Differential square-zero extensions and Picard-Vessiot theory

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Motivation: Differential Deformation Theory

Long-term goal: understand differential families of differential algebraic varieties:

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & S \end{array}$$

Here $\mathcal{X}$ is a family of differential algebraic varieties parameterized by S. We think of it as a family of differential deformations of X.

In algebraic geometry, deformation theory is applied to construct "nice" families (moduli spaces) of algebraic varieties. One can sometimes use moduli spaces to prove finiteness results, etc.

The first step in deformation theory is to study first-order infinitesimal deformations (i.e., small neighborhoods of X in $\mathcal{X}$), or equivalently: square-zero extensions of rings.

Square-zero extensions

If A is a C-algebra, a *square-zero extension* of A is an exact seq

$$(R, \phi) : \quad 0 \to I \to R \xrightarrow{\phi} A \to 0$$

where R is a C-algebra, ϕ is a C-algebra homomorphism, and the kernel I is a finitely generated ideal such that $I^2 = 0$.

This makes I into an A-module: for $a \in A$ and $x \in I$, set $ax := rx$ for any $r \in \phi^{-1}(a)$. [For any other $r' \in \phi^{-1}(a)$ we have $rx = r'x$.]

Also: (R, ϕ) is a square-zero extension of A by (the A-module) I.

The *trivial square-zero extension* of A by I is $A \oplus I$ with multiplication given by $(a, x) \cdot (b, y) = (ab, ay + bx)$.

Split square-zero extensions

A *morphism of square-zero extensions* $R_1 \to R_2$ of A by I is $\eta \in \mathrm{Hom}_{C\text{-alg}}(R_1, R_2)$ making the following diagram commutative

$$\begin{array}{ccccc} I & \longrightarrow & R_1 & \xrightarrow{\phi_1} & A \\ \| & & \downarrow{\scriptstyle \eta} & & \| \\ I & \longrightarrow & R_2 & \xrightarrow{\phi_2} & A \end{array}$$

Any such η is an isomorphism, and if η' is another one then $\eta - \eta' \in \mathrm{Der}_C(A, I)$.

We say that (R, ϕ) is *split* if there exists a C-algebra section $\sigma : A \to R$ such that $\phi \circ \sigma = \mathrm{id}_A$. If $\sigma' : A \to R$ is another splitting, then $\sigma - \sigma' \in \mathrm{Det}_C(A, I)$.

A square-zero extension of A by I is split if and only if it is isomorphic to the trivial square-zero extension of A by I:

$$A \oplus I \to R : (a, x) \mapsto \sigma(a) + x.$$

Differential square-zero extensions

If A is a Δ-ring with subring of constants C, a *differential square-zero extension*[1] of A is an exact sequence

$$(R, \phi) : \quad 0 \to I \to R \xrightarrow{\phi} A \to 0$$

where R is a Δ-C-algebra, ϕ is a Δ-C-algebra homomorphism, and the kernel I is a finitely generated ideal such that $I^2 = 0$.

This makes I into a Δ-A-module, and we also say that (R, ϕ) is a differential square-zero extension of A by (the Δ-A-module) I.

A *Δ-morphism of differential square-zero extensions* is defined as before, with $\eta : R_1 \to R_2$ a Δ-homorphism.

The Δ-trivial extension is $A \oplus I$ with $\delta(a, x) = (\delta(a), \delta(x)) \; \forall \delta \in \Delta$.

We say that (R, ϕ) is *Δ-split* if there exists a Δ-C-algebra section $\sigma : A \to R$ such that $\phi \circ \sigma = \mathrm{id}_A$.

(R, ϕ) is Δ-split if and only if it is Δ-isomorphic to Δ-trivial $A \oplus I$.

[1] Magid 2016

Δ-structures on split square-zero extensions

Suppose A is a Δ-ring and I is a Δ-A-module. We wish to classify the differential square-zero extensions of A by I that are algebraically split: $R = A \oplus I$.

