A NOTE ON CANONICAL BASES IN PARTIAL DIFFERENTIAL FIELDS

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This note is intended as a supplement for the slides I gave at a November 2012 Kolchin seminar talk. I rework some of the definitions and explain some of the model-theoretic notation in differential algebraic terms.

1. Bounding the Kolchin polynomial of a relative canonical base

In this section, we will discuss tuples which generate differential fields over which a given type does not fork (what might be called a relative canonical base or a relative field of definition). Versions of this lemma appeared in preprints of my paper about indecomposability. Thanks very much to P. Cassidy and W. Sit for numerous discussions of the result. Following suggestions by the members of the Kolchin seminar, I have reworked some of the definitions and proofs of the result in order to make them more accessible to non-model theorists; this theorem is equivalent to the one which can be found in a copy of my thesis at my Berkeley webpage `http://math.berkeley.edu/people/faculty/james-freitag`, but as I have mentioned, the notation and definitions have been reworked. Throughout the note, K is a differentially closed field of characteristic zero with derivations $\Delta = \{\delta_1, \dots, \delta_m\}$ which commute.

We remind the reader that the Kolchin polynomials are ordered by eventual domination. In what follows, we will write the Kolchin polynomial of a type p in the following canonical form

$$\omega_p(t) = \sum_{0 \leq i \leq m} a_i \binom{t+i}{i}.$$

The following definition will be useful for the statements of the remaining results in the section.

Definition 1.1. Let $p \in S(k_1)$ and $q \in S(k_2)$ where $k_1 \leq k_2$ are differential fields and q is an extension of p. Let $\omega_p(t) = \sum_{0 \leq i \leq m} a_i \binom{t+i}{i}$ and $\omega_q(t) = \sum_{0 \leq i \leq m} d_i \binom{t+i}{i}$. We say that p and q are *n-equivalent* if $a_i = b_i$ for all $i \geq n$. In this case, we also write $\omega_p(t) \equiv_n \omega_q(t)$.

Remark 1.2. This new notion, n-equivalence, is a measure of forking; for instance, 0-equivalence is equivalent to nonforking. 1-equivalent means that the forking only changes the constant term of the Kolchin polynomial. The notion is only meaningful for $n \leq m$.

We consider a simple example then $m = 1$. Let a, b be singletons considered over the field $\mathbb{Q}$. Suppose $\delta^2(b) = 0$ and a is generic over $\mathbb{Q}$. Now, consider the differential field $\mathbb{Q}(c)$ where $\delta(b) = c$ (note that $\delta(c) = 0$). Then $tp(a, b/\mathbb{Q}(c))$ is a forking extension of $tp(a, b/\mathbb{Q})$. The type are 1-equivalent in this case, because all of the forking only affects the constant term of the Kolchin polynomial.

Theorem 1.3. *Suppose that $p(x) \in S(K)$. Then, suppose, for some differential subfield $A \subseteq K$ and $n \in \mathbb{N}$ that $\omega_{p|_A}(t) \equiv_n \omega_p(t)$. Then there is a tuple $\bar{c} \in K$ such that $\omega_p(t) = \omega_{p|_{\mathbb{Q}\langle \bar{c} \rangle}}(t)$ and $\omega_{\bar{c}/A}(t) < \binom{t+n}{n}$.*

A diagram of the pertinent field extensions is given and discussed following the theorem. It may be helpful in tracking some of the developments of the theorem.

Proof. Let $\langle b_k \rangle_{k \in \mathbb{N}}$ be a Morley sequence over K in the type of p.

Remark 1.4. In model theoretic terms, this simply means that each of the tuples b_k satisfy the first order formulae over K (specified by the type p) and that $b_k \mathop{\smile\!\!\!\!\!\!\mid}_K b_1, \ldots b_{k-1}$. When we look at the sequence over smaller differential subfields $A \subset K$, the first criterion is preserved - knowing the first order type over a larger set means we know it over a smaller set. The second criterion is not preserved - now, $b_k \mathop{\smile\!\!\!\!\!\!\mid}_A b_1, \ldots b_{k-1}$ is not known - perhaps $b_1, \ldots b_k$ tell us additional information about b_k over A.

In differential algebraic terms, this simply means that each of the tuples b_k generate isomorphic differential fields over K (specified by the type p) and that $K\langle b_k \rangle$ is algebraically disjoint from $K\langle b_1, \ldots b_{k-1} \rangle$ over K. When we look at the sequence over smaller differential subfields $A \subset K$, the first criterion is preserved - knowing the isomorphism type of the differential field extension generated by b_k over a field means we know it over a subfield. The second criterion is not preserved - there is no reason that $A\langle b_k \rangle$ needs to be algebraically disjoint from $A\langle b_1, \ldots b_{k-1} \rangle$ over A.

