On the nature of the generating series of walks in the quarter plane

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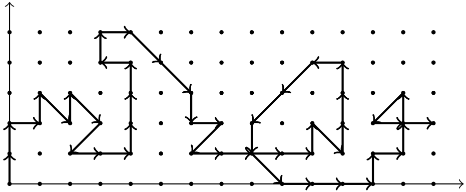
Abstract

- Consider the walks in the quarter plane starting from $(0,0)$ with steps in a fixed set

$$\mathcal{D} \subset \{\leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow\}.$$

- Example with possible directions

$$\mathcal{D} \subset \{\leftarrow, \uparrow, \rightarrow, \searrow, \downarrow, \swarrow\}.$$



Abstract

- Let $f_{\mathcal{D},i,j,k}$ equals the number of walks in $\mathbb{N}^2$ starting from $(0,0)$ ending at (i,j) in k steps in $\mathcal{D}$.
- Generating series: $F_{\mathcal{D}}(x,y,t) := \sum_{i,j,k} f_{\mathcal{D},i,j,k} x^i y^j t^k$.
- Classification problem: when $F_{\mathcal{D}}(x,y,t)$ is algebraic, holonomic, differentially algebraic?
- Today, we are able to classify in which cases $F_{\mathcal{D}}$ is algebraic (resp. holonomic).

$\rightarrow$ O. Bernardi, A. Bostan, M. Bousquet-Mélou, F. Chyzak, G. Fayole, M. van Hoeij, R. Iasnogorodski, M. Kauers, I. Kurkova, V. Malyshev, M. Mishna, K. Raschel, B. Salvy...

Definition

- *Let $f \in \mathbb{C}((x))$. We say that f is differentially algebraic if $\exists n \in \mathbb{N}$, $P \in \mathbb{C}(x)[X_0, \ldots, X_n]$ such that*

$$P(f, f', \ldots, f^{(n)}) = 0.$$

- *Otherwise we say that f is differentially transcendent.*

1. Classification of the walks
2. Elliptic functions
3. Transcendence of the generating functions
4. Algebraic cases

The kernel of the walk

Identify directions in $\mathcal{D}$ by $(i,j), i,j \in \{-1,0,1\}$.
Consider

$$S_{\mathcal{D}}(x,y) = \sum_{(i,j)\in\mathcal{D}} x^i y^j,$$

and the kernel of the walk is

$$K_{\mathcal{D}}(x,y,t) := xy(1 - tS_{\mathcal{D}}(x,y)).$$

Example

$$\mathcal{D} = \{\leftarrow, \uparrow, \searrow\} = \{(-1,0),(0,1),(1,-1)\}.$$

$$S_{\mathcal{D}}(x,y) = x^{-1} + y + xy^{-1},$$

$$K_{\mathcal{D}}(x,y,t) := xy - t(y + xy^2 + x^2).$$

The functional equation of the walk

The generating series $F_{\mathcal{D}}(x, y, t)$ and the kernel $K_{\mathcal{D}}(x, y, t)$ satisfy the following equation

$$K_{\mathcal{D}}(x, y, t)F_{\mathcal{D}}(x, y, t) = \\ xy - K_{\mathcal{D}}(x, 0, t)F_{\mathcal{D}}(x, 0, t) - K_{\mathcal{D}}(0, y, t)F_{\mathcal{D}}(0, y, t) \\ + K_{\mathcal{D}}(0, 0, t)F_{\mathcal{D}}(0, 0, t).$$

Group of the walk

Fix $t \notin \overline{\mathbb{Q}}$. Consider the algebraic curve

$$E_t := \{(x,y) \in \mathbb{P}_1(\mathbb{C})^2 | K_{\mathcal{D}}(x,y,t) = 0\}.$$

Consider the involutions

$$\begin{array}{rccl} \iota_1 := & E_t & \to & E_t \\ & (x,y) & \mapsto & \left(x, \frac{\sum_{(i,-1)\in\mathcal{D}} x^i}{y\sum_{(i,1)\in\mathcal{D}} x^i}\right) \\ \iota_2 := & E_t & \to & E_t \\ & (x,y) & \mapsto & \left(\frac{\sum_{(-1,j)\in\mathcal{D}} y^j}{x\sum_{(1,j)\in\mathcal{D}} y^j}, y\right). \end{array}$$

We attach to $\mathcal{D}$ the group of the walk

$$G_t := \langle \iota_1, \iota_2 \rangle.$$

Reduction to an elliptic case

Over the 2^8 possible walks, only 79 need to be studied.

