Examples as Avatars

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Abstract

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Throughout its development, differential algebraic geometry has received its impetus from interesting examples that both instruct and point to new paths to explore.

Three possible languages for the variational formalism: the classical, geometric, and differential algebra

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The language of differential algebra is better suited for expressing invariant properties of differential equations, and puts at the disposal of the investigator the extensive apparatus of commutative algebra, differential algebra and algebraic geometry. The numerous "explicit formulas" for the solutions of the classical and newest differential equations have good interpretations in this language; the same may be said for conservation laws.

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However, the language of differential algebra which has been traditional since the work of Ritt does not contain the means for describing changes of the functions (dependent variables) and the variables x_i *(independent variables), and for clarifying properties which are invariant under such changes. This is one of the main reasons for the embryonic state of so-called "Bäcklund transformations" in which there has been a recent surge of interest.*

Notation etc.

All rings have characteristic 0. $\mathbb{C}$ = field of complex nos.
Commuting derivation operators: $\Delta = \{\delta_1, \cdots, \delta_m\}$,
$\mathcal{F} = \mathbb{C}(x_1, \ldots, x_m)$, $\delta_i \mid \mathcal{F} = \partial_{x_i}$

Definition

Δ-field $\mathcal{F}^\dagger$ is differentially closed $\Longleftrightarrow$ A set of Δ-poly eqs / $\mathcal{F}^\dagger$ with a soln rational over an extension Δ-field of $\mathcal{F}^\dagger$ has a solution rational over $\mathcal{F}^\dagger$.

$\mathcal{F}^\dagger$ =diff closure $(\mathcal{F})$, $\mathbb{C} = \left(\mathcal{F}^\dagger\right)^\Delta$

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$y = (y_1, \ldots, y_n)$: Δ-indeterminates.
$y^{(r)}$ = derivatives of the y_i of order $\leq r$.
$\mathcal{F}^\dagger \{y\} = \cup_{r \in \mathbb{N}} \mathcal{F}^\dagger \left[y^{(r)} \right]$: Δ-polynomial ring.

Kolchin topology

Definitions

$X \subseteq \mathbb{A}^n$ is *Kolchin closed* (a Δ-variety) if it is the set of solns of a finite set of Δ-poly eqs.

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Let $F_1, \ldots, F_r$ be in $\mathcal{F}^\dagger\{y\}$. The Δ-ideal $[F_1, \ldots, F_r]$ generated by $F_1, \ldots, F_r$ is the ideal generated by $F_1, \ldots, F_r$ and their derivatives of all orders.

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Differential rational functions

X: Δ-variety in $\mathbb{A}^n$
$\mathfrak{p}$: its defining Δ-ideal.
$\overline{y} = (\overline{y_1}, \ldots, \overline{y_n})$, $\overline{y_j} = y_j \bmod \mathfrak{p}$. $\mathcal{F}^\dagger\{\overline{y}\}$ = ring of Δ-*poly fcns* on X. Its quotient field $\mathcal{F}^\dagger\langle X\rangle$ is the field of Δ-*rational functions* on X.

A Δ-*rational map* $\varphi : X \dashrightarrow \mathbb{A}^p$ is a p-tupleof Δ-rational functions on X. Dom T is a Kolchin-open dense subset U of X.

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Theorem

(Chevalley-Seidenberg ConstructabilityTheorem) $T(U)$ contains a Kolchin-open subset of its Kolchin closure.

Example

$\mathcal{F} = \mathbb{C}(x)$, $\Delta = \left\{\delta = \frac{d}{dx}\right\}$. $\mathsf{P}_{II}(c)$ is the second order ordinary differential equation

$$y'' = 2y^3 + xy + c, \qquad c \in \mathbb{C}.$$

It defines a Δ-subvariety $X(c)$ of the affine line $\mathbb{A}^1$.

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It defines a Δ-subvariety $X(c)$ of the affine line $\mathbb{A}^1$.Spose $c \neq \frac{1}{2}$.

$$T(y) := -y - \frac{c - \frac{1}{2}}{\partial_x y - y^2 - \frac{x}{2}}.$$

$T(X(c)) = X(c-1)$. It is a morphism of Δ-varieties.