By the characterization of forking in $DCF_{0,m}$ this simply means that for all $j \in \mathbb{N}$,

$$\omega_p(t) = \omega_{b_j/K}(t) = \omega_{b_j/K\langle b_0, \ldots, b_{j-1} \rangle}(t)$$

As noted in the above remark, we do not know, however, that the same holds over the differential subfield $A \subseteq K$. The sequence is still necessarily A-indiscernible, that is $tp(b_j/A)$ does not depend on k. It is not necessarily A-independent (that is, it may be that $b_i \not\mathop{\smile\!\!\!\!\!\!\mid}_A b_j$). In general, we simply know that $\omega_{b_j/A\langle b_0, \ldots, b_{j-1} \rangle}$ is a decreasing sequence of polynomials, again, ordered by eventual domination. By the well-orderedness of Kolchin polynomials we know that the sequence is eventually constant [2]. Alternatively, this fact can be seen by noting the superstability of $DCF_{0,m}$ and the fact that decreases in Kolchin polynomial correspond to forking extensions (in this case, one is not really invoking the entire strength of Sit's theorem, because we are assuming the sequence is of a particular nature). So, for the rest of the proof, we fix a k such that if $n \geq k$, the sequence is constant. That is, above k, we know that we have a Morley sequence over $A\langle b_0, \ldots, b_{k-1} \rangle$ in the type of p. Now, fix a model $K' \models DCF_{0,m}$ with K' containing K and $\{b_0, \dots, b_{k-1}\}$. We let p' be the (unique) nonforking extension of p to K'.

We can get elements $\bar{c} \subseteq acl(A\langle b_0, \dots, b_{k-1}\rangle)$ such that p' does not fork over $\bar{c}$. In fact, by [1] (page 132) and the fact that $DCF_{0,m}$ eliminates imaginaries, we can assume that $\bar{c} \in K$.

Remark 1.5. This is an instance of a general stability-theoretic result - one can find the canonical base of a type (in differential algebraic terms, the field of definition of variety corresponding to the type) from the algebraic closure of an indiscernible sequence in the type itself. I do not know of other instance of using this technique in differential field in the manner I am using it (to bound ranks). Certainly indiscernible sequences have been utilized extensively (via Kolchin, Shelah, etc. for various purposes, but notably to prove the nonminimality of differential closure).

We know that

$$\omega_{p|_A}(t) = f(t) + h(t)$$

where

$$f(t) = \sum_{i=n}^{m} c_i \binom{t+i}{i}$$

and

$$h(t) = \sum_{i=0}^{n-1} c_i \binom{t+i}{i}.$$

By assumption, $\omega_{p|_A}(t) \equiv_n \omega_p(t)$. Thus, $f(t) \leq \omega_p(t)$. By construction $\langle b_i \rangle$ was an indiscernible sequence, so if we define $\bar{b} := (b_0, \dots, b_{k-1})$, then,

$$k \cdot f(t) \leq \omega_{\bar{b}/K}(t)$$

Then we know that

$$kf(t) \leq \omega_{\bar{b}/A\langle\bar{c}\rangle}(t) = \omega_{\bar{b}/K}(t) \tag{1}$$

So, for all $i = 0, 1, \dots, k-1$, we have that

$$\omega_{b_i/A\langle b_0,\dots b_{i-1}\rangle}(t) \leq \omega_{p|_A}(t) = f(t) + h(t)$$

Clearly,

$$\omega_{\bar{b}/A}(t) \leq kf(t) + kh(t) \tag{2}$$

By assumption, $\bar{c} \in acl(A\langle\bar{b}\rangle)$ so $\omega_{\bar{b}/A}(t) = \omega_{\bar{b}\bar{c}/A}(t)$. Then

$$\omega_{\bar{b}/A\langle\bar{c}\rangle}(t) + \omega_{\bar{c}/A}(t) \leq \omega_{\bar{c}\bar{b}/A}(t) = \omega_{\bar{b}/A}(t). \tag{3}$$

The left inequality is always true for tuples - at specific values of t, one is calculating the transcendence degree of the same tuple, $\{\theta(\bar{c}\bar{b}) \,|\, \theta \in \Theta(t)\}$, but with a bigger base field on the left hand side during the calculation of the transcendence degree of $\{\theta(\bar{b}) \mid \theta \in \Theta(t)\}$. The field extensions on the left side of the inequality are:

$$\begin{array}{c} \mathbb{Q}\langle A, \bar{c}\rangle(\theta(\bar{b}) \mid \theta \in \Theta(t)) \\ \Big| \\ \mathbb{Q}\langle A, \bar{c}\rangle \end{array} \qquad \begin{array}{c} \mathbb{Q}\langle A\rangle(\theta(\bar{c}) \mid \theta \in \Theta(t)) \\ \Big| \\ \mathbb{Q}\langle A\rangle \end{array}$$

And the one on the right side is:

$$\begin{array}{c} \mathbb{Q}\langle A\rangle(\theta(\bar{b}\bar{c}) \mid \theta \in \Theta(t)) \\ \Big| \\ \mathbb{Q}\langle A\rangle \end