- $\forall t, \#G_t < \infty$ for 23 walks.

  → A. Bostan, M. Bousquet-Mélou, M. Kauers, M. Mishna

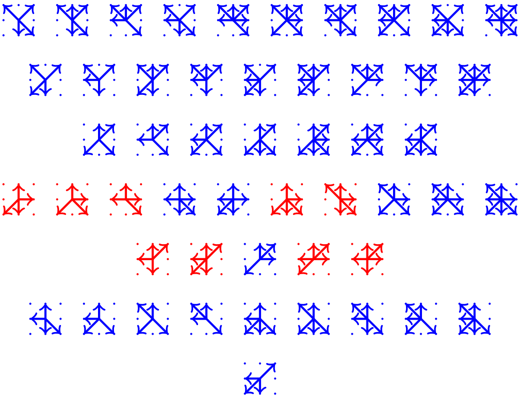
- $\exists t, \#G_t = \infty$ for 56 walks.
  - E_t has genus zero for 5 walks.
  - E_t has genus one for 51 walks.

  → I. Kurkova, K. Raschel

From now we fix $t \notin \overline{\mathbb{Q}}$ such that $\#G_t = \infty$ and assume that E_t has genus one.

E_t is an elliptic curve

Main result



Theorem (D-H-R-S 2017)

In 42 cases, $x \mapsto F_{\mathcal{D}}(x, 0, t), y \mapsto F_{\mathcal{D}}(0, y, t)$ are diff. tr.

In 9 cases, $x \mapsto F_{\mathcal{D}}(x, 0, t), y \mapsto F_{\mathcal{D}}(0, y, t)$ are diff. alg.

Elliptic functions

- $\mathcal{M}er(E_t)$ = meromorphic function on E_t.
- $\exists \omega_{1,t} \in \mathrm{i}\mathbb{R}_{>0}, \omega_{2,t} \in \mathbb{R}_{>0}$, such that

$$\mathcal{M}er(E_t) = \{f(\omega) \in \mathcal{M}er(\mathbb{C}) | f(\omega) = f(\omega+\omega_{1,t}) = f(\omega+\omega_{2,t})\}.$$

- We define the Weierstrass function:

$$\wp_t(\omega) = \frac{1}{\omega^2} + \sum_{p,q \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(\omega + p\omega_{1,t} + q\omega_{2,t})^2} - \frac{1}{(p\omega_{1,t} + q\omega_{2,t})^2}.$$

- $\mathcal{M}er(E_t) = \mathbb{C}(\wp_t(\omega), \partial_\omega \wp_t(\omega))$.

Analytic continuation

Proposition (Kurkova, Raschel)

The series $x \mapsto F_{\mathcal{D}}(x,0,t)$, $y \mapsto F_{\mathcal{D}}(0,y,t)$ admit multivalued meromorphic continuation on the elliptic curve E_t.

- Let $\widetilde{F}_{x,\mathcal{D}}(\omega)$ (resp. $\widetilde{F}_{y,\mathcal{D}}(\omega)$) be the meromorphic continuation of $F_{\mathcal{D}}(x,0,t)$ (resp. $F_{\mathcal{D}}(0,y,t)$), we will see as meromorphic functions on $\mathbb{C}$.
- $\exists$ explicit $f \in \mathbb{C}(X)$ (resp. $g \in \mathbb{C}(X), \omega_{3,t} \in \mathbb{R}_{>0}$) such that $x = f(\wp_t(\omega))$ (resp. $y = g(\wp_t(\omega - \omega_{3,t}/2))$).

Theorem (Kurkova, Raschel)

The function $\widetilde{F}_{x,\mathcal{D}}(\omega)$ (resp. $\widetilde{F}_{y,\mathcal{D}}(\omega)$) is not holonomic.