Example – A transformation of the affine line

$\Delta = \{\delta_1, \ldots, \delta_m\}$
The logarithmic derivative map: $\ell\delta_i : \mathbb{A}^1 \dashrightarrow \mathbb{A}^1 \quad \ell\delta_i y = \frac{\delta_i y}{y}$

Linear differential algebraic groups

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All Δ-groups in this talk are Δ-subgroups of $GL(n, \mathcal{F}^{\dagger})$.
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Definition

Δ-group homomorphism: group homomorphism and morphism of Δ-varieties.

The dimension polynomial

$X :=$ Δ-subvariety of $\mathbb{A}^n$, with defining Δ-ideal $\mathfrak{p}$, and field $\mathcal{F}^\dagger \langle X \rangle$ of Δ-rational functions.
$\Delta\text{-dim}\ (X) = \Delta\text{-tr deg}_{\mathcal{F}^\dagger}\ (\mathcal{F}^\dagger \langle X \rangle)$
$=$ max # Δ- alg indep generators of $\mathcal{F}^\dagger \langle X \rangle / \mathcal{F}^\dagger$.

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Classical differential equations theory

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$\mathcal{F}^{\dagger}\langle X\rangle = \varinjlim_{r\in\mathbb{N}} \mathcal{F}^{\dagger}\left(\overline{y}^{(r)}\right)$, where $\overline{y}^{(r)}$ is the family of derivatives of $\overline{y}$ of order $\leq r$. It is an inductive limit of finitely generated fields.

Dim poly d_X: a numerical polynomial.
$d_X(\overline{y}) = \text{tr deg}_{\mathcal{F}^\dagger}\mathcal{F}^\dagger\left(\overline{y}^{(r)}\right), \quad r \gg 0.$

$$d_X = \sum_{j=0}^{m} a_m \binom{X+m}{m}, a_m \in \mathbb{Z}.$$

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$X \subsetneq Y \Longrightarrow d_X < d_Y$
d_X not a diff birat invariant.d and a_d are diff. birat. invariants

Example 1 – Turbulence into Heat

$\mathcal{F} = \mathbb{C}(x, t)$
$\Delta = \{\partial_x, \partial_t\}$
$\mathcal{F}^\dagger =$ diff closure of $\mathbb{C}(x, t)$

$$H = \left\{ u \in \mathbb{G}_a : \partial_x^2 y - \partial_t y = 0 \right\}.$$

H is a Δ-subgroup of the additive group $\mathbb{G}_a$.

Dim poly of Heat variety

Defining diffpoly (H):

$$h = \partial_x^2 y - \partial_t y.$$

$$d_H = 2\binom{X+1}{1} - 1 = 2X + 1.$$

Algorithm: Kolchin, *Differential algebra and algebraic groups*, Lemma 16, p. 51 and Theorem 6, p. 115.

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Δ-type $(H) = 1$.
Typ Δ-dim $(H) = 2$.
Generically, the solns heat eq depend on 2 **arbitrary functions** of t.

A power series expansion

Near the origin, $f(t)$ and $g(t)$, are restrictions to $x = 0$ of y and $\partial_x y$.

$$\begin{aligned} y(x,t) &= y(t,0) + \partial_x y(t,0)x + \partial_x^2 y(t,0)\frac{x^2}{2!} + \partial_x^3 y(t,0)\frac{x^3}{3!}\dots \\ &= f(t) + g(t)x + f'(t)\frac{x^2}{2!} + g'(t)\,\frac{x^3}{3!}\dots \end{aligned}$$

The ring $\mathcal{F}^{\dagger}\{\overline{y}\}$of Δ-polynomial functions on H is the ∂_x-polynomial ring

$$\mathcal{F}^{\dagger}\left[\overline{y}, \partial_x \overline{y}, \partial_x^2 \overline{y}, \ldots\right]$$

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Fact

Every proper Δ-subgroup of the Heat variety H has Δ-type 0 (is a finite-dimensional $\mathbb{C}$-vector space).

The Cole-Hopf transformation

$\ell\partial_x : H \dashrightarrow \mathbb{A}^1$ is called the *Cole-Hopf* transformation. Dom $(\ell\partial_x)$ is the Kolchin dense open set $H \backslash \{0\}$. What is the image of $H \backslash \{0\}$ under the Cole-Hopf transformation? Let $v \in \mathbb{A}^1$. Describe the fiber $(\ell\partial_x)^{-1}(v)$ in H.

