient map $\rho_n : \mathscr{A}_n(G/K) \to G\backslash\mathscr{A}_n(G/K)$.

Our differential constructible map χ may be taken to be $\rho_n \circ \nabla : (G/K) \to G\backslash\mathscr{A}_n(G/K)$ for $n \gg 0$.

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The generalized logarithm as a differential analytically constructible function

To say that $\chi : X \to G \backslash \mathscr{A}_n(G/K)$ is differential analytically constructible means that there there is an analytically constructible function $\widetilde{\chi} : \mathscr{A}_n(X) \to G \backslash \mathscr{A}_n(G/K)$ for which $\chi = \widetilde{\chi} \circ \nabla$.

Peterzil-Starchenko GAGA

Theorem (Peterzil-Starchenko)

If X is a complex algebraic variety and $Y \subseteq X(\mathbb{C})$ is an o-minimally definable, analytically constructible set, then Y is algebraically constructible.

Corollary

If there is some set $F \subseteq (G/K)(\mathbb{C})$ for which $\pi \upharpoonright F$ is o-minimally definable and surjective onto $X(\mathbb{C})$, then χ is differentially algebraic.

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The standard o-minimal structure for these purposes is $\mathbb{R}_{\mathrm{an,exp}}$, in which one is allowed all polynomials over the reals, the real exponential function, and real analytic functions restricted to compact boxes (and any other function built from these).

- $\exp_A : \mathbb{C}^g \to A(\mathbb{C})$ where A is an abelian variety of dimension g
- $j : \mathfrak{h} \to \mathbb{A}^1(\mathbb{C})$, the analytic j-function expressing $\mathbb{A}^1 = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{h}$
- More generally, theta functions and covering maps associated to moduli spaces of abelian varieties and for their universal families.

When does the Peterzil-Starchenko theorem apply?

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