Lemma

- *$F_{\mathcal{D}}(x,0,t)$ is diff. tr. $\Leftrightarrow \widetilde{F}_{x,\mathcal{D}}(\omega)$ is diff. tr.*
- *$F_{\mathcal{D}}(0,y,t)$ is diff. tr. $\Leftrightarrow \widetilde{F}_{y,\mathcal{D}}(\omega)$ is diff. tr.*

Functional equation evaluated on E_t

The meromorphic continuation satisfy

$$\begin{aligned}
\tau\left(\widetilde{F}_{x,\mathcal{D}}(\omega)\right) &= \widetilde{F}_{x,\mathcal{D}}(\omega) && +y(-\omega)\left(x(\omega+\omega_{3,t}) - x(\omega)\right), \\
\tau\left(\widetilde{F}_{y,\mathcal{D}}(\omega)\right) &= \widetilde{F}_{y,\mathcal{D}}(\omega) && +x(\omega)(y(-\omega) - y(\omega)),
\end{aligned}$$

where $\tau := h(\omega) \mapsto h(\omega + \omega_{3,t})$.
These are two difference equations and we may use difference Galois theory.

Some consequences of difference Galois theory

Let $b := x(\omega)(y(-\omega) - y(\omega))$.

Proposition (D-H-R-S 2017)

The function $\widetilde{F}_{y,\mathcal{D}}$ is diff. alg. iff there exist an integer $n \geq 0$, $c_0, \dots, c_{n-1} \in \mathbb{C}$ and $h \in \mathcal{M}er(E_t)$ such that

$$\partial_\omega^n(b) + c_{n-1}\partial_\omega^{n-1}(b) + \cdots + c_1\partial_\omega(b) + c_0 b = \tau(h) - h.$$

Corollary

$\widetilde{F}_{x,\mathcal{D}}$ is diff. alg. $\Leftrightarrow$ $\widetilde{F}_{y,\mathcal{D}}$ is diff. alg.

Corollary

Assume that b has a pole $\omega_0 \in \mathbb{C}$, such that, for all $0 \neq k \in \mathbb{Z}$, $\tau^k(\omega_0)$ not a pole of b. Then, $\widetilde{F}_{y,\mathcal{D}}$ is diff. tr.

Poles of b

We now see b as a function $\mathbb{P}_1(\mathbb{C})^2 \supset E_t \to \mathbb{P}_1(\mathbb{C})$. The set of poles of b is contained in

$$\{\underbrace{(\infty, \alpha_1), (\infty, \alpha_2)}_{\text{Poles of } x(\omega)}, \underbrace{(\beta_1, \infty), (\beta_2, \infty)}_{\text{Poles of } y(\omega)}, \underbrace{(\beta_1, \gamma_1), (\beta_2, \gamma_2)}_{\text{Poles of } y(-\omega)}\}.$$

Lemma

- *In the poles of x, α_1, α_2 are roots of $\sum_{(1,j)\in\mathcal{D}} y^{j+1}$.*
- *In the poles of y, β_1, β_2 are roots of $\sum_{(i,1)\in\mathcal{D}} x^{i+1}$.*

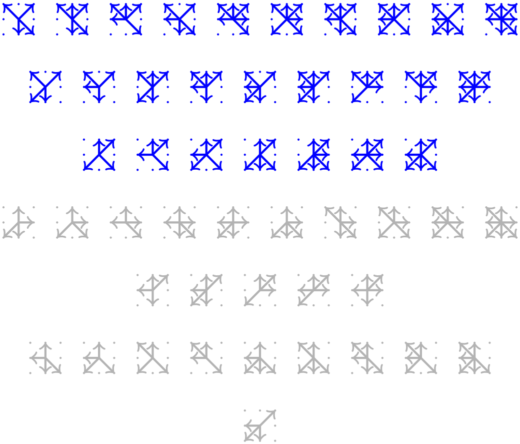
The base field

Lemma

Let $\mathbb{Q}(t) \subset L \subset \mathbb{C}$ *field ext. Let* $P \in E_t$*. Then*

$$P \in \mathbb{P}_1(L)^2 \Leftrightarrow \tau(P) \in \mathbb{P}_1(L)^2 \Leftrightarrow \iota_1(P) \in \mathbb{P}_1(L)^2 \Leftrightarrow \iota_2(P) \in \mathbb{P}_1(L)^2.$$

