Differential Algebraic Geometry, Part II

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October 19, 2007

Basic assumptions.

All rings are associative, commutative, with 1, and contain the field $\mathbb{Q}$ of rational numbers. Inparticular, we eliminate the 0 ring.
$\Delta = \{\delta_1, \ldots, \delta_m\}$ commuting derivation operators.
$\Theta = \left\{\theta = \delta_1^{i_1} \cdots \delta_m^{i_m} : i = (i_1, \ldots, i_m) \in \mathbb{N}^m\right\}$.
$\mathcal{R}$: Δ-ring. We reserve the symbols $y, y_1, \ldots, y_n$ for Δ-indeterminates or families of such.

Declaration of Future Intent: Kolchin topology – affine varieties.

Kovacic approach: Points are prime Δ-ideals. Define

$$X = \mathrm{Diffspec}\mathcal{R}$$

to be the set of prime Δ-ideals of $\mathcal{R}$. $\forall\ S \subseteq \mathcal{R}$,

$$V(S) = \{\mathfrak{p} \in X : S \subseteq \mathfrak{p}\}.$$

$V \subseteq X$ is *closed* if

$$\exists S \subseteq \mathcal{R} : V = V(S).$$

Weil-Kolchin approach: Kolchin closed sets: Zero sets of Δ-polynomial ideals. Points: n-tuples, coordinates in a differentially closed Δ-field. Coordinate ring $\mathcal{R}$: finitely Δ-generated over a Δ-field,and reduced.

Differential ideals, part II.

Lemma

Let $a, b \in \mathcal{R}, \theta \in \Theta$, *ord* $\theta = d$. *Then,* $a^{d+1}\theta b \in [ab]$.

Proof.

We argue by induction on d. The base case $d = 0$ is clearly true. So, suppose ord $\theta = d \geq 1$. Write $\theta = \delta\theta'$.Then, ord $\theta' \leq d - 1$. Therefore, by the induction assumption, $a^d\theta' b \in [ab]$.Therefore,

$$\begin{aligned} & a(da^{d-1}\delta a\theta' b + a^d\delta\theta' b) \\ = \; & d\delta a \; a^{d-1}\theta' b + a^{d+1}\delta\theta' b \in [ab]. \end{aligned}$$

It follows that

$$a^{d+1}\theta b \in [ab].$$

□

Corollary

Let $a, b \in \mathcal{R}, \theta \in \Theta$*. Then,* $a\theta b$ *and* $b\theta a$ *are in* $\sqrt{[ab]}$*.*

Proof.

If ord $\theta = d$, $a^{d+1}\theta b \in [ab]$. Therefore, $(a\theta b)^{d+1} \in [ab]$. So, $a\theta b \in \sqrt{[ab]}$. Now, interchange a and b. □

Definition

Let $\mathfrak{i}$ be a Δ-ideal of $\mathcal{R}$, and let $a \in \mathcal{R}$.

$$\mathfrak{i} : a^{\infty} = \{ b \in \mathcal{R} : \exists n \in \mathbb{N} \text{ with } a^n b \in \mathfrak{i} \} .$$

Lemma

If $\mathfrak{i}$ is a Δ-ideal of $\mathcal{R}$, and $a \in \mathcal{R}$. Then $\mathfrak{i} : a^{\infty}$ is a Δ-ideal of $\mathcal{R}$ containing $\mathfrak{i}$, and is radical if $\mathfrak{i}$ is radical.

Proof.

Clearly, $\mathfrak{i} : a^{\infty}$ is an ideal of $\mathcal{R}$ containing $\mathfrak{a}$. Let $b \in \mathfrak{i} : a^{\infty}$, and $\delta \in \Delta$. $\exists n \in \mathbb{N}$ such that $a^n b \in \mathfrak{i}$. By the last lemma, $a^{2n} \delta b \in [a^n b] \subset \mathfrak{i}$. Therefore, $\delta b \in \mathfrak{i}$. Spose $\mathfrak{i}$ is radical. Let $x \in \mathcal{R}$ and spose $x^p \in \mathfrak{i} : a^{\infty}$ for some positive integer p. Then, $\exists n \in \mathbb{N}$ such that $a^n x^p \in \mathfrak{i}$. Therefore, $ax \in \sqrt{\mathfrak{i}} = \mathfrak{i}$. □

Definition

Let $\mathfrak{i}$ be a radical Δ-ideal of $\mathcal{R}$, and let $a \in \mathfrak{i}$. Then, $\mathfrak{i} : a = \{b \in \mathcal{R} : ab \in \mathfrak{i}\}$.