$v \in \ell\partial_x \left(H \setminus \{0\}\right) \Longleftrightarrow \exists u \neq 0$ in H such that $\partial_x u - vu = 0$. u satisfies the differential equations

$$\begin{aligned} h &= \partial_t u + \partial_x^2 u = 0, \\ g &= \partial_x u - vu = 0, \end{aligned}$$

and the inequation

$$u \neq 0.$$

The defining Δ-ideal $\mathcal{F}\langle v\rangle\{y\}$ of the fiber is the prime Δ-ideal:

$$\begin{aligned} \mathfrak{p} &= [h, g]. \\ h &= \partial_t y + \partial_x^2 y, \quad g = \partial_x y - vy \end{aligned}$$

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Fix an orderly ranking of the derivative operators, with $\partial_x < \partial_t$.
Rosenfeld algorithm (*pseudoreduction*) $\longrightarrow$ *autoreduced* set of generators of $\mathfrak{p}$: Each poly is free of derivatives of the leader of the other.

The autoreduced set is:

$$
\begin{aligned}
& \{g, r\}, \\
g &= \partial_x y - vy, \\
r &= \partial_t y + \left(v^2 + \partial_x v\right) y.
\end{aligned}
$$

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$$\partial_t \partial_x y = \partial_x \partial_t y$$

Rosenfeld coherence:

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Rosenfeld coherence:

$$\begin{aligned} S &= \partial_x r - \partial_t g \in \mathfrak{p}. \\ &= vr + \left(v^2 + \partial_x v\right) g + b(v)y, \quad \text{where} \\ b(v) &= \partial_x^2 v + \partial_t v + 2v\partial_x v. \end{aligned}$$

Rosenfeld coherence condition

$$b(v) = \partial_x^2 v + \partial_t v + 2v\partial_x v.$$

$b\,(v) = 0 \Longrightarrow S$ is in the ideal generated by g, r. The leaders of g, r are $< \partial_t \partial_x y$, the lowest common derivative of the leaders. Thus, the set $\{g, r\}$ is a coherent.autoreduced set Thus it is a characteristic set of $\mathfrak{p}$.

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$b\left(v\right) = 0 \Longrightarrow S$ is in the ideal generated by g, r. The leaders of g, r are $< \partial_t \partial_x y$, the lowest common derivative of the leaders. Thus, the set $\{g, r\}$ is a coherent.autoreduced set Thus it is a characteristic set of $\mathfrak{p}$. Therefore, $y \notin \mathfrak{p}$. Therefore, there is a nonzero element in the fiber over v in H.

If $b(v) \neq 0$, $\mathfrak{p} = [y]$. 0 is the only solution of the given system of equations and inequation. Thus, $\ell\partial_x (H \setminus \{0\}) =$

$$\left\{ v \in \mathbb{G}_a : 0 = b(v) = \partial_x^2 v + \partial_t v + 2v\partial_x v \right\}.$$

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The differential equation $b(v) = 0$ is called the *Burgers equation* (without external force) (Johannes Martinus Burgers–1895-1981, Harry Bateman, 1914). It models turbulence that is not sensitive to initial conditions. Denote the Δ-variety defined by b by B.

The Integrability conditions

$\exists u \in H$ such that $g(u) = r(u) = 0 \Longleftrightarrow$

$$\begin{aligned} \ell \partial_x u &= v \\ \ell \partial_t u &= -(v^2 + \partial_x v) \end{aligned}$$

The Integrability conditions

$\exists u \in H$ such that $g(u) = r(u) = 0 \Longleftrightarrow$

$$\begin{aligned} \ell\partial_x u &= v \\ \ell\partial_t u &= -(v^2 + \partial_x v) \\ &\Longleftrightarrow \partial_t v = -\partial_x(v^2 + \partial_x v) \Longleftrightarrow b(v) = 0. \end{aligned}$$

So, the Burgers equation is the integrability condition for the existence of a nonzero pre-image of v.

The principal bundle defined by Cole-Hopf

Let $v \in B$. The fiber over v in H. Let $u \in H$ be such that $\ell\partial_x u = v$.

$$\ell\partial_x^{-1}(v) = \mathbb{G}_m(\mathbb{C})\, u.$$

So, the Heat variety covers the Burgers variety – $\ell\partial_x$ defines a principal bundle for the multiplicative Δ-group of constant field $\mathbb{C}$.

