

**KOLCHIN SEMINAR IN  
DIFFERENTIAL ALGEBRA**

**Differential Dimension Polynomials**

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The concept of dimension polynomial was first introduced by Hilbert at the end of the 19-th century as a generalization of the following observation. Consider the ring of polynomials  $A = K[X_1, \dots, X_m]$  in several variables over a field  $K$ . Then  $A$  is a direct sum  $\bigoplus_{r=0}^{\infty} A^{(r)}$  where  $A^{(r)}$  is the vector  $K$ -space generated by all monomials  $X_1^{i_1} X_2^{i_2} \dots X_m^{i_m}$  of total degree  $r$  (that is,  $i_1 + i_2 + \dots + i_m = r$ ). Clearly,  $\dim_K A^{(r)}$  is equal to the number of such monomials (they form a basis of  $A^{(r)}$ ), so  $\dim_K A^{(r)}$  is the number of non-negative integers  $(i_1, \dots, i_m)$  with  $i_1 + i_2 + \dots + i_m = r$ . It is a well-known fact that that this number is  $\binom{r+m-1}{m-1} = \frac{(r+m-1)(r+m-2)\dots(r+1)}{(m-1)!} = \frac{r^{m-1}}{(m-1)!} + o(r^{m-1})$  where  $o(r^{m-1})$  is a polynomial of  $r$  whose degree is less than  $m - 1$ .

### **Theorem 1 (Hilbert).**

Let  $A = K[X_1, \dots, X_m] = \bigoplus_{r=0}^{\infty} A^{(r)}$  be a graded polynomial ring over a field  $K$  and  $M = \bigoplus_{r=0}^{\infty} M^{(r)}$  a graded  $A$ -module. (It means that  $M =_A M$  is represented as a direct sum of vector  $K$ -spaces  $M^{(r)}$  with  $A^{(r)}M^{(s)} \subseteq M^{(r+s)}$  for all  $r, s$ .)

Then there exists a polynomial  $\phi(t)$  with rational coefficients such that  $\phi(r) = \dim_K M^{(r)}$  for all sufficiently large  $r$  (that is, there exists  $r_0$  such that the last equality holds for all  $r \geq r_0$ ; in this case we also write that the equality holds for all  $r \gg 0$ ).

Polynomial  $\phi(t)$  is called the Hilbert polynomial of the graded module  $M$ . It is a *numerical polynomial*, that is, a polynomial with rational coefficients which takes integer values for all sufficiently large values of argument.

Hilbert theorem and a similar result on filtered modules (where one considers the ascending chain of vector  $K$ -subspaces  $A_r = \{f \in A \mid \deg f \leq r\}$  of the polynomial ring  $A$  and an ascending chain  $\{M_r\}$  of vector  $K$ -spaces of  $M$  with the condition  $A_r M_s \subseteq M_{r+s}$ ) introduce numerical polynomials that play a very important role in commutative algebra and algebraic geometry. The development of the corresponding technique in differential algebra was initiated in the 60-th by E. Kolchin and continued by J. Johnson, W. Sit, and a number of other mathematicians.

We begin the review of the theory of numerical polynomials associated with the of differential algebraic structures with the J. Johnson's theorem on dimension polynomial of a differential module.

In what follows,  $\mathbf{Z}$ ,  $\mathbf{N}$ , and  $\mathbf{Q}$  denote the sets of integers, non-negative integers and rational numbers, respectively.

Let  $K$  be a differential field of zero characteristic with a basic set  $\Delta = \{\delta_1, \dots, \delta_m\}$ , that is, a field  $K$  together with the mutually commuting derivations  $\delta_i : K \rightarrow K$ .

Let  $\Theta$  be the free commutative semigroup generated by the set  $\Delta$ , and for any nonnegative integer  $r$ , let  $\Theta(r)$  denote the set of all elements  $\theta = \delta_1^{k_1} \dots \delta_m^{k_m} \in \Theta$  such that  $ord \theta = \sum_{i=1}^m k_i \leq r$ .

An expression  $\sum_{\theta \in \Theta} a_\theta \theta$ , where all  $a_\theta \in K$  and only finitely many elements  $a_\theta$  are different from 0, is called a differential (or  $\Delta$ -) operator over  $K$ .  $\sum_{\theta \in \Theta} a_\theta \theta = \sum_{\theta \in \Theta} b_\theta \theta$  if and only if  $a_\theta = b_\theta$  for all  $\theta \in \Theta$

The set  $D$  of all  $\Delta$ -operators is a ring with the natural structure of a left  $R$ -module where  $\delta a = a\delta + \delta(a)$  for any  $a \in R$ ,  $\delta \in \Delta$ . (This equality comes from the action of this operator on  $K$ :  $(\delta a)(x) = \delta(ax) = a\delta(x) + \delta(a)x = (a\delta + \delta(a))x$  for any  $x \in K$ .) The ring  $D$  is called **the ring of differential (or  $\Delta$ -) operators over  $K$** .

The order of a  $\sigma$ -operator  $A = \sum_{\theta \in \Theta} a_\theta \theta \in D$  is defined as  $ord A = \max\{ord \theta \mid a_\theta \neq 0\}$ , and the ring  $D$  is considered as a filtered ring with the ascending filtration  $(D_r)_{r \in \mathbf{Z}}$  such that  $D_r = \{A \in D \mid ord A \leq r\}$  for any  $r \in \mathbf{N}$  and  $D_r = 0$  for  $r < 0$ .

With the above notation, a left  $D$ -module is said to be a **differential  $K$ -module** or a  $\Delta$ - $K$ -*module*. Thus, a vector  $K$ -space  $M$  is called a  $\Delta$ - $K$ -module, if the elements of  $\Delta$  act on  $M$  as commuting endomorphisms of the additive group of  $M$  and  $\delta(ax) = a\delta x + \delta(a)x$  for any  $x \in M, a \in K$ . We say that a  $\Delta$ - $K$ -module  $M$  is finitely generated, if it is finitely generated as a left  $D$ -module.