Lemma

If $\mathfrak{i}$ is a radical Δ-ideal of $\mathcal{R}$, and $a \in \mathcal{R}$, then $\mathfrak{i} : a$ is a radical Δ-ideal of $\mathcal{R}$.

Proof.

Clearly, $\mathfrak{i}$ is an ideal of $\mathcal{R}$. Let $b \in \mathfrak{i} : a$ and $\delta \in \Delta$. Then, $a^2\delta b \in [ab] \subset \mathfrak{i}$. Since $\mathfrak{i}$ is radical, $a\delta b \in \mathfrak{i}$. So, $\mathfrak{i} : a$ is a Δ-ideal. As in the last proof, it is radical. □

Corollary

Let $\mathfrak{i}$ *be a radical* Δ*-ideal of* $\mathcal{R}$*, and let* $A \subset \mathcal{R}$*. Let* $\mathfrak{t} = \{t \in \mathcal{R} : At \subset \mathfrak{i}\}$*. Then,* $\mathfrak{t}$ *is a radical* Δ*-ideal of* $\mathcal{R}$*.*

Proof.

$\mathfrak{t} = \bigcap_{a \in A} \mathfrak{i} : a$. For every $a \in A$, $\mathfrak{i} : a$ is a radical Δ-ideal of $\mathcal{R}$. We saw in Part I that the intersection of a set of radical Δ-ideals is a radical Δ-ideal. □

Lemma

Let S, T be subsets of $\mathcal{R}$. Then,

$$\sqrt{[S]}\sqrt{[T]} \subseteq \sqrt{[ST]}$$

Proof.

Set $\mathfrak{i} = \sqrt{[ST]}$. By the corollary, $\mathfrak{t} = \{t \in \mathcal{R} : St \subset \mathfrak{i}\}$ is a radical Δ-ideal, clearly containing T, hence containing $\sqrt{[T]}$. So, $S\sqrt{[T]} \subset \sqrt{[ST]}$. But, $\mathfrak{s} = \{s : s\sqrt{[T]} \subset \mathfrak{i}\}$is a radical Δ-ideal containing S, hence containing $\sqrt{[S]}$. Thus, $\sqrt{[S]}\sqrt{[T]} \subset \sqrt{[ST]}$. □

Corollary

Let S, T be subsets of $\mathcal{R}$. Then

$$\sqrt{[ST]} = \sqrt{[S]} \cap \sqrt{[T]}.$$

Proof.

$\sqrt{[ST]} \subseteq \sqrt{[S]} \cap \sqrt{[T]}$. Let $a \in \sqrt{[S]} \cap \sqrt{[T]}$. Then, there exist positive integers k, l such that $a^k \in [S]$, and $a^l \in [T]$. Thus, $a^{k+l} \in \sqrt{[S]}\sqrt{[T]} \subseteq \sqrt{[ST]}$. □

Maximal ideals

Unless otherwise indicated, in this section, and the next, $\mathcal{R}$ is not necessarily differential. Reference: Bourbaki, *Commutative Algebra*, Ch. II, section 6, p. 73.

Definition

A subset Σ of $\mathcal{R}$ containing 1, and closed under multiplication, is called a multiplicative set in $\mathcal{R}$.

Lemma

Let $\mathfrak{a}$ be an ideal in $\mathcal{R}$, and let Σ be a multiplicative set such that $\Sigma \cap \mathfrak{a}$ is empty ($\mathfrak{a}$ avoids Σ). There exists an ideal $\mathfrak{m}$ of $\mathcal{R}$ containing $\mathfrak{a}$, and maximal with respect to avoiding Σ. Furthermore, $\mathfrak{m}$ is prime.

Corollary

$\mathcal{R}$ has maximal ideals, and all of them are prime.

Proof.

Set Σ equal to 1, and $\mathfrak{a}$ equal to 0. □

Theorem

Let $\mathcal{R}$ be a Δ-ring, and let $\mathfrak{a}$ be a Δ-ideal of $\mathcal{R}$. Let Σ be a multiplicative set in $\mathcal{R}$ such that $\Sigma \cap \mathfrak{a}$ is empty. There exists a Δ-ideal of $\mathcal{R}$ containing $\mathfrak{a}$ and maximal with respect to avoiding Σ. Furthermore, m is prime.

Proof.