Generic case



Theorem (D-H-R-S 2017)

Assume that $\{\alpha_1, \alpha_2, \beta_1, \beta_2\} \cap (\mathbb{C} \setminus \mathbb{Q}(t)) \neq \varnothing$. *Then,* $\widetilde{F}_{x,\mathcal{D}}, \widetilde{F}_{y,\mathcal{D}}$ *are differentially transcendent.*

Sketch of proof in the case 

- The poles of b are $\{(\infty, \pm\mathrm{i}), (\pm\mathrm{i}, \infty), (\pm\mathrm{i}, \pm\mathrm{i}t + t)\}$.
- Involution $\sigma \in \mathrm{Gal}(\mathbb{Q}(\mathrm{i}, t)|\mathbb{Q}(t))$. Then $\sigma \circ \tau = \tau \circ \sigma$.

Definition

Let $P, Q \in E_t$. We say that $P \sim Q$ if $\exists k \in \mathbb{Z}$ such that $\tau^k(P) = Q$.

Lemma

$(\infty, \mathrm{i}) \not\sim (\infty, -\mathrm{i})$.

Proof.

Assume that $\tau^k(\infty, \mathrm{i}) = (\infty, -\mathrm{i})$. We have $\tau^k(\infty, -\mathrm{i}) = (\infty, \mathrm{i})$ and $\tau^{2k}(\infty, \mathrm{i}) = (\infty, \mathrm{i})$. No fixed point by τ implies $k = 0$. Contradiction. □

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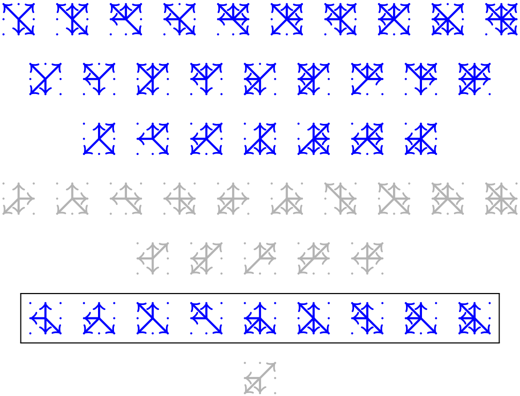
Triple pole case ($\nearrow, \rightarrow \notin \mathcal{D}$)



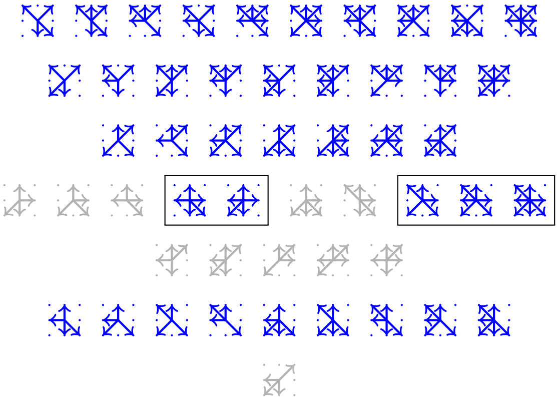
Triple pole case $(\nearrow, \rightarrow \notin \mathcal{D})$

- (∞, ∞) double pole of x.
- (∞, ∞) simple pole of y.
- (∞, ∞) only triple pole of b.

Corollary

Assume that $\nearrow, \rightarrow \notin \mathcal{D}$. Then, $\widetilde{F}_{x,\mathcal{D}}, \widetilde{F}_{y,\mathcal{D}}$ are diff. tr.

Double pole case ($\nearrow \notin \mathcal{D}$)



Double pole case ($\nearrow \notin \mathcal{D}$)

- (∞, ∞) simple pole of x, resp y.
- $(\infty, \star)$ simple pole of x, resp. $y(-\omega)$.
- (∞, ∞), $(\infty, \star)$ are only double poles of b.