A filtration of a  $\Delta$ - $K$ -module  $M$  is an ascending chain  $(M_r)_{r \in \mathbf{Z}}$  of vector  $K$ -subspaces of  $M$  such that  $D_r M_s \subseteq M_{r+s}$  for all  $r, s \in \mathbf{Z}$ ,  $M_r = 0$  for all sufficiently small  $r \in \mathbf{Z}$ , and  $\bigcup_{r \in \mathbf{Z}} M_r = M$ . A filtration  $(M_r)_{r \in \mathbf{Z}}$  is **excellent** if all  $M_r$  ( $r \in \mathbf{Z}$ ) are finitely generated over  $K$  and there exists  $r_0 \in \mathbf{Z}$  such that  $M_r = D_{r-r_0} M_{r_0}$  for any  $r \in \mathbf{Z}, r \geq r_0$ .

**Example.** If  $M = \sum_{i=1}^p Dx_i$ , then  $(M_r = \sum_{i=1}^p D_r x_i)_{r \in \mathbf{Z}}$  is an excellent filtration of  $M$ .

**Theorem 2 (J. Johnson, 1969).** *Let  $K$  be a differential field with a basic set  $\Delta = \{\delta_1, \dots, \delta_m\}$  and let  $(M_r)_{r \in \mathbf{Z}}$  be an excellent filtration of a  $\Delta$ - $K$ -module  $M$ .*

*Then there exists a polynomial  $\chi(t) \in \mathbf{Q}[t]$  such that*

- (i)  $\chi(r) = \dim_K M_r$  for all  $r \gg 0$ .
- (ii)  $\deg \chi(t) \leq m$  and the polynomial  $\chi(t)$  can be written as  $\chi(t) = \sum_{i=0}^m a_i \binom{t+i}{i}$  where  $a_0, a_1, \dots, a_m \in \mathbf{Z}$ .
- (iii) The degree  $d$  and the coefficients  $a_m$  and  $a_d$  do not depend on the excellent filtration  $(M_r)_{r \in \mathbf{Z}}$ .

(iv) *The coefficient  $a_m$  is equal to the maximal number of elements  $x_1, \dots, x_k \in M$  such that the set  $\{\theta x_i | \theta \in \Theta, 1 \leq i \leq k\}$  is linearly independent over  $K$ . (Such elements  $x_1, \dots, x_k$  are said to be differentially (or  $\Delta$ -) linearly independent over  $K$ .)*

The polynomial  $\chi(t)$  is called the *differential dimension polynomial* of  $M$  associated with the filtration  $(M_r)_{r \in \mathbf{Z}}$ . The numbers  $d$ ,  $a_m$  and  $a_d$  are called the **differential dimension**, **differential type**, and **typical differential dimension** of  $M$ , respectively; they are denoted by  $\Delta\text{-dim}_K M$ ,  $\Delta\text{-type}_K M$ , and  $\Delta\text{-tdim}_K M$ .

**Sketch of the proof.** Notice that the ring  $D' = \bigoplus_{r \in \mathbf{Z}} D_r/D_{r-1}$  is isomorphic to the polynomial ring  $K[X_1, \dots, X_m]$  ( $D'$  is generated over  $K$  by the elements  $\bar{\delta}_i = \delta_i + D_0 \in D_1/D_0$ ,  $i = 1, \dots, m$ . Clearly,  $\bar{\delta}_i$  commute and  $\bar{\delta}_i a = a \bar{\delta}_i$  for any  $a \in K$ , so the isomorphism between  $D'$  and  $K[X_1, \dots, X_m]$  can be obtained by matching  $\bar{\delta}_i$  with  $X_i$ .)

The direct sum  $\bigoplus_{r \in \mathbf{Z}} M_r/M_{r-1}$  can be considered as a  $D'$ -module (with  $(A + D_{r-1})(x + M_{s-1}) = Ax + M_{r+s-1}$  for any  $A + D_{r-1} \in D_r/D_{r-1}$  and  $x + M_{s-1} \in M_s/M_{s-1}$ ). By the Hilbert Theorem, there is a polynomial  $\phi(t) \in \mathbf{Q}[t]$  of degree at most  $m - 1$  such that  $\phi(r) = \dim_K(M_r/M_{r-1})$  for  $r \gg 0$ . Since  $\dim_K M_r = \sum_{s \leq r} \dim(M_s/M_{s-1})$ , one can now obtain the main statement of the theorem from the following well-known result:

Let  $f(t) = a_n t^n + \cdots + a_0$ , ( $a_i \in \mathbf{Q}$ ) be a numerical polynomial with  $f(s) \in \mathbf{Z}$  for all  $s \in \mathbf{Z}$ ,  $s \geq s_0$ . Then there is a numerical polynomial  $g(t) = \frac{1}{n+1} a_n t^{n+1} + \dots$  such that  $g(r) = \sum_{i=s_0+1}^r f(i)$  for all  $r > s_0$ .

There is a close relation between differential and Krull dimensions of a differential module.

Let  $M$  be a module over a commutative ring  $R$ ,  $U$  a family of  $R$ -submodules of  $M$ , and  $\mathcal{B}_U = \{(N, N') \in U \times U \mid N' \subseteq N\}$ . Also, let  $\overline{\mathbf{Z}} = \mathbf{Z} \cup \{\infty\}$  ( $a < \infty$  for  $a \in \mathbf{Z}$ ). Then there is a unique map  $\mu_U : \mathcal{B}_U \rightarrow \overline{\mathbf{Z}}$  such that

- (i)  $\mu_U(N, N') \geq -1$  for all  $(N, N') \in \mathcal{B}_U$ ;
- (ii) If  $d \in \mathbf{N}$ , then  $\mu_U(N, N') \geq d$  if and only if  $N \neq N'$  and there is an infinite chain  $N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N'$  with  $N_i \in U$  and  $\mu_U(N_{i-1}, N_i) \geq d - 1$  for  $i = 1, 2, \dots$ .

Then  $\sup\{\mu_U(N, N') | (N, N') \in \mathcal{B}_U\}$  is called the **type** of the  $R$ -module  $M$  **over the family**  $U$ ; it is denoted by  $\text{type}_U M$ . The least upper bound of the lengths  $p$  of chains  $N_0 \supseteq N_1 \supseteq \dots \supseteq N_p$  such that  $N_i \in U$  ( $0 \leq i \leq p$ ) and  $\mu_U(N_{i-1}, N_i) = \text{type}_U M$  for  $i = 1, \dots, p$  is called the **dimension of**  $M$  **over**  $U$ ; it is denoted by  $\text{dim}_U M$ .