Let $\mathfrak{J}(\mathfrak{a})$ be the set of all Δ-ideals of $\mathcal{R}$ containing $\mathfrak{a}$ and avoiding the multiplicative set. $\mathfrak{J}(\mathfrak{a})$ is not empty. Order $\mathfrak{J}(\mathfrak{a})$ with respect to inclusion $\subseteq$. Let $\mathfrak{T}$ be a totally ordered subset of $\mathfrak{J}(\mathfrak{a})$. The union of the ideals in $\mathfrak{T}$ is in $\mathfrak{J}(\mathfrak{a})$. Therefore, $\mathfrak{J}(\mathfrak{a})$ has a maximal element $\mathfrak{m}$ by Zorn's Lemma. $\sqrt{\mathfrak{m}}$ is a Δ-ideal, and also avoids Σ. Therefore, $\mathfrak{m}$ is a radical Δ-ideal. Spose $a, b \in \mathcal{R}$, and $ab \in \mathfrak{m}$, but neither a nor b is in $\mathfrak{m}$. Then, $\sqrt{[\mathfrak{m}, a]}$ contains an element s of Σ, and $\sqrt{[\mathfrak{m}, b]}$ contains an element t of Σ. Therefore,

$$st \in \sqrt{[\mathfrak{m}, a]}\sqrt{[\mathfrak{m}, b]} \subseteq \sqrt{[\mathfrak{m}, ab]} = \mathfrak{m}.$$

□

Corollary

Let $\mathcal{R}$ be a Δ-ring, and let $S \subseteq \mathcal{R}$. Let $a \in \mathcal{R}$. Then, $a \in \sqrt{[S]} \Leftrightarrow$ every prime Δ-ideal containing S also contains a. In particular, $1 \in \sqrt{[S]} \Longleftrightarrow S$ is not contained in any prime Δ-ideal containing S.

Proof.

$a \in \sqrt{[S]} \implies a$ is in every prime Δ-ideal containing S. For the converse, spose $a \notin \sqrt{[S]}$. $\Sigma := \left\{a^k : k \in \mathbb{N}\right\}$. There is a prime Δ-ideal containing $\sqrt{[S]}$ but not a. For the second statement, let $a = 1$. $\square$

Theorem

Let $\mathcal{R}$ be a Δ-ring, and let S be a subset of $\mathcal{R}$ such that $1 \notin [S]$. Then, $\sqrt{[S]}$ is the intersection of all the prime Δ-ideals containing S.

Proof.

$1 \notin \sqrt{[S]}$. So, $\sqrt{[S]}$ is a proper ideal of $\mathcal{R}$. If $a \notin \sqrt{[S]}$, there is a prime Δ-ideal of $\mathcal{R}$ containing S but not a. □

Corollary

$\mathcal{R}$ has a maximal Δ-ideal, and all of them are prime.

Proof.

Set Σ equal to 1, and $\mathfrak{a}$ equal to 0 in the theorem. □

Note: A maximal Δ-ideal $\mathfrak{m}$ of $\mathcal{R}$ need not be maximal. There may be ideals of $\mathcal{R}$ containing $\mathfrak{m}$.
For example, set $\mathcal{R} = \mathbb{Q}[x], \delta = \frac{d}{dx}$, $\mathfrak{m} = (0)$.

Corollary

Every Δ-ring $\mathcal{R}$ contains a prime Δ-ideal.

Definition

$\mathcal{R}$ is Δ-*simple* if (0) is a maximal Δ-ideal.

Corollary

If $\mathcal{R}$ is Δ-simple, then it is an integral domain.

Proof.

(0) is a prime Δ-ideal. □

Corollary

A Δ-ideal $\mathfrak{a}$ of a Δ-ring $\mathcal{R}$ is a maximal Δ-ideal if and only if $\mathcal{R}/\mathfrak{a}$ is Δ-simple.

Minimal prime ideals.

Definition

The set of nilpotent elements of $\mathcal{R}$ is called the *nil radical* $\mathfrak{n}(\mathcal{R})$ of $\mathcal{R}$.

$\mathfrak{n}(\mathcal{R}) = \sqrt{(0)}$.

Definition

A ring is *reduced* if it has no nonzero nilpotent elements. (0) is a radical Δ-ideal.

$\mathcal{R}/\mathfrak{n}(\mathcal{R})$ is reduced.

- $\mathcal{R}$ is an integral domain $\iff (0)$ is the only minimal prime ideal.
- Every minimal prime ideal of $\mathcal{R}$ contains $\mathfrak{n}\,(\mathcal{R})$.
- $\mathcal{R}$ is reduced $\iff \mathfrak{n}\,(\mathcal{R})$ is (0).

Lemma

If $\mathfrak{p}$ is a minimal prime ideal of $\mathcal{R}$, then for all $x \in \mathfrak{p}$, there exists $s \in \mathcal{R} \backslash \mathfrak{p}$ such that sx is nilpotent.

Proof.