Lemma

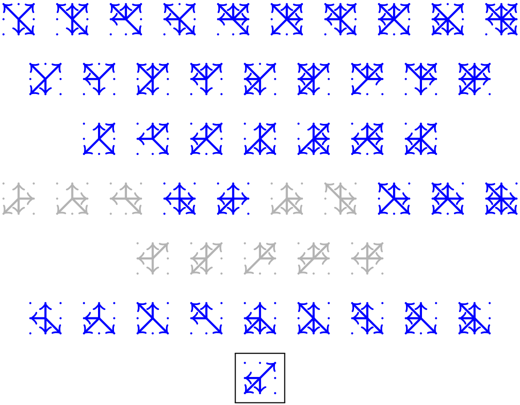
If $(\infty, \infty) \sim (\infty, \star)$*, then* $\exists k \in \mathbb{Z}$*,* $j \in \{1, 2\}$ *s.t.*

$$\iota_j \circ \tau^k(\infty, \infty) = \tau^k(\infty, \infty).$$

Corollary

Assume that $\mathcal{D} \in \Big\{$  $\Big\}$*. Then,* $\widetilde{F}_{x,\mathcal{D}}, \widetilde{F}_{y,\mathcal{D}}$ *are diff. tr.*

A symmetric case:



A symmetric case: 

There are 3 simple poles: $(\infty, 0)$, $(0, \infty)$, and $(0, -1)$.

Lemma

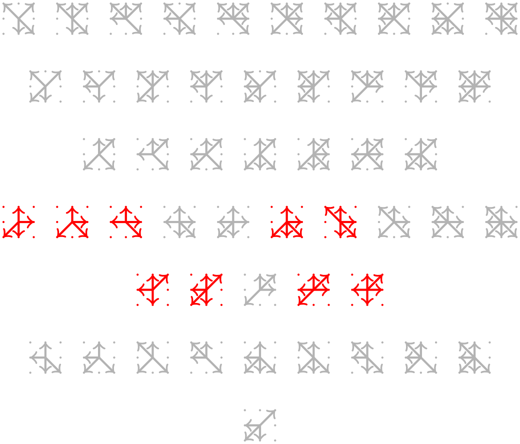
If $(\alpha, \beta) \sim (\beta, \alpha)$, $\alpha, \beta \in \mathbb{P}_1(\mathbb{Q}(t))$, *then* $\exists \gamma \in \mathbb{P}_1(\mathbb{Q}(t))$, *s.t.*

$$K_{\mathcal{D}}(\gamma, \gamma, t) = 0.$$

Corollary

The series $\widetilde{F}_{x,\mathcal{D}}, \widetilde{F}_{y,\mathcal{D}}$ *are diff. tr.*

Algebraic cases



Orbit of the poles, case 

Polar divisor of b	$(-1, \frac{t}{t+1}) + (\infty, 0) + (-1, \infty)$
τ-Orbit of one of the poles of b	$(-1, \frac{t}{t+1}) \downarrow \tau\ (0, \infty) \downarrow \tau\ (\infty, 0) \downarrow \tau\ (0, 0) \downarrow \tau\ (-1, \infty)$

In 8 cases, every poles of b are on the same orbit

A criteria of algebraicity

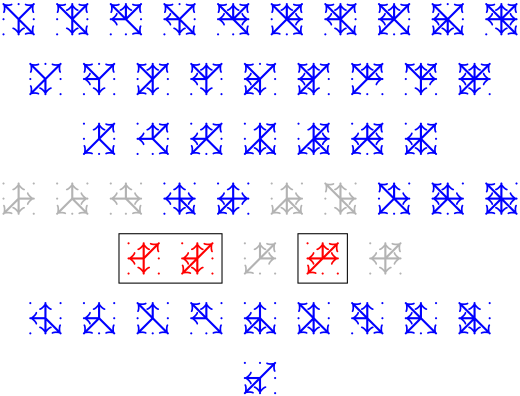
Proposition (D-H-R-S 2017)

The function $\widetilde{F}_{y,\mathcal{D}}$ is diff. alg. iff for all poles ω_0 of b, we have that

$$h(\omega) = \sum_{i=1}^{s} b(\omega + n_i \omega_{3,t})$$

is analytic at ω_0 where $\omega_0 + n_1\omega_{3,t}, \ldots, \omega + n_s\omega_{3,t}$ are the poles of b that belong to $\omega_0 + \mathbb{Z}\omega_{3,t}$.