If $\Delta = \{\delta_1, \ldots, \delta_n\}$, a computation shows that

$$\delta_i^R(a, x) = (\delta_i^A(a), \delta_i^I(x) + h_i(a)),$$

where $h_i \in \mathrm{Der}_C(A, I)$. The commutativity $\delta_i^R \delta_j^R = \delta_j^R \delta_i^R$ for each i and j is equivalent to the integrability condition

$$\delta_i^I h_j - h_j \delta_i^A = \delta_j^I h_i - h_i \delta_j^A.$$

Consider the Δ-A-module structure on $D := \mathrm{Der}_C(A, I)$ given by

$$\delta_i^D(h) = \delta_i^I h - h \delta_i^A.$$

The integrability condition then becomes $\delta_i^D(h_j) = \delta_j^D(h_i)$.

Splittings of differential square-zero extensions

Proposition

Suppose that $\mathcal{D}_A = A \cdot \Delta \subseteq \mathrm{Der}_C(A, A)$ *is locally free of finite rank and* A *is* Δ*-simple. Then any differential square-zero extension of* A *is algebraically split.*

We restrict our attention to Δ-rings A satisfying the hypotheses of the Proposition, and consider a split (not Δ-split) differential square zero extension $R = A \oplus I$ with differential structure given by

$$\delta_i^R(a, x) = (\delta_i^A(a), \delta_i^I(x) + h_i(a))$$

for each $\delta_i \in \Delta$ as before. If $\sigma : A \to R : a \mapsto (a, g(a))$ is any other algebraic splitting (i.e., $g \in \mathrm{Der}_C(A, I)$), we have $\delta_i^R \sigma = \sigma \delta_i^A$:

$$(\delta_i^A(a), \delta_i^I(g(a)) + h_i(a)) = \delta_i^R(\sigma(a)) = \sigma(\delta_i^A(a)) = (\delta_i^A(a), g(\delta_i^A(a)))$$

if and only if $g\delta_i^A - \delta_i^I g = -\delta_i^D(g) = h_i$.

Δ-splittings and Picard-Vessiot extensions

Lemma

The differential square-zero extension $R = A \oplus I$ with differential structure given by $h_1, \ldots, h_n \in \mathrm{Der}_C(A, I)$ is Δ-split if and only if $\exists\ g \in \mathrm{Der}_C(A, I)$ such that $\delta_i^D(g) = h_i$ for each $i = 1, \ldots, n$.

If R is not Δ-split, we wish to find a Δ-ring extension $A \to B$ such that $R \otimes_A B = B \oplus (I \otimes_A B)$ has a differential structure given by $\tilde{h}_i \in \mathrm{Der}_C(B, I \otimes_A B)$ extending h_i on A and such that there exists $\tilde{g} \in \mathrm{Der}_C(B, I \otimes_A B)$ with $\delta_i^D(\tilde{g}) = \tilde{h}_i$.

This B should be a "Picard-Vessiot extension" for $\{\delta_i^D(g) = h_i\}$.

Technical difficulty: in general there is no predetermined choice for what the $\tilde{h}_i$ and $\tilde{g}$ should be, because we lack a map $\mathrm{Der}_C(A, I) \to \mathrm{Der}_C(B, I \otimes_A B)$.

N.B.: Since the trivial splitting $A \to R : a \mapsto (a, 0)$ is not a Δ-hom, the tensor product Δ-structure on $R \otimes_A B$ is incorrect.

Generalized differential rings (1)

We consider a more general definition of differential ring[2]: a pair $(A, \mathcal{D}_A)$ where A is a ring and $\mathcal{D}_A \subset \mathrm{Der}(A, A)$ is an A-submodule closed under the Lie bracket $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$.

