A density theorem for parameterized differential Galois theory

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In this talk, we are interested in the differential equations on the form:

$$\partial_z Y(z,t) = A(z,t)Y(z,t), \qquad (1)$$

where $t = t_1, \dots, t_n$ is a parameter, $A \in \mathrm{M}_m(\mathcal{O}_U(\{z\}))$ and $\mathcal{O}_U(\{z\})$ is a ring we will define later.

We want to define a parameterized differential Galois group for (1) and, analogously to the density theorem of Ramis, give a set of topological generators.

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We want to define a parameterized differential Galois group for (1) and, analogously to the density theorem of Ramis, give a set of topological generators.

1. Parameterized Hukuhara-Turrittin theorem
2. Parameterized differential Galois theory
3. Density theorem
4. Applications
5. References

Let U be a polydisc of $\mathbb{C}^n$. Let

- $\mathcal{M}_U$ be the field fo meromorphic function on U.
- $\hat{K}_U := \mathcal{M}_U((z))$.
- $\mathcal{O}_U(\{z\}) := \{\sum f_i(t)z^i \in \hat{K}_U | \forall t \in U, z \mapsto \sum f_i(t)z^i$ is a germ of meromorphic function$\}$.
- $K_U := \mathrm{Frac}(\mathcal{O}_U(\{z\}))$.

Proposition

There exist $U' \subset U$ and $\nu \in \mathbb{N}^*$, such that we have an invertible matrix solution of (1) of the form:

$$F(z,t) := \hat{H}(z,t) e^{L(t)\log(z)} e^{Q(z,t)},$$

where

- $\hat{H}(z,t) \in \mathrm{GL}_m(\hat{K}_{U'}[z^{1/\nu}])$.
- $L(t) \in \mathrm{M}_m(\mathcal{M}_{U'})$.
- $Q(z,t) := \mathrm{Diag}(q_i(z,t))$, *with* $q_i(z,t) \in z^{-1/\nu}\mathcal{M}_{U'}[z^{-1/\nu}]$.

Remark

If we restrict U, we may assume that $U' = U$.

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Remark

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Let $\Delta_t := \{\partial_{t_1}, \ldots, \partial_{t_n}\}$. We still consider (1) with the fundamental solution $F(z,t)$ given by the parameterized Hukuhara-Turrittin theorem.
Let $\widetilde{K_U}$ be the (∂_z, Δ_t)-differential field generated by K_U and the entries of $F(z,t)$.

Proposition

$\widetilde{K_U}|K_U$ is a parameterized Picard-Vessiot extension of (1), i.e, the field of the ∂_z-constants of $\widetilde{K_U}$ is $\mathcal{M}_U$.

Let $\mathrm{Aut}_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}|K_U\right)$ be the group of field automorphism of $\widetilde{K_U}$ that commutes with all the derivations and that let K_U invariant. Let us consider the representation:

$$\begin{array}{rccc} \rho_F: & \mathrm{Aut}_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}|K_U\right) & \longrightarrow & \mathrm{GL}_m(\mathcal{M}_U) \\ & \varphi & \longmapsto & F^{-1}\varphi(F). \end{array}$$

Theorem

The image of ρ_F is a differential subgroup of $\mathrm{GL}_m(\mathcal{M}_U)$, i.e, there exist $P_1,\dots,P_k$, Δ_t-differential polynomials in coefficients in $\mathcal{M}_U$, such that:

$$(A_{i,j}) \in \rho_F\left(\mathrm{Aut}_{\partial_z}^{\Delta_t}\right) \Longleftrightarrow P_1(A_{i,j}) = \cdots = P_k(A_{i,j}) = 0.$$

We define the Kolchin topology as the topology of $\mathrm{GL}_m(\mathcal{M}_U)$ in which closed sets are zero sets of differential algebraic polynomials in coefficients in $\mathcal{M}_U$.

Proposition

Let G be a subgroup of $\mathrm{Aut}_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}|K_U\right)$. If $\widetilde{K_U}^{G} = K_U$, then G is dense for Kolchin topology in $\mathrm{Aut}_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}|K_U\right)$.

We still consider (1) with parameterized Picard-Vessiot extension $\widetilde{K_U}|K_U$ and with Galois group $\mathrm{Aut}^{\Delta_t}_{\partial_z}\left(\widetilde{K_U}|K_U\right)$.