**Theorem 3 (J. Johnson, 1969).** *Let  $K$  be a differential ( $\Delta$ -),  $M$  a finitely generated vector  $\Delta$ - $K$ -module, and  $U$  the family of all  $\Delta$ - $K$ -submodules of  $M$ .*

- (i) *If  $\Delta\text{-dim}_K M > 0$ , then  $\text{type}_U M = m$  and  $\text{dim}_U M = \Delta\text{-dim}_K M$ .*
- (ii) *If  $\Delta\text{-dim}_K M = 0$ , then  $\text{type}_U M < m$ .*

Another application of Theorem 2 is a new proof of the following theorem on differential dimension polynomial due to E. Kolchin.

Let a  $\Delta$ -field  $K$  ( $Char K = 0, \Delta = \{\delta_1, \dots, \delta_m\}$ ), the semigroup of power products  $\Theta$  and  $\Theta(r) = \{\theta \in \Theta \mid ord \theta \leq r\}$  ( $r \in \mathbf{N}$ ) be as before.

Let  $L = K\langle \eta_1, \dots, \eta_n \rangle$  be a differential ( $\Delta$ -) field extension of  $K$  generated by a finite set  $\eta = \{\eta_1, \dots, \eta_n\}$ . (As a field,  $L = K(\{\theta\eta_j \mid \theta \in \Theta, 1 \leq j \leq n\})$ .) The following fundamental result describes the growth of the transcendence degree of fields  $K(\Theta(r)\eta_1 \cup \dots \cup \Theta(r)\eta_n)$  ( $r = 1, 2, \dots$ ) over  $K$ . (Recall that if  $F$  is a subfield of a field  $G$ , then elements  $b_1, \dots, b_k \in G$  are said to be algebraically independent over  $F$  if there is no nonzero polynomial  $f$  in  $k$  variables with coefficients in  $F$  such that  $f(b_1, \dots, b_k) = 0$ . The maximal number of algebraically independent over  $F$  elements of  $G$  is called the transcendence degree of  $G$  over  $F$  and denoted by  $trdeg_F G$ .)

### **Theorem 4 (Kolchin, 1964).**

With the above conventions, there exists a polynomial  $\omega_{\eta|K}(t) \in \mathbf{Q}[t]$  such that

(i)  $\omega_{\eta|K}(r) = \text{trdeg}_K K(\{\theta\eta_j | \theta \in \Theta(r), 1 \leq j \leq n\})$  for all sufficiently large  $r \in \mathbf{Z}$ ;

(ii)  $\deg \omega_{\eta|K} \leq m$  and  $\omega_{\eta|K}(t)$  can be written as  $\omega_{\eta|K}(t) = \sum_{i=0}^m a_i \binom{t+i}{i}$  where  $a_0, \dots, a_m \in \mathbf{Z}$ ;

(iii)  $d = \deg \omega_{\eta|K}$ ,  $a_m$  and  $a_d$  do not depend on the choice of the system of  $\Delta$ -generators  $\eta$  of the extension  $L/K$  (clearly,  $a_d \neq a_m$  iff  $d < m$ , that is  $a_m = 0$ ). Moreover,  $a_m$  is equal to the differential transcendence degree of  $L$  over  $K$ , i. e., to the maximal number of elements  $\xi_1, \dots, \xi_k \in L$  such that the set  $\{\theta\xi_i | \theta \in \Theta, 1 \leq i \leq k\}$  is algebraically independent over  $K$ .

**Sketch of the J.Johnson's proof of the Kolchin Theorem.** It is easy to check that the vector  $L$ -space  $Der_K L$  of all  $K$ -linear derivations of the field  $L$  into itself becomes a  $\Delta$ - $L$ -module if one defines the actions of elements of  $\Delta$  on  $Der_K L$  as follows:  
 $\delta(\partial) = \delta \circ \partial - \partial \circ \delta$  for any  $\delta \in \Delta, \partial \in Der_K L$ .  
Also, setting  $\delta(\phi) = \delta \circ \phi - \phi \circ \delta$ , one can consider a structure of a  $\Delta$ - $L$ -module on the dual vector  $L$ -space  $(Der_K L)^* = Hom_L(Der_K L, L)$  and on the vector  $L$ -space of differentials  $\Omega_K(L)$ . (It is generated over  $L$  by all elements  $d\zeta \in (Der_K L)^*$  ( $\zeta \in L$ ) such that  $d\zeta(D) = D(\zeta)$  for any  $D \in Der_K L$ ; one can easily check that  $\delta(d\zeta) = d\delta(\zeta)$  for any  $\delta \in \Delta, \zeta \in L$ .)

Now, using classical results on continuation of derivations from a field to an overfield, one can prove the following proposition that, together with Theorem 2, implies the Kolchin theorem.

**Proposition 5 (J. Johnson, 1969).**

*With the above notation, let  $\Omega_K(L)_r$  ( $r \in \mathbf{N}$ ) denote the vector  $L$ -subspace of  $\Omega_K(L)$  generated by the set  $\{d\delta(\eta_i) \mid \gamma \in \Theta(r), 1 \leq i \leq s\}$  and let  $\Omega_K(L)_r = 0$  for  $r < 0$ . Then*

(i)  *$(\Omega_K(L))_r)_{r \in \mathbf{Z}}$  is an excellent filtration of the  $\Delta$ - $L$ -module  $\Omega_K(L)$ .*

(ii)  *$\dim_K \Omega_K(L)_r = \text{trdeg}_K K(\{\delta\eta_j \mid \gamma \in \Theta(r), 1 \leq j \leq s\})$  for all  $r \in \mathbf{N}$ .*

(iii) *The differential dimension polynomial  $\omega_{\eta|K}(t)$  is equal to the differential dimension polynomial of  $\Omega_K(L)$  associated with the filtration  $(\Omega_K(L))_r)_{r \in \mathbf{Z}}$ .*

