Moser-based algorithms over Univariate and Bivariate (Differential) Fields

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Abstract

Let K be a commutative field of characteristic zero equipped with a derivation δ, that is a map $\delta : K \to K$ satisfying

$$\delta(f+g) = \delta f + \delta g \quad \text{and} \quad \delta(fg) = \delta(f)g + f\delta(g) \quad \text{for all } f, g \in K.$$

We denote by $\mathcal{C}$ its field of constants, V a K-vector space of dimension n, and A an element in $K^{n\times n}$. Set $\Delta = \delta - A$, then Δ is a δ- differential operator acting on V, that is, a $\mathcal{C}$-linear endomorphism of V satisfying the Leibniz condition:

$$\forall f \in K,\ v \in V\ \Delta(fv) = \delta(f)v + f\Delta(v).$$

A well known example in the univariate case is the linear singular system of differential equations (see, e.g., [3, 19, 12]) for which $\delta = x\frac{d}{dx}$ and $K = \mathbb{C}((x))$, the field of formal Laurent power series in x over the field of complex numbers. Let Y be an unknown n-dimensional column vector, then $\Delta Y = 0$ is rewritten as

$$x\frac{dY}{dx} = A(x)Y = x^{-p}(A_0 + A_1x + A_2x^2 + \dots)Y. \tag{1}$$

where $A_0 := A(0)$, referred to as the leading coefficient matrix, has a nonzero rank r. The non-negative integer p is called the *Poincaré rank* and its reduction to the minimal integer value, the *true Poincaré rank*, is essential in the classification of singularities (see, e.g., [15, 17]). If p is null then system (1) is said to be *regular singular*, that is, in any small sector, its solutions grow at most as an algebraic function. Otherwise, it is *irregular singular*. While Levelt investigated in [15] the existence of stationary sequences of free lattices, Moser defined two rational numbers:

$$m(A) = \max\,(0, p + \frac{r}{n}) \quad \text{and} \quad \mu(A) = \min\,\{\, m(T^{-1}\Delta T) \mid T \in GL(V)\}. \tag{2}$$

It follows that system (1) is regular whenever $\mu(A) \leq 1$. For $m(A) > 1$, Moser proved that $m(A) > \mu(A)$ if and only if the polynomial

$$\theta(\lambda) := \lim_{x\to 0}\ x^r\ det(\lambda I + \frac{A_0}{x} + A_1) \tag{3}$$

vanishes identically in λ. In this case, system (1) (resp. $A(x)$) is said to be Moser-reducible and $m(A)$ can be diminished by applying a coordinate transformation $T \in GL(V)$ of the form

$$T = (P_0 + P_1x)\ diag(1, \dots, 1, x, \dots, x)$$

where P_0, P_1 are constant matrices and $det(P_0) \neq 0$ [17, Theorems 1 and 2, pg 381].

Based on Moser's reduction criteria (3), Barkatou-Pfluegel [8] developed efficient algorithms which constitute a substantial portion of the formal reduction of system (1), that is the algorithmic procedure that computes the transformation w.r.t. which the matrix presentation of the operator is in canonical form, so that formal solutions can be constructed (see, e.g., Barkatou and/or Pfluegel [4, 6, 18]). Moser's notion of regularity and the mentioned algorithms, were generalized as well to linear functional matrix equations by Barkatou-Broughton-Pfluegel

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[9]. This gave rise to the package ISOLDE [7] which is written in the computer algebra system Maple and dedicated to the symbolic resolution of linear functional matrix equations, a particular case of which is system (1).
Another prominent example in the univariate case is for which δ is the zero map and $K = \mathbb{C}((\epsilon))$. Thereby, Δ is just a linear operator in the standard way and $A(\epsilon)$ is the widely studied perturbation of the constant matrix A_0 by an order ϵ, (see, e.g., Kato [14], Baumgartel [10]). Jeannerod-Pflugel [13] borrowed (3) from the theory of differential systems to investigate efficient algorithmic resolution of the perturbed algebraic eigenvalue-eigenvector problem.
However, although Moser-based algorithms, that is algorithm based on (3), have proved their efficiency and utility in the univariate case so far, they have not been considered yet for operators over bivariate fields. This is the interest of this talk.

In particular, Barkatou developed in [5] a Moser-based algorithm for the differential systems associated to $\delta = x\frac{d}{dx}$ and $K = \mathbb{Q}(x)$, the field of rational functions in x. We show that this algorithm can be well-generalized to two widely studied generalizations of system (1) over bivariate differential fields, namely the singularly-perturbed linear differential systems [2, 20] and completely integrable Pfaffian systems in two variables with normal crossings [1, 11].

Moreover, since Moser-based and Levelt's algorithms serve the same utility, i.e. rank reduction of system (1), it is natural to question their comparison. We also discuss results in this direction.

Keywords
Linear systems of partial differential equations, Singularly-Perturbed Linear Differential Systems, Formal solutions, Moser-based reduction, Computer algebra.

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