Definition

We define $\hat{m} \in \mathrm{Aut}^{\Delta_t}_{\partial_z}\left(\widetilde{K_U}|K_U\right)$ *by:*

- $\hat{m}|_{\hat{K}_U} = \mathrm{Id}$.
- $\hat{m}(z^{\alpha}) = e^{2i\pi\alpha} z^{\alpha}$.
- $\hat{m}(\log(z)) = \log(z) + 2i\pi$.
- $\forall q \in z^{-1/\nu}\mathcal{M}_U[z^{-1/\nu}]$,

$$\hat{m}(e^q) = e^{\hat{m}(q)}.$$

Definition

We define the exponential torus as the subgroup of $\mathrm{Aut}_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}|K_U\right)$ *of elements* τ *that satisfies:*

- $\tau|_{\hat{K}_U} = \mathrm{Id}$.
- $\tau(z^{\alpha}) = z^{\alpha}$.
- $\tau(\log(z)) = \log(z)$.
- *There exists* α_{τ}*, a character of* $z^{-1/\nu}\mathcal{M}_U[z^{-1/\nu}]$*, such that* $\forall q \in z^{-1/\nu}\mathcal{M}_U[z^{-1/\nu}]$,

$$\tau(e^{q}) = \alpha_{\tau}(q)e^{q}.$$

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Let us consider

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$$z^2\partial_z Y(z) + Y(z) = z. \tag{2}$$

Let $d \neq (2\mathbb{Z}+1)\pi$. The following function is solution of (2)

$$\mathcal{S}^d(h) = \int_0^{\infty e^{id}} f(\zeta) e\left(-\frac{\zeta}{z}\right) d\zeta,$$

where $f(\zeta) = \frac{1}{1+\zeta}$

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$\mathcal{S}^d(h)$ *is* 1*-Gevrey asymptotic to h:*

$$\left| \mathcal{S}^d(h) - \sum_{n\geq 0}^{N} (-1)^n n! z^{n+1} \right| \leq A^{N+1}(N+1)!|z|^{N+1},$$

where $A > 0$ *is a constant.*

Let $d \in \mathbb{R}$ and let $k \in \mathbb{Q}^{>0}$.

- $\hat{\mathcal{B}}_k\left(\sum a_n z^n\right) = \sum \frac{a_n}{\Gamma(n/k)}\zeta^n$.
- $\mathcal{L}_1^d\left(f\right)\left(z\right) = \displaystyle\int_0^{\infty e^{id}} \frac{f(\zeta)}{ze\left(\frac{\zeta}{z}\right)} d\zeta$.
- $\mathcal{L}_k^d := \rho_k \circ \mathcal{L}_1^d \circ \rho_{1/k}$.

Let us consider $h(z) \in \mathbb{C}[[z]]$ solution of a linear differential equation in coefficients germs of meromorphic functions.

- There exist $\Sigma \subset \mathbb{R}$ (set of singular directions), $(\kappa_1, \dots, \kappa_r) \in (\mathbb{Q}^{>0})^r$, such that if $d \notin \Sigma$,

$$\mathcal{S}^d(h) = \mathcal{L}^d_{\kappa_r} \circ \cdots \circ \mathcal{L}^d_{\kappa_1} \circ \hat{\mathcal{B}}_{\kappa_1} \circ \cdots \circ \hat{\mathcal{B}}_{\kappa_r}(h),$$

  is a germ of analytic solution on the sector $\arg(z) \in]d - \pi/2\kappa_r, d + \pi/2\kappa_r[$.
- Σ is finite modulo $2\pi\mathbb{Z}$.
- The map $h \mapsto \mathcal{S}^d(h)$ induces a morphism of differential field.

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Let us consider system $\partial_z Y(z) = A(z)Y(z)$ with coefficients that are germs of meromorphic functions. Let $\hat{H}(z)e^{L\log(z)}e^{Q(z)}$ be the Hukuhara-Turrittin solution of $\partial_z Y(z) = A(z)Y(z)$.

There exist $\Sigma \subset \mathbb{R}$ finite modulo 2π, $\varepsilon > 0$, such that if $d \notin \Sigma$, $\mathcal{S}^d\left(\hat{H}(z)\right) e^{L\log(z)}e^{Q(z)}$ is solution of $\partial_z Y(z) = A(z)Y(z)$ with entries that are germs of meromorphic functions on the sector $\arg(z) \in]d - \pi/2\varepsilon, d + \pi/2\varepsilon[$.

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For $d \in \mathbb{R}$, let $\mathrm{St}^{\mathrm{d}} \in \mathrm{GL}_m(\mathbb{C})$ such that:

$$\mathcal{S}^{d^-}\left(\hat{H}(z)\right)e^{L\log(z)}e^{Q(z)} = \mathcal{S}^{d^+}\left(\hat{H}(z)\right)e^{L\log(z)}e^{Q(z)}\mathrm{St}^{\mathrm{d}},$$

where $d - \pi/2\varepsilon < d^- < d < d^+ < d + \pi/2\varepsilon$ and $[d^-, d[\cup]d, d^+] \cap \Sigma = \varnothing$.