The polynomial  $\omega_{\eta|K}$  defined in the Kolchin theorem is called the **differential dimension polynomial of the  $\Delta$ -field extension  $K \subseteq L$  associated with the system of  $\Delta$ -generators  $\eta$ .**

Let  $R = K\{y_1, \dots, y_n\}$  be the ring of differential polynomials over  $K$ . (Recall that this is a ring of polynomials in the countable set of variables  $\theta y_i$  ( $\theta \in \Theta$ ,  $1 \leq i \leq n$ ) treated as a differential ( $\Delta$ -) ring where the action of each  $\delta_i$  is extended from  $K$  by setting  $\delta_i(\theta y_j) = (\delta_i \theta)y_j$ .) Clearly, the ring homomorphism  $\pi : K\{y_1, \dots, y_n\} \rightarrow L = K\langle \eta_1, \dots, \eta_n \rangle$ , that leaves the elements of  $K$  fixed and sends  $\theta y_i$  to  $\theta \eta_i$ , commutes with the actions of  $\delta_j$  (such a homomorphism is called *differential* or a  *$\Delta$ -homomorphism*) and  $P = \text{Ker } \pi$  is a prime differential ( $\Delta$ -) ideal of  $R$ .

It is called a **defining  $\Delta$ -ideal** of the extension  $K \subseteq L$  (or of the  $n$ -tuple  $\eta$  over  $K$ ). The original proof of the Kolchin theorem, as well as most of the methods of computation of differential dimension polynomials, is based on the consideration of a characteristic set of the defining ideal.

Conversely, if  $P$  is any prime  $\Delta$ -ideal in  $R = K\{y_1, \dots, y_n\}$ , then the quotient field of the integral domain  $R/P$  can be naturally viewed as a  $\Delta$ -field extension of  $K$  with a finite set of  $\Delta$ -generators  $\eta_i = y_i + P$ , the canonical images of the  $\Delta$ -indeterminates  $y_1, \dots, y_n$  in  $R/P$ . By the Kolchin theorem, one can associate with  $P$  a numerical polynomial  $\omega_{\eta|K}(t)$  (also denoted by  $\omega_P(t)$ ) called a *differential dimension polynomial of the  $\Delta$ -ideal  $P$* .

A system of algebraic differential equations over  $K$  is a system of the form

$$f_i(y_1, \dots, y_n) = 0 \quad (i \in I)$$

where  $\{f_i\}_{i \in I} \subseteq R$ ; by a solution we mean an  $n$ -tuple with coordinates in some differential field extension of  $K$  that annuls all  $f_i$ .

Let  $P$  be the radical differential ideal generated by  $\{f_i | i \in I\}$  in  $R$  (that is,  $P$  is the radical of the ideal generated by  $\{\theta f_i | \theta \in \Theta, i \in I\}$ ). If  $P$  is prime (this is always the case if the system is linear), then one can associate with  $P$  the numerical polynomial  $\omega_P(t)$  called a *differential dimension polynomial of the system of differential equations*. This polynomial, as it was first noticed by A. Mikhalev and Pankratev (1980), is an algebraic form of the A. Einstein's concept of strength of a system of differential equations.

A. Einstein defined the strength of a system of partial differential equations governing a physical field in his work *The Meaning of Relativity. Appendix II. Generalization of gravitation theory.* Princeton, 1953, pp. 133 - 165]:

”... the system of equations is to be chosen so that the field quantities are determined as strongly as possible. In order to apply this principle, we propose a method which gives a measure of strength of an equation system. We expand the field variables, in the neighborhood of a point  $\mathcal{P}$ , into a Taylor series (which presupposes the analytic character of the field); the coefficients of these series, which are the derivatives of the field variables at  $\mathcal{P}$ , fall into sets according to the degree of differentiation. In every such degree there appear, for the first time, a set of coefficients which would be free for arbitrary choice if it were not that the field must satisfy a system of differential equations. Through this system of differential equations (and its derivatives with respect to the coordinates) the number of coefficients is restricted, so that in each degree a smaller number of coefficients is left free for arbitrary choice. The set of numbers of ”free” coefficients for all degrees of differentiation is then a measure of the ”weakness” of the system of equations, and through this, also of its ”strength”.”

Taking into account the A. Einstein's approach, one can say that for  $r \gg 0$ , the value  $\omega(r)$  of a differential dimension polynomial  $\omega(t)$  of a system of algebraic differential equations is the number of Taylor coefficients of order  $\leq r$  of a solution that can be chosen arbitrarily.

Thus,  $\omega(t)$  can be viewed as a measure of strength, that is why the problem of computation of differential dimension polynomials is important not only for the study of differential algebraic structures, but also for the study of equations of mathematical physics. Most of the known methods of computation of such polynomials (as well as the original proof of the Kolchin theorem) are based on constructing characteristic sets of the corresponding prime differential ideals and expressing the differential dimension polynomials of these ideals as sums of certain numerical polynomials associated with the leaders of elements of characteristic sets. However, in many cases (in particular, in the case of a system of linear differential equations) the differential dimension polynomial can be found by application of Proposition 5 and construction of a free resolution of the corresponding module of differentials.

To illustrate this approach, notice, first that the differential dimension polynomial  $\omega_D(t)$  associated with the filtration  $(D_r)_{r \in \mathbf{Z}}$  of the ring of  $\Delta$ -operators  $D$  over a differential field  $K$  with a basic set  $\Delta = \{\delta_1, \dots, \delta_m\}$  is equal to  $\binom{t+m}{m}$ . Indeed, for all  $r \geq 0$ ,  $\dim_K D_r = \text{Card}\{\delta_1^{k_1} \dots, \delta_m^{k_m} \mid \sum_{i=1}^m k_i \leq r\} = \binom{r+m}{m}$ .

Let  $F_k$  be a free left  $D$ -module of rank  $k$  with free generators  $f_1, \dots, f_k$ . Then for every  $l \in \mathbf{Z}$ , one can consider an excellent filtration  $((F_k^l)_r)_{r \in \mathbf{Z}}$  of the module  $F_k$  such that  $(F_k^l)_r = \sum_{i=1}^k D_{r-l} f_i$  for every  $r \in \mathbf{Z}$ . We obtain a filtered  $\Delta$ - $K$ -module denoted by  $F_k^l$ . The expression of  $\omega_D(t)$  imply the following expressions for the differential dimension polynomial  $\chi_M(t)$  of  $M = F_k^l$ :

$$\chi_M(t) = k \binom{t + m - l}{m}. \quad (1)$$

A finite direct sum of such filtered  $\Delta$ - $K$ -modules is called a *free filtered  $\Delta$ - $K$ -module*. The corresponding differential dimension polynomial is a sum of polynomials of the form (1).