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So, the Heat variety covers the Burgers variety – $\ell\partial_x$ defines a principal bundle for the multiplicative Δ-group of constant field $\mathbb{C}$. We turn to our second example: another well-known principal bundle.

The *irreducible Painlevé equations* are 6 ordinary differential equations of *second order* and defined over $\mathbb{C}(x)$ of the form

$$y'' = F\left(x, y, y', c\right),$$

where F is a rational function, and c is a constant.

The equations satisfy:

1. *Painlevé Property*: the absence of movable singularities. The locations of the singularities of the solutions (apart from poles) depend only on the coefficients of F.

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2. The equation cannot be integrated algebraically, or transformed into a simpler equation or the defining first order differential equation of an elliptic function.

Property 2: *generically*, the solutions are *essentially transcendental*
Paul Prudent Painlevé and his student B. Gambier from 1900 to 1906 (about 100 years ago).

Painlevé I

$\Delta = \{\delta\}$, $\mathcal{F} = \mathbb{C}(x), \delta = \frac{d}{dx}; \mathcal{F}^{\dagger} =$ diff closure $(\mathcal{F})$

$$\text{PI:} \quad y'' = 6y^2 + x$$

All solutions are essentially transcendental.
The variety has no rational solutions and no Δ-subvariety of typical dim 1.

Theorem

Let $W \subset \mathbb{A}^1$ be defined by PI. $\exists$ Δ-subvariety T of typical dimension 4 of $\mathbb{G}_m$ and a surjective morphism $\mu : T \longrightarrow W$ whose fibers are principal homogeneous spaces (cosets) of a Δ-subgroup G of $\mathbb{G}_m$.

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$$\frac{d}{dx}\left(\frac{y'}{y}\right) = \frac{d}{dx}\left(\ell \frac{d}{dx} y\right) = 0.$$

$$\mu(\tau) = -\frac{d}{dx}\left(\frac{\tau'}{\tau}\right), \tau \in T.$$

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$$G = \left\{ y = e^{c_1 x + c_0} \mid c_0, c_1 \in \mathbb{C} \right\}.$$

T has defining differential equation

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Definition

The elements of T are called the τ-functions associated with PI.

Surjectivity: μ is dominant. $\mu(T)$ contains a Kolchin-open dense subset U of W. $W \setminus U$ must be empty.

Let V, W be irreducible Δ-varieties of the same type. Let $\varphi : V \longrightarrow W$ be a dominant Δ-rational morphism. Is the following generalization of Sit's theorem true generically?

$$\text{typdim } \varphi^{-1}(w) + \text{ typdim } W = \text{ typdim } V$$

Definitions

An algebraic group (Δ-group) is *simple* if every proper normal algebraic subgroup (Δ-subgroup) is finite. Equivalently, its Lie algebra is simple.

A *Chevalley group* is a simple algebraic group that is defined over the field $\mathbb{Q}$ of rational numbers.

Theorem

Let Γ be a simple Δ-group.

1. $\exists$ *Chevalley group H such that Γ is isomorphic to a Zariski dense Δ-subgroup G of H.*
2. *If $G \neq H$, $\exists$ A in the Lie algebra of matrices of H such that G is the Δ-subgroup of H defined by the equation*

$$\begin{aligned} YAY^{-1} + Y' \cdot Y^{-1} &= A. \\ Y' &= [A, Y] \quad \textit{Lax eqn} \end{aligned}$$

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Note: The Lax eqn is the defining equation of G, Lie G, and the assoc.subalg of $\mathrm{M}\left(n, \mathcal{F}^{\dagger}\right)$ generated by G.

Gauge action

Let $\mathcal{F} = \mathbb{C}\left(x\right)$, $\quad \mathcal{F}^{\dagger} =$ diff closure of $\mathcal{F}$. Then, $\left(\mathcal{F}^{\dagger}\right)^{\Delta} = \mathbb{C}$.
The Δ- group $SL(2, \mathcal{F}^{\dagger})$ acts by affine transformations on its Lie algebra of matrices $sl\left(2, \mathcal{F}^{\dagger}\right)$:

$$A \longmapsto ZAZ^{-1} + Z' \cdot Z^{-1}.$$

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The *gauge action* is transitive.
The proper Zariski dense Δ-subgroups of $SL\left(2, \mathcal{F}^{\dagger}\right)$ are the isotropy groups of matrices in $sl\left(2, \mathcal{F}^{\dagger}\right)$ under the gauge action.