A morphism of differential rings $(A, \mathcal{D}_A) \to (B, \mathcal{D}_B)$ is a pair (φ, D_φ), where $\varphi : A \to B$ is a ring homomorphism and $D_\varphi : \mathcal{D}_A \otimes_A B \to \mathcal{D}_B$ is a B-module isomorphism compatible with the Lie bracket and such that for every $a \in A$ and $\delta \in \mathcal{D}_A$ we have

$$\varphi(\delta(a)) = D_\varphi(\delta)(\varphi(a)).$$

If A is a Δ-ring we take $\mathcal{D}_A := A \cdot \Delta \subset \mathrm{Der}(A, A)$.

If $\mathcal{D}_A$ is locally free of finite rank, A is simple and noetherian, and C_A is algebraically closed, there is a Picard-Vesiot theory[3] for A-finitely generated differential A-modules.

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[2] Gillet-Gorchinskiy-Ovchinnikov 2013

[3] André 2001

Generalized differential rings (2)

Suppose that $(A, \mathcal{D}_A)$ is a simple differential ring with $\mathcal{D}_A = \mathrm{Der}_C(A, A)$ locally free of finite rank. Then

$$\mathrm{Der}_C(A, I) \simeq \mathrm{Hom}_A(\Omega^1_{A/C}, I) \simeq (\Omega^1_{A/C})^\vee \otimes_A I \simeq \mathcal{D}_A \otimes_A I.$$

If we consider the Δ-structures:

- on $\mathrm{Der}_C(A, I)$ given by $\delta_i^D(h) = \delta_i^I h - h\delta_i^A$;
- on $\mathcal{D}_A \otimes_A I$ given by $\delta_i^\otimes(d \otimes x) = [\delta_i^A, d] \otimes x + d \otimes \delta_i^I(x)$;

then $\mathrm{Der}_C(A, I) \simeq \mathcal{D}_A \otimes_A I$ is a differential isomorphism.

For any $(\varphi, D_\varphi) : (A, \mathcal{D}_A) \to (B, \mathcal{D}_B)$ the B-isomomorphism $D_\varphi : \mathcal{D}_A \otimes_A B \to \mathcal{D}_B$ gives

$$\mathrm{Der}_C(A, I) \to \mathcal{D}_A \otimes_A I \to \mathcal{D}_B \otimes_B (I \otimes_A B) \to \mathrm{Der}_C(B, I \otimes_A B).$$

Since D_φ is compatible with the Lie bracket, the composite $\mathrm{Der}_C(A, I) \to \mathrm{Der}_C(B, I \otimes_A B)$ is a differential homomorphism.

Main result

Suppose $(A, \mathcal{D}_A)$ is a simple and noetherian differential ring with $\mathcal{D}_A = \mathrm{Der}_C(A, A) = A \cdot \Delta$ locally free of finite rank and $C = A^\Delta$ algebraically closed.

Lemma

Let M be a Δ-A-module and $m_1, \dots m_n \in M$ such that $\delta_i^M m_j = \delta_j^M m_i$, where $\Delta = \{\delta_1, \dots, \delta_n\}$. Then there exists a Picard-Vessiot extension P of A and an element $\mu \in M \otimes_A P$ such that $\delta_i(\mu) = m_i \otimes 1 \in M \otimes_A P$.

Theorem

For any split differential square-zero extension $A \oplus I$ there exists a Picard-Vessiot extension P of A such that $P \oplus (I \otimes_A P)$ is Δ-split.

Proof.

Find a PV extension P of A and $g \in \mathrm{Der}_C(A, I) \otimes_A P$ such that $\delta_i^D(g) = h_i \otimes 1$. Use $\psi : \mathrm{Der}_C(A, I) \otimes_A P \to \mathrm{Der}_C(P, I \otimes_A P)$ to define Δ-structure on $P \oplus (I \otimes_A P)$ given by $\psi(h_i \otimes 1)$. Then $\sigma : p \mapsto (p, -\psi(g)(p))$ is a Δ-splitting $P \to P \oplus (I \otimes_A P)$. □

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