If  $L = K\langle\eta_1, \dots, \eta_n\rangle$ , then  $\Omega_K(L)$  is generated over  $D$  by the finite set  $\{d\eta_1, \dots, d\eta_n\}$ , and there exists a finite resolution

$$0 \rightarrow M_q \xrightarrow{d_{q-1}} M_{q-1} \rightarrow \dots \xrightarrow{d_0} M_0 \xrightarrow{\rho} \Omega_K(L) \rightarrow 0$$

where each  $M_i$  is a free filtered  $\Delta$ - $K$ -module and  $\rho$ ,  $d_0, \dots, d_{q-1}$  are  $\Delta$ -homomorphisms of filtered  $D$ -modules. In this case  $\omega_{\eta|K}(t)$  can be expressed as

$\omega_{\eta|K}(t) = \sum_{i=1}^q (-1)^i \chi_{M_i}(t)$ , so we can find  $\omega_{\eta|K}(t)$  as a sum of polynomials of the form (1).

**Example 1.** *Wave equation:*

$$\delta_1^2 y + \delta_2^2 y + \delta_3^2 y - \delta_4^2 y = 0.$$

The differential dimension polynomial of this equation is the differential dimension polynomial of the  $\Delta$ -field extension  $K\langle\eta\rangle \supset K$  where  $\eta$  is the canonical image of  $y$  in the quotient field  $L$  of  $K\{y\}/[\delta_1^2 y + \delta_2^2 y + \delta_3^2 y - \delta_4^2 y]$ . The resolution of the module of differentials is

$$0 \rightarrow F_1^2 \rightarrow F_1^0 \rightarrow \Omega_K(L) \rightarrow 0$$

whence  $\omega_{\eta|K}(t) = \binom{t+4}{4} - \binom{t+2}{4} = 2\binom{t+3}{3} - \binom{t+2}{2}$ .

**Example 2.** *Equations for electromagnetic field given by the potential:*

$$\sum_{j=1}^4 (\delta_i^2 y_i - \delta_i \delta_j y_j) = 0 \quad (i = 1, 2, 3, 4)$$

$$\sum_{j=1}^4 \delta_j y_j = 0.$$

In this case the resolution of the module of differentials is

$$0 \rightarrow F_1^3 \rightarrow F_4^2 \bigoplus F_1^1 \rightarrow F_4^0 \rightarrow \Omega_K(L) \rightarrow 0,$$

so the differential dimension polynomial

$$\begin{aligned} \omega(t) &= 4 \binom{t+4}{4} + \binom{t+1}{4} - \left[ 4 \binom{t+2}{4} + \binom{t+3}{4} \right] \\ &= 6 \binom{t+3}{3} - \binom{t+2}{2} - (t+1). \end{aligned}$$

In 1975 W. Sit proved that the set  $W$  of all differential dimension polynomials associated with differential field extensions is well-ordered with respect to the natural order:  $f(t) \leq g(t)$  if and only if  $f(r) \leq g(r)$  for  $r \gg 0$ . It follows that every finitely generated differential ( $\Delta$ -) field extension  $K \subset L$  has a system of differential generators  $\eta = \{\eta_1, \dots, \eta_n\}$  such that for any other system of  $\Delta$ -generators  $\zeta$  of  $L$  over  $K$ , one has  $\omega_{\eta|K}(t) \leq \omega_{\zeta|K}(t)$ . In this case we say that  $\omega_{\eta|K}(t)$  is the **minimal differential dimension polynomial** of the extension, it is denoted by  $\omega_{L|K}(t)$ .

There is no algorithm of computation of minimal differential dimension polynomials. The following result is one of the few known facts about such polynomials.

**Theorem 6 (A. Mikhalev, E. Pankratev, 1980).**

Let us consider the set  $W_m$  of all differential dimension polynomials associated with differential field extensions whose basic set consists of  $m$  derivations and that have type  $m - 1$  and typical differential dimension  $r$ . Then  $\binom{t+m}{m} - \binom{t+m-r}{m}$  is the minimal polynomial in  $W_m$ .

As a consequence of the last theorem we obtain that the differential dimension polynomial  $\omega_{\eta|K}(t) = \binom{t+4}{4} - \binom{t+2}{4} = 2\binom{t+3}{3} - \binom{t+2}{2}$  obtained for the wave equation (Example 1) is the minimal polynomial of the corresponding differential field extension  $K\langle\eta\rangle \supset K$ .

In what follows,  $K$  denotes a differential field of zero characteristic whose basic set  $\Delta$  is a union of  $p$  disjoint finite sets ( $p \geq 1$ ):  $\Delta = \Delta_1 \cup \cdots \cup \Delta_p$ , where  $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$  ( $i = 1, \dots, p$ ). In other words, we fix a partition of the set  $\Delta$ .

For any  $\theta = \delta_{11}^{k_{11}} \cdots \delta_{1m_1}^{k_{1m_1}} \delta_{21}^{k_{21}} \cdots \delta_{pm_p}^{k_{pm_p}} \in \Theta$ , we define the order of the element  $\theta$  with respect to  $\Delta_i$  as follows:  $ord_i \theta = \sum_{j=1}^{m_i} k_{ij}$  ( $i = 1, \dots, p$ ).

If  $r_1, \dots, r_p \in \mathbf{N}$ , we set  $\Theta(r_1, \dots, r_p) = \{\theta \in \Theta \mid ord_i \theta \leq r_i \text{ for } i = 1, \dots, p\}$  and  $\Theta(r_1, \dots, r_p)\xi = \{\theta(\xi) \mid \theta \in \Theta(r_1, \dots, r_p)\}$  for any  $\xi \in F$ .