Uni-orbit, simple pole case



Uni-orbit, simple pole case

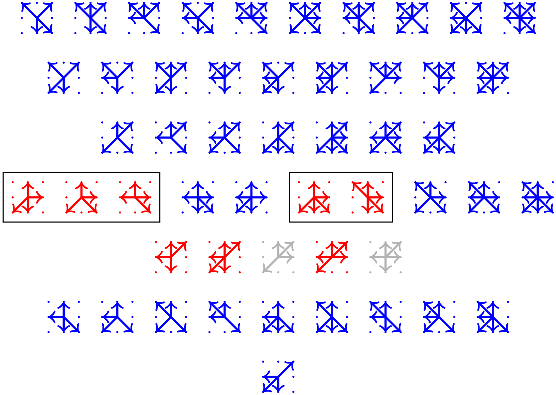
Lemma

$b \in \mathcal{M}er(E_t) \Longrightarrow$ sum of residues of b is zero.

Corollary

Assume that $\mathcal{D} \in \{$  $\}$. Then, every poles of b are on the same orbit and are simple. Consequently, $\widetilde{F}_{x,\mathcal{D}}, \widetilde{F}_{y,\mathcal{D}}$ are diff. alg.

Uni-orbit, double pole case



Uni-orbit, double pole case

Lemma

If $b = \sum_{\ell \geq k} \frac{c_\ell}{(\omega - \omega_0)^\ell}$*, then* $b = \sum_{\ell \geq k} \frac{(-1)^{\ell+1} c_\ell}{(\omega + \omega_0)^\ell}$.

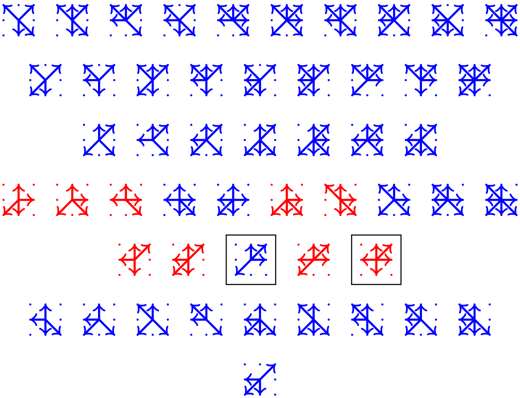
Sketch of proof.

We use $b(-\omega) = -b(\omega)$. □

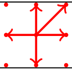
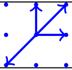
Corollary

Assume that $\mathcal{D} \in \{$  $\}$*. Then, every poles of* b *are on the same orbit and are at most double.*
Consequently, $\widetilde{F}_{x,\mathcal{D}}, \widetilde{F}_{y,\mathcal{D}}$ *are diff. alg.*

Bi-orbit case



Bi-orbit case

Walk			
Polar divisor of b [Residue]	$(-1, \frac{t}{2t+1}), [\alpha]$ $+(\infty, 0), [-\alpha]$		$+(\infty, -1), [-\alpha]$ $+(-1, \infty), [\alpha]$
τ-Orbit of the poles	$(-1, \frac{t}{2t+1})$ $\downarrow \tau$ $(0, \infty)$ $\downarrow \tau$ $(\infty, 0)$	$\not\sim$	$(\infty, -1)$ $\downarrow \tau$ $(0, 0)$ $\downarrow \tau$ $(-1, \infty)$
Walk			
Polar divisor of b [Residue]	$(-1, \frac{-t}{t+1}), [\alpha]$ $+(-1, \infty), [\alpha]$		$+(\infty, -1), [-\alpha]$ $+(\infty, 0), [-\alpha]$
τ-Orbit of the poles	$(-1, \frac{-t}{t+1})$ $\downarrow \tau$ $(0, \infty)$ $\downarrow \tau$ $(-1, \infty)$	$\not\sim$	$(\infty, -1)$ $\downarrow \tau$ $(\infty, 0)$

Conclusion and perspectives

- Mix of algebra and analysis allows us to treat every cases.
- In the differentially algebraic cases, explicit computation of the telescoper should lead to the expression of the differential equations.
- We now should be able to treat the genus zero case.