Let us consider (1). Let $\hat{H}(z,t)e^{L(t)\log(z)}e^{Q(z,t)}$ be parameterized Hukuhara-Turrittin solution. If we restrict U, we may assume that:

- There exist $(d_i(t))$ continuous in t and finite modulo $2\pi\mathbb{Z}$, that satisfies $d_i(t) < d_{i+1}(t)$.
- $(\kappa_{1,i,j},\dots,\kappa_{r,i,j}) \in \left(\mathbb{Q}^{>0}\right)^r$,

such that for all $d(t)$ continuous in t that does not interect $\Sigma_t := \bigcup d_i(t)$, we have an analytic solution of (1):

$$\mathcal{S}^{d(t)}\left(\hat{H}(z,t)\right)e^{L(t)\log(z)}e^{Q(z,t)} :=$$
$$\mathcal{L}^{d(t)}_{\kappa_{r,i,j}} \circ \dots \circ \mathcal{L}^{d(t)}_{\kappa_{1,i,j}} \circ \hat{\mathcal{B}}_{\kappa_{1,i,j}} \circ \dots \circ \hat{\mathcal{B}}_{\kappa_{r,i,j}}\left(\hat{H}_{i,j}(z,t)\right)e^{L(t)\log(z)}e^{Q(z,t)},$$

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For $d(t) \in \Sigma_t$ continuous in t, let $t \mapsto \mathrm{St}^{\mathrm{d(t)}} \in \mathrm{GL}_m(\mathcal{M}_U)$ such that for all $t_0 \in U$, $\mathrm{St}^{\mathrm{d(t_0)}} \in \mathrm{GL}_m(\mathbb{C})$, is the Stokes matrix in the direction t_0 of

$$\partial_z Y(z, t_0) = A(z, t_0) Y(z, t_0).$$

Proposition

$\mathrm{St}^{\mathrm{d(t)}} \in \mathrm{Aut}_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}|K_U\right).$

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Let us consider (1). Let $\widetilde{K_U}|K_U$ be the parameterized Picard-Vessiot extension and $\mathrm{Aut}_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}|K_U\right)$ be the Galois group.

Theorem (D)

The group generated by the (parameterized) monodromy matrix, the exponential torus and the Stokes matrices is dense for Kolchin topology in $\mathrm{Aut}_{\partial_z}^{\Delta_t}\left(\widetilde{K_U}|K_U\right)$.

Let us consider $\partial_z Y(z,t) = A(z,t)Y(z,t)$ with $A \in \mathrm{M}_m(\mathcal{M}_U(z))$. We can define, $\widetilde{\mathcal{M}_U(z)}|\mathcal{M}_U(z)$, parameterized Picard-Vessiot extension and $\mathrm{Aut}^{\Delta_t}_{\partial_z}\left(\widetilde{\mathcal{M}_U(z)}|\mathcal{M}_U(z)\right)$ the Galois group.

Theorem (D)

The group generated by the (parameterized) monodromy matrix, the exponential torus and the Stokes matrices of all the singularities is dense for Kolchin topology in $\mathrm{Aut}^{\Delta_t}_{\partial_z}\left(\widetilde{\mathcal{M}_U(z)}|\mathcal{M}_U(z)\right)$.

Definition

Let $\partial_{t_0} := \partial_z$. *Let* $A_0 \in \mathrm{M}_m\Big(\mathcal{M}_U(z)\Big)$. *We say that the linear differential equation* $\partial_{t_0} Y = A_0 Y$ *is completely integrable if there exist* $A_1, \ldots, A_n \in \mathrm{M}_m\Big(\mathcal{M}_U(z)\Big)$ *such that, for all* $0 \leq i, j \leq n$,

$$\partial_{t_i} A_j - \partial_{t_j} A_i = A_i A_j - A_j A_i.$$

Theorem (Cassidy/Singer 1 $\Leftrightarrow$ 2, D 1 $\Leftrightarrow$ 3)

We have the equivalence:

1. $\partial_{t_0} Y = A_0 Y$ *is completely integrable.*
2. *The parameterized differential Galois group of* $\partial_{t_0} Y = A_0 Y$ *is conjugated to an algebraic subgroup of* $\mathrm{GL}_m(\mathbb{C})$.
3. *The topological generator of the parameterized differential Galois group of* $\partial_{t_0} Y = A_0 Y$ *are conjugated to constant matrices.*

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Definition

We say that k is a so-called universal Δ_t-field, if it has characteristic 0 and for any Δ_t-field $k_0 \subset k$, Δ_t-finitely generated over $\mathbb{Q}$, and any Δ_t-finitely generated extension k_1 of k_0, there is a Δ_t-differential k_0-isomorphism of k_1 into k.

Theorem ($\Rightarrow$ D, $\Leftarrow$ Mitschi/Singer)

Let G be a differential subgroup of $\mathrm{GL}_m(k)$. Then, G is the global parameterized differential Galois group of some equation having coefficients in $k(z)$ if and only if G contains a finitely generated subgroup that is Kolchin-dense in G.

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