As before,  $L = K\langle \eta_1, \dots, \eta_n \rangle$  is a  $\Delta$ -field extension of  $K$  generated by the set  $\eta = \{\eta_1, \dots, \eta_n\}$ .

## Theorem 7.

(i) With the above conventions, there exists a polynomial  $\Phi_\eta(t_1, \dots, t_p) \in \mathbf{Q}[t_1, \dots, t_p]$  such that

$$\Phi_\eta(r_1, \dots, r_p) = \text{trdeg}_K K\left(\bigcup_{j=1}^n \Theta(r_1, \dots, r_p)\eta_j\right)$$

for all sufficiently large  $(r_1, \dots, r_p) \in \mathbf{N}^p$

(i. e., there exist  $s_1, \dots, s_p \in \mathbf{N}$  such that the last equality holds for all elements  $(r_1, \dots, r_p) \in \mathbf{N}^p$  with  $r_1 \geq s_1, \dots, r_p \geq s_p$ );

(ii)  $\text{deg}_{t_i} \Phi_\eta \leq m_i$  ( $1 \leq i \leq p$ ), so that  $\text{deg} \Phi_\eta \leq m$  and the polynomial  $\Phi_\eta(t_1, \dots, t_p)$  can be represented as

$$\Phi_\eta = \sum_{i_1=0}^{m_1} \cdots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \cdots \binom{t_p + i_p}{i_p}$$

where  $a_{i_1 \dots i_p} \in \mathbf{Z}$  for all  $i_1, \dots, i_p$ .

For any permutation  $(j_1, \dots, j_p)$  of the set  $\{1, \dots, p\}$ , we define the lexicographic order  $\leq_{j_1, \dots, j_p}$  on  $\mathbf{N}^p$  as follows:  $(r_1, \dots, r_p) \leq_{j_1, \dots, j_p} (s_1, \dots, s_p)$  iff either  $r_{j_1} < s_{j_1}$  or there exists  $k \in \mathbf{N}$ ,  $1 \leq k \leq p - 1$ , such that  $r_{j_\nu} = s_{j_\nu}$  for  $\nu = 1, \dots, k$  and  $r_{j_{k+1}} < s_{j_{k+1}}$ .

If  $\Sigma \subseteq \mathbf{N}^p$ , then  $\Sigma'$  denotes the set  $\{e \in \Sigma \mid e \text{ is a maximal element of } \Sigma \text{ with respect to one of the } p! \text{ lexicographic orders } \leq_{j_1, \dots, j_p}\}$ .

**Example 3.**

Let  $\Sigma = \{(3, 0, 2), (2, 1, 1), (0, 1, 4), (1, 0, 3), (1, 1, 6), (3, 1, 0), (1, 2, 0)\} \subseteq \mathbf{N}^3$ .

Then  $\Sigma' = \{(3, 0, 2), (3, 1, 0), (1, 1, 6), (1, 2, 0)\}$ .

**Theorem 8.** Let  $K$  be a differential field whose basic set of derivation operators  $\Delta$  is a union of  $p$  disjoint finite sets ( $p \geq 1$ ):  $\Delta = \Delta_1 \cup \cdots \cup \Delta_p$ , where  $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$  ( $i = 1, \dots, p$ ). Let  $L = K\langle \eta_1, \dots, \eta_n \rangle$  be a  $\Delta$ -field extension of  $K$  with the finite set of  $\Delta$ -generators  $\eta = \{\eta_1, \dots, \eta_n\}$  and  $\Phi_\eta(t_1, \dots, t_p)$

$$= \sum_{i_1=0}^{m_1} \cdots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \cdots \binom{t_p + i_p}{i_p}$$

the corresponding dimension polynomial.

Let  $E_\eta = \{(i_1, \dots, i_p) \in \mathbf{N}^p \mid 0 \leq i_k \leq m_k$   
 $(k = 1, \dots, p) \text{ and } a_{i_1 \dots i_p} \neq 0\}$ .

Then  $d = \deg \Phi_\eta$ ,  $a_{m_1 \dots m_p}$ , elements

$(k_1, \dots, k_p) \in E'_\eta$ , the corresponding coefficients  $a_{k_1 \dots k_p}$  and the coefficients of the terms of total degree  $d$  do not depend on the choice of the system of  $\Delta$ -generators  $\eta$ .

### Example 4.

Let  $F$  be a differential field with a basic set  $\Delta = \{\delta_1, \delta_2\}$ ,  $G = F\langle\eta\rangle$  and the defining equation on the  $\Delta$ -generator  $\eta$  is as follows.

$$\delta_1^a \delta_2^b \eta + \delta_2^{a+b} \eta = 0 \quad (2)$$

where  $a, b \in \mathbf{N}$ . In other words,  $G$  is  $\Delta$ -isomorphic to the quotient field  $Q(F\{y\}/P)$  where  $P$  is the linear  $\Delta$ -ideal of  $F\{y\}$  generated by  $f(y) = \delta_1^a \delta_2^b y + \delta_2^{a+b} y$ .

It can be shown that

$$\omega_{\eta/F}(t) = (a+b)t - \frac{(a+b)(a+b-3)}{2}$$

and

$$\Phi_{\eta}(t_1, t_2) = (a+b)t_1 + at_2 + 2a + b - ab - a^2.$$

Comparing the classical Kolchin polynomial  $\omega_{\eta/F}(t)$  and the polynomial  $\Phi_{\eta}(t_1, t_2)$  we see that  $\omega_{\eta/F}(t)$  carries two differential birational invariants, its degree 1 and the leading coefficient  $a + b$ , while  $\Phi_{\eta}(t_1, t_2)$  carries three such invariants, its total degree 1,  $a + b$ , and  $a$  (therefore,  $a$  and  $b$  are uniquely determined by the polynomial  $\Phi_{\eta}$ ). In other words, if  $\xi = \{\xi_1, \dots, \xi_q\}$  is any other system of  $\Delta$ -generators of the  $\Delta$ -extension  $G/F$ , then

$$\omega_{\xi/F}(t) = (a + b)t + c_1 \quad \text{and}$$

$$\Phi_{\xi}(t_1, t_2) = (a + b)t_1 + at_2 + c_2$$

for some integers  $c_1$  and  $c_2$ . Thus,  $\Phi_{\eta}(t_1, t_2)$  gives both parameters  $a$  and  $b$  of the equation (2) while  $\omega_{\eta/F}(t)$  gives just the sum of the parameters.