Let $\mathfrak{p}$ be a minimal prime ideal of $\mathcal{R}$, and let $x \in \mathfrak{p}$. Let $\Sigma' = \left\{ sx^k : s \in \mathcal{R} \backslash \mathfrak{p}, k \in \mathbb{Z} > 0 \right\}$. Σ' is a multiplicative set. If $0 \notin \mathfrak{t}'$, there exists a maximal ideal $\mathfrak{p}'$ avoiding Σ'. $\mathfrak{p}'$ is prime. Let $\Sigma = \mathcal{R} \backslash \mathfrak{p}$. Spose $\mathfrak{p}' \cap \Sigma \neq \emptyset$. Let $s \in \mathfrak{p}' \cap \Sigma$. Then, $sx \in \mathfrak{p}'$. This is a contradiction. Thus,.$\mathfrak{p}' \subseteq \mathfrak{p}$. By the minimality of $\mathfrak{p}$, $\mathfrak{p}' = \mathfrak{p}$. But, $x \in \mathfrak{p}'$. Contradiction. So, $0 \in \Sigma'$. Thus, $\exists k \in \mathbb{Z} > 0$ and $s \in \mathcal{R} \backslash \mathfrak{p}$ such that $sx^k = 0$. therefore, $s^k \in \mathcal{R} \backslash \mathfrak{p}$, and clearly $s^k x^k = (sx)^k = 0$. □

Corollary

Let $\mathfrak{p}$ be a minimal prime ideal of $\mathcal{R}$. Then, every element of $\mathfrak{p}$ is a zero divisor of $\mathcal{R}$.

Proof.

Let $a \in \mathfrak{p}$. Then, there exists $s \in \mathcal{R} \backslash \mathfrak{p}$ and a positive integer k such that $(sa)^k = s^k a^k = 0$. Since $s^k \in \mathcal{R} \backslash \mathfrak{p}$, $s^k \neq 0$. Let n be the smallest positive integer such that $s^k a^n = 0$. If $n = 1$, we are done. If not, $\left(s^k a^{n-1}\right) \neq 0$, but $\left(s^k a^{n-1}\right) a = 0$. Thus, a is a zero divisor in $\mathcal{R}$. □

Query: Does every ring $\mathcal{R}$ contain a minimal prime ideal?

Theorem

1. *Every prime ideal* $\mathfrak{q}$ *of a ring* $\mathcal{R}$ *contains a minimal prime ideal* $\mathfrak{p}$.
2. $\mathfrak{n}(\mathcal{R})$ *is the intersection of all the prime ideals of* $\mathcal{R}$*, and of all the minimal primes of* $\mathcal{R}$.

Proof.

1. It suffices to show that the set $\mathfrak{Q}$ of prime ideals contained in $\mathfrak{q}$, ordered by inclusion $\supseteq$, is inductive. So, let $\mathfrak{T}$ be a totally ordered subset of $\mathfrak{Q}$. The intersection $\mathfrak{p}_0$ of the prime ideals in $\mathfrak{T}$ is also prime. For, let $x, y \in \mathcal{R}$, $x \notin \mathfrak{p}_0$ and $y \notin \mathfrak{p}_0$, since $\mathfrak{T}$ is totally ordered, there is a prime ideal $\mathfrak{p} \in \mathfrak{T}$ with x and y in $\mathfrak{p}$. Thus, $xy \notin \mathfrak{p}$, hence not in $\mathfrak{p}_0$ Thus, $\mathfrak{p}_0 \in \mathfrak{Q}$. So, by Zorn's Lemma, $\mathfrak{Q}$ has a minimal element. □

Proof.

2. Note that $\mathfrak{n}(\mathcal{R})$ is contained in every prime ideal of $\mathcal{R}$. Conversely, let x be a non-nilpotent element of $\mathcal{R}$. The set Σ of all x^k, $k \in \mathbb{N}$, is a multiplicative set in $\mathcal{R}$, not containing 0. Therefore, there is a prime ideal of $\mathcal{R}$ not containing x. So, the nil radical of $\mathcal{R}$ is the intersection of all the prime ideals of $\mathcal{R}$. By the first statement, every prime ideal contains a minimal prime ideal. So, $\mathfrak{n}(\mathcal{R})$ is the intersection of the minimal primes of $\mathcal{R}$. □

Theorem

If $\mathcal{R}$ is a Δ-ring, then every minimal prime ideal $\mathfrak{p}$ of $\mathcal{R}$ is a Δ-ideal. (Keigher, Prime differential ideals in differential rings, 1977, Proposition 1.5, p. 242).

Proof.

Next week. □