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The proper Zariski dense Δ-subgroups of $SL\left(2, \mathcal{F}^{\dagger}\right)$ are the isotropy groups of matrices in $sl\left(2, \mathcal{F}^{\dagger}\right)$ under the gauge action.We turn now to our third example.

Configurations of four points on the projective line

This is the story of configurations of 4 points on $\mathbb{P}^1(\mathcal{F}^\dagger)$ (Emile Picard)

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Configurations of four points on the projective line

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A Δ-variety V in $\mathbb{A}^1\left(\mathcal{F}^\dagger\right)$ is a Riccati variety if it is the set of solutions of

$$y' = a_0 + a_1 y + a_2 y^2, \;\; a_0, a_1, a_2 \in \mathcal{F}^\dagger.$$

Configurations of four points on the projective line

This is the story of configurations of 4 points on $\mathbb{P}^1(\mathcal{F}^\dagger)$ (Emile Picard) Riccati varieties on the projective line $\mathbb{P}^1(\mathcal{F}^\dagger)$ behave like circles on the projective line $\mathbb{P}^1(\mathbb{C})$.
A Δ-variety V in $\mathbb{A}^1\left(\mathcal{F}^\dagger\right)$ is a Riccati variety if it is the set of solutions of

$$y' = a_0 + a_1 y + a_2 y^2, \;\; a_0, a_1, a_2 \in \mathcal{F}^\dagger.$$

$y : \frac{y_1}{y_0}$ The homogenization of the Riccati equation is

$$y_1' y_0 - y_1 y_0' = a_0 y_0^2 + a_1 y_1 y_0 + a_2 y_1^2$$

$\infty = (1, 0)$. The *affine variety* V is Kolchin closed in $\mathbb{P}^1(\mathcal{F}^\dagger)$ if and only if $a_2 \neq 0$.

The representing matrix

Definition

The matrix

$$A = \begin{pmatrix} (\frac{1}{2})a_1 & a_0 \\ -a_2 & -(\frac{1}{2})a_1 \end{pmatrix}.$$

represents the Riccati equation

$$y' = a_0 + a_1 y + a_2 y^2.$$

Example

Example

Let

$$A = \begin{pmatrix} 0 & \frac{x}{2} \\ -1 & 0 \end{pmatrix}.$$

A represents the Riccati equation

$$y' = \frac{x}{2} + y^2.$$

The projective representation of the special linear group

$\mathbb{P}^1\left(\mathcal{F}^\dagger\right) = \mathbb{A}^1\left(\mathcal{F}^\dagger\right) \cup \infty.$

$$\begin{aligned} a + \infty &= \infty + a = \infty, \quad \text{if } a \in \mathbb{A}^1\left(\mathcal{F}^\dagger\right) \\ a \cdot \infty &= \infty \cdot a = \infty \quad \text{if } a \neq 0 \\ \frac{a}{0} &= \infty \ \text{ if } a \neq 0 \ \ \frac{a}{\infty} = 0 \ \text{ if } a \neq \infty \end{aligned}$$

Definition

$PSL(2,\mathcal{F}^\dagger)$ is the group of projective linear transformations of $\mathbb{P}^1(\mathcal{F}^\dagger)$. Let $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in SL(2,\mathcal{F}^\dagger)$.

$$\begin{aligned}
\text{Let } u &\in PSL(2,\mathcal{F}^\dagger), \qquad u \neq \infty, -\frac{d}{c}\\
T(u) &= \frac{\alpha u + \beta}{\gamma u + \delta}\\
T(\infty) &= \frac{\alpha}{\gamma} \quad T\left(-\frac{\delta}{\gamma}\right) = \infty.
\end{aligned}$$

The triple-transitivity of the projective special linear group.

Let u_1, u_2, u_3 be distinct points in $\mathbb{P}^1\left(\mathcal{F}^\dagger\right)$. $\exists$ a unique projective linear transformation λ mapping u_1, u_2, u_3 to $0, 1, \infty$.

$$\lambda\left(u\right) = \frac{\left(u - u_1\right)\left(u_2 - u_3\right)}{\left(u_1 - u_2\right)\left(u_3 - u\right)} \text{if } u_1, u_2, u_3 \neq \infty.$$

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$$\begin{aligned}
\lambda\left(u\right) &= -\left(\frac{u_2 - u_3}{u_3 - u}\right) \text{if } u_1 = \infty, \\
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\lambda\left(u\right) &= -\left(\frac{u - u_1}{u_1 - u_2}\right) \text{if } u_3 = \infty.
\end{aligned}$$

If $u_1, u_2, u_3 \neq \infty$, the determinant of any matrix representing $\lambda$is $(u_1 - u_2)(u_2 - u_3)(u_3 - u_1) \neq 0$

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Corollary

Let u_1, u_2, u_3 and v_1, v_2, v_3 be triples of distinct points in $\mathbb{P}^1(\mathcal{F}^\dagger)$. . There is a unique projective linear transformation λ mapping u_1, u_2, u_3 to v_1, v_2, v_3.

Corollary

$PSL\left(2, \mathcal{F}^{\dagger}\right)$ is not 4-transitive.

Corollary

Let $u,\ u_1, u_2, u_3$ and $v,\ v_1, v_2, v_3$ be quadruples of points in $\mathbb{P}^1(\mathcal{F}^{\dagger})$, with u_1, u_2, u_3 and $v_1, v_2 v_3$ distinct. There is a projective linear transformation mapping $u,\ u_1, u_2, u_3$ to $v,\ v_1, v_2, v_3$ iff their cross-ratios are equal.

Corollary

Cross-ratio is an invariant of the projective linear group.

The Metaphor

Theorem

Let u_1, u_2, u_3 be distinct points in $\mathbb{P}^1(\mathbb{C})$. There is a unique circle or line C containing u_1, u_2, u_3. The circle is the curve in the complex plane defined by the condition

$$\frac{(u - u_1)(u_2 - u_3)}{(u_1 - u_2)(u_3 - y)} = x \in \mathbb{P}^1(\mathbb{R}), \quad \mathbb{R} \textit{ the field of real numbers.}$$

$PSL(2, \mathbb{C})$ permutes the circles in $\mathbb{P}^1(\mathbb{C})$. Let $\mathbb{R}$ be the field of real numbers. $PSL(2, \mathbb{R})$ is the stabilizer of $\mathbb{P}^1(\mathbb{R})$.

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Riccati eq – simplest NMS eq.

Fix a *fundamental system* u_1, u_2, u_3 of disinct solutions of a Riccati equation. A function $u \neq u_1, u_2, u_3$ lies on the same Riccati variety $\Longleftrightarrow$ their cross-ratio is a constant $\Longleftrightarrow$

$$\exists c \in \mathbb{C} : \frac{(u - u_1)(u_2 - u_3)}{(u_1 - u_2)(u_3 - u)} = c. \qquad \text{(Picard)}$$

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This is called a *non-linear superposition principle*. Thus, the only singularities of u are poles.

Note: By Lie's Theorem, a non-linear superposition principle $\overrightarrow{u} = F(\overrightarrow{u_1}, \ldots, \overrightarrow{u_r}, c_1, \ldots, c_s)$ ($\overrightarrow{u} = (u_1, \ldots, u_n)$) for the solution set of a first order ordinary ($\delta = \frac{d}{dt}$) vector differential equation exists if and only if the equation is in the form

$$\overrightarrow{u}' = z_1(t)\overrightarrow{u_1} + \cdots + z_r(t)\overrightarrow{u_r},$$

where the vector fields $\overrightarrow{u_1}, \cdots, \overrightarrow{u_r}$ generate a finite-dimensional Lie algebra. If $n = 1$, as it is here, this Lie algebra is a proper subalgebra (over $\mathbb{R}$ or $\mathbb{C}$ for Lie) of the Lie algebra of vector fields on a 1-manifold. Thus, it is the Lie algebra

$$sl_2 = \left\{ z_1(t)\frac{d}{du} + z_2(t)\, u\frac{d}{du} + z_3(t)\, u^2\frac{d}{du} \right\},$$

giving us the Riccati equation.

$$\frac{du}{dt} = z_1(t) + z_2(t)\, u + z_3(t)\, u^2.$$

Theorem

Let u_1, u_2, u_3 be distinct points in $\mathbb{P}^1\left(\mathcal{F}^\dagger\right)$. There is a unique Riccati variety V containing u_1, u_2, u_3. The Riccati variety is the Δ-variety defined by the condition

$$\frac{(u-u_1)(u_2-u_3)}{(u_1-u_2)(u_3-u)} = \frac{(u_2-u_3)\,u + u_1(u_3-u_2)}{(u_2-u_1)\,u + u_3(u_1-u_2)} \in \mathbb{P}^1(\mathbb{C}).$$

The space of Riccati varieties

Identify the Riccati variety V:

$$y' = a_0 + a_1 y + a_2 y^2$$

with $\begin{pmatrix} a_0 & a_1 & a_2 \end{pmatrix}$, the coordinate vector of the representing matrix $A = \begin{pmatrix} (\frac{1}{2})a_1 & a_0 \\ -a_2 & -(\frac{1}{2})a_1 \end{pmatrix}$ with respect to an appropriate basis of $\mathrm{sl}\left(2, \mathcal{F}^{\dagger}\right)$ of matrices in $\mathrm{sl}\left(2, \mathbb{C}\right)$.
Set $\begin{pmatrix} a_0 & a_1 & a_2 \end{pmatrix} = R\left(a_0, a_1, a_2\right)$.

Let

$$Z = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha\delta - \beta\gamma = 1.$$

Let $\begin{pmatrix} b_0 & b_1 & b_2 \end{pmatrix} = R(b_0, b_1, b_2)$. The gauge action by Z transforms $R(a_0, a_1, a_2)$ into $R(b_0, b_1, b_2)$

$$\begin{pmatrix} b_0 & b_1 & b_2 \end{pmatrix}^t$$

$$= \begin{pmatrix} \alpha^2 & -\alpha\beta & \beta^2 \\ -2\alpha\gamma & \alpha\delta + \beta\gamma & -2\beta\delta \\ \gamma^2 & -\delta\gamma & \delta^2 \end{pmatrix} \begin{pmatrix} a_0 & a_1 & a_2 \end{pmatrix}^t$$

$$+ \begin{pmatrix} \alpha\beta' - \alpha'\beta & \delta\alpha' - \delta'\alpha + \beta\gamma' - \beta'\gamma & \gamma\delta' - \gamma'\delta \end{pmatrix}^t.$$

The stabilizers of Riccati varieties

Theorem

$PSL(2,\mathcal{F}^{\dagger})$ permutes the Riccati varieties in $\mathbb{P}^1\left(\mathcal{F}^{\dagger}\right)$ by an action induced by the its gauge action of $SL\left(2,\mathcal{F}^{\dagger}\right)$ on its Lie algebra.

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Theorem

$G \subset SL\left(2, \mathcal{F}^{\dagger}\right)$ is the stabilizer of a Riccati variety $\Longleftrightarrow$it is a proper Zariski dense Δ-subgroup $\Longleftrightarrow$ G is the isotropy group of the matrix A representing the equation.

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Corollary

The stabilizers of Riccati varieties are conjugate to $SL\left(2, \mathbb{C}\right)$

The classical solutions of PII

We turn to an application of the gauge action on Riccati varieties. Deformation equations suggest the presence of a differential algebraic group.

$$\mathrm{P}_{II}(c): \quad y'' = 2y^3 + xy + c.$$

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Let $X(c)$ be the variety it defines.

Theorem

The Painlevé variety $X\left(\frac{1}{2}\right)$ has a Δ-subvariety $V\left(\frac{1}{2}\right)$ of typical dimension 1. It is a Riccati variety, defined by the equation

$$y' = y^2 + (\frac{1}{2})x.$$

The Δ-rational map

$$z \longmapsto -\frac{z'}{z}$$

transforms the Airy equation

$$z'' + \frac{x}{2}z = 0$$

into the equation

$$y' = y^2 + (\frac{1}{2})x.$$

So, the elements of $V(\frac{1}{2})$ are called *Airy solutions* of $\mathrm{P}_{II}(\frac{1}{2})$.

Corollary

The Riccati variety $V\left(\frac{1}{2}\right)$ *is a homogeneous space, under the gauge action of* $SL\left(2, \mathcal{F}^{\dagger}\right)$*, of the isotropy group* G *of the matrix*

$$A = \begin{pmatrix} 0 & \frac{x}{2} \\ -1 & 0 \end{pmatrix}.$$