## References

- [1] Balaba, I. N. Dimension polynomials of extensions of difference fields. *Vestnik Moskov. Univ.*, Ser. I, Mat. Mekh., 1984, no. 2, 31 - 35. (Russian)
- [2] Balaba, I. N. Calculation of the dimension polynomial of a principal difference ideal. *Vestnik Moskov. Univ.*, Ser. I, Mat. Mekh., 1985, no. 2, 16 - 20. (Russian)
- [3] Balaba, I. N. Finitely generated extensions of difference fields. *VINITI* (Moscow, Russia), 1987, no. 6632-87. (Russian)
- [4] Brenti, F. Hilbert Polynomials in Combinatorics, *Journal of Algebraic Combinatorics*, 7 (1998), 127 - 156.
- [5] Caboara, M.; Dedominicis, G., and Robbiano, L. Multigraded Hilbert functions and Buchberger algorithm. *Proceedings of ISSAC 96*, 72 - 85 (1996).
- [6] Some properties of the lattice points and their application to differential algebra. *Comm. Algebra*, 15 (1987), no. 12, 2625 - 2632.
- [7] Some upper bounds for the multiplicity of an autoreduced subset of  $N^m$  and their applications. *Algebraic algorithms and error correcting codes* (Grenoble, 1985), 306 - 315. Lecture Notes in Comp. Sci., 229. Springer, Berlin, 1986.
- [8] Carra Ferro, G. Some Remarks on Differential Hilbert polynomials in two variables. *Preprint presented at IMACS-ACA 99 Conference*. Madrid, 1999.
- [9] Carra Ferro, G. Hilbert polynomials in two variables and bifiltered ideals. *Zero-dimensional Schemes and Applications* (Naples, 2000) *Queen's Papers in Pure and Appl. Math.*, Queen's Univ., Kingston, ON, 2002.
- [10] Carra Ferro, G.; Sit, W. On term-orderings and rankings. *Computational Algebra* (Fairfax, VA, 1993), 31 - 77. Lecture Notes in Pure and Appl. Math., 151, Dekker, New York, 1994.
- [11] Cohn, R. M. Order and dimension. *Proc. Amer. Math. Soc.*, 87 (1983), no. 1, 1 - 6.
- [12] Dzhavadov, G. A. Characteristic Hilbert polynomial for differential-difference modules and its applications. *VINITI* (Moscow, Russia), 1979, no. 1992-79. (Russian)
- [13] Einstein, A. The Meaning of Relativity. Appendix II (Generalization of gravitation theory), 4th ed. Princeton, 1953, 133 - 165.
- [14] Eisenbud, D. Commutative Algebra with View toward Algebraic Geometry. *Springer-Verlag*, New York, 1995.

- [15] Johnson, Joseph L. Differential dimension polynomials and a fundamental theorem on differential modules. *Amer. J. Math.*, 91 (1969), no. 1, 239 - 248.
- [16] Johnson, Joseph L. Kähler differentials and differential algebra. *Ann. of Math.* (2), 89 (1969), 92 - 98.
- [17] Johnson, Joseph L. A notion on Krull dimension for differential rings. *Comment. Math. Helv.*, 44 (1969), 207 - 216.
- [18] Johnson, Joseph L. Kähler differentials and differential algebra in arbitrary characteristic. *Trans. Amer. Math. Soc.*, 192 (1974), 201 - 208.
- [19] Johnson, Joseph L. A notion of regularity for differential local algebras. *Contribution to algebra. Collection of papers dedicated to Ellis Kolchin.* Acad. Press, New York, 1977, 211 - 232.
- [20] Johnson, Joseph L.; Sit, W. On the differential transcendence polynomials of finitely generated differential field extensions. *Amer. J. Math.*, 101 (1979), 1249 - 1263.
- [21] Kolchin E. R. The notion of dimension in the theory of algebraic differential equations. *Bull Amer. Math.Soc.*, 70 (1964), 570 - 573.
- [22] Kolchin, E. R. Some problems in differential algebra. *Proc. Int'l Congress of Mathematicians* (Moscow - 1966), Moscow, 1968, 269 - 276.
- [23] Kolchin, E. R. Differential Algebra and Algebraic Groups. *Academic Press*, New York - London, 1973.
- [24] Kondrateva, M. V. Description of the set of minimal differential dimension polynomials. *Vestnik Mosk. Univ.*, Ser. I (1988), no. 1, 35 - 39. (Russian)
- [25] Kondrateva, M. V. The minimal dimension polynomial of a field extension defined by a system of linear differential equations. *Math. Zametki*, 45 (1989), no. 3, 80 - 86. (Russian)
- [26] Kondrateva, M. V.; Pankratev, E. V. A recursive algorithm for the computation of Hilbert polynomial. *Lect. Notes Comp. Sci.*, 378 (1990), 365-375. Proc. EUROCAL 87, Springer-Verlag.
- [27] Kondrateva, M. V.; Pankratev, E. V. Algorithms of computation of characteristic Hilbert polynomials. *Packets of Applied Programs. Analytic Transformations*, 129 - 146. Nauka, Moscow, 1988. (In Russian)
- [28] Kondrateva, M. V.; Pankratev, E. V.; Serov, R. E. Computations in differential and difference modules. *Proceedings of the Internat. Conf. on the Analytic Computations and their Applications in Theoret. Physics*, Dubna, 1985, 208 - 213. (In Russian)

- [29] Kondrateva, M. V.; Levin, A. B.; Mikhalev, A. V.; Pankratev, E. V. Computation of dimension polynomials. *Internat. J. of Algebra and Comput.*, 2 (1992), no. 2, 117 - 137.
- [30] Kondrateva, M. V.; Levin, A. B.; Mikhalev, A. V.; Pankratev, E. V. Differential and Difference Dimension Polynomials. *Kluwer Academic Publishers.*, Dordrecht, 1998.
- [31] Levin, A. B. Characteristic polynomials of filtered difference modules and difference field extensions. *Uspehi Mat. Nauk*, 33 (1978), no. 3, 177 - 178 (Russian). English transl.: *Russian Math. Surveys*, 33 (1978), no.3, 165-166.
- [32] Levin, A. B. Characteristic polynomials of inversive difference modules and some properties of inversive difference dimension. *Uspehi Mat. Nauk*, 35 (1980), no. 1, 201 - 202 (Russian). English transl.: *Russian Math. Surveys*, 35 (1980), no. 1, 217 - 218.
- [33] Levin, A. B. Characteristic polynomials of difference modules and some properties of difference dimension. *VINITI* (Moscow, Russia), 1980, no. 2175-80. (Russian)
- [34] Levin, A. B. Type and dimension of inversive difference vector spaces and difference algebras. *VINITI* (Moscow, Russia), 1982, no. 1606-82. (Russian)
- [35] Levin, A. B. Characteristic polynomials of  $\Delta$ -modules and finitely generated  $\Delta$ -field extensions. *VINITI* (Moscow, Russia), 1985, no. 334-85. (Russian)
- [36] Levin, A. B. Inversive difference modules and problems of solvability of systems of linear difference equations. *VINITI* (Moscow, Russia), 1985, no. 335-85. (Russian)
- [37] Levin, A. B. Computation of Hilbert polynomials in two variables. *J. Symbolic Comput.*, 28 (1999), 681 - 709.
- [38] Levin, A. B. Characteristic polynomials of finitely generated modules over Weyl algebras. *Bull. Austral. Math. Soc.*, 61 (2000), 387 - 403.
- [39] Levin, A. B. Reduced Grobner bases, free difference-differential modules and difference-differential dimension polynomials. *J. Symbolic Comput.*, 29 (2000), 1 - 26.
- [40] Levin, A. B. On the set of Hilbert polynomials. *Bull. Austral. Math. Soc.*, 64 (2001), 291 - 305.
- [41] Levin, A. B. Multivariable dimension polynomials and new invariants of differential field extensions. *Internat. J. Math. and Math. Sci.*, 27, no. 4, 201 - 213.
- [42] Levin, A. B.; Mikhalev, A. V. The differential dimension polynomial and the strength of a system of differential equations. *Computable invariants in the theory of algebraic systems (collection of papers)*, Novosibirsk, 1987, 58 - 66. (Russian)

- [43] Levin, A. B.; Mikhalev, A. V. Difference-differential dimension polynomials. *VINITI* (Moscow, Russia), 1988, no. 6848-B88. (Russian)
- [44] Levin, A. B.; Mikhalev, A. V. Dimension polynomials of filtered G-modules. *VINITI* (Moscow, Russia), 1988, no. 6849-B88. (Russian)
- [45] Levin, A. B.; Mikhalev, A. V. Dimension polynomials of filtered G-modules and finitely generated G-field extensions. *Algebra (collection of papers)*. Moscow State University, Moscow, 1989, 74 - 94.
- [46] Levin, A. B.; Mikhalev, A. V. Dimension polynomials of differential modules. *Abelian Groups and Modules*, 9 (1989), 51 - 67. (Russian)
- [47] Levin, A. B.; Mikhalev, A. V. Type and dimension of finitely generated vector G-spaces. *Vestnik Mosk. Univ., Ser. I, Mat. Mekh.*, 1991, no. 4, 72-74 (Russian). English transl.: *Moscow Univ. Math. Bull*, 46, no. 4, 51 - 52.
- [48] Levin, A. B.; Mikhalev, A. V. Dimension polynomials of difference-differential modules and of difference-differential field extensions. *Abelian Groups and Modules*, 10 (1991), 56 - 82. (Russian)
- [49] Levin, A. B.; Mikhalev, A. V. Dimension polynomials of filtered differential G-modules and extensions of differential G-fields. *Contemp.Math.*, 131 (1992), Part 2, 469 - 489.
- [50] Levin, A. B.; Mikhalev, A. V. Type and dimension of finitely generated G-algebras. *Contemp. Math.*, 184 (1995), 275 - 280.
- [51] F. S. Macaulay, Some properties of enumeration in the theory of modular systems, *Proc. London Math. Soc.* (2), 26 (1927), 531-555.
- [52] Mikhalev, A. V.; Pankratev, E. V. Differential dimension polynomial of a system of differential equations. *Algebra (collection of papers)*. Moscow State Univ., Moscow, 1980, 57 - 67. (Russian)
- [53] Mikhalev, A. V.; Pankratev, E. V. Differential and Difference Algebra. *Algebra, Topology, Geometry*, 25, 67 - 139. *Itogi Nauki i Tekhniki, Akad. Nauk SSSR*, 1987 (Russian). English transl.: *J. Soviet. Math.*, 45 (1989), no. 1, 912 - 955.
- [54] Mikhalev, A. V.; Pankratev, E. V. Computer Algebra. Calculations in Differential and Difference Algebra. *Moscow State Univ.*, Moscow, 1989. (Russian)
- [55] Mora, F.; Moller, H. M. (1983). The computation of the Hilbert function. *Lect Notes Comp. Sci.* **162**, 157 - 167.

- [56] Pankratev, E. V. Computations in differential and difference modules. *Symmetries of partial differential equations*, Part III. *Acta Appl. Math.*, 16 (1989), no. 2, 167 - 189.
- [57] Robbiano, L. and Valla, G. Hilbert series of bigraded algebras. *Bolletino U. M. I.* (8), 1-B, 521-540 (1998).
- [58] Sit, W. Typical differential dimension of the intersection of linear differential algebraic groups. *J. algebra*, 32 (1974), no. 3, 476 - 487.
- [59] Sit, W. Well-ordering of certain numerical polynomials. *Trans. Amer. Math. Soc.*, 212 (1975), 37 - 45.
- [60] Sit, W. Differential dimension polynomials of finitely generated extensions. *Proc. Amer. Math. Soc.*, 68 (1978), no. 3, 251 - 257.
- [61] Sit, W. Some comments on term-ordering in Gröbner basis computations. *ACM SIGSAM Bulletin*, 23 (1989), no. 2, 34 - 38.
- [62] Zukerman, I. A new measure of a partial differential field extension. *Pacific J. Math.*, 15 (1965), no. 1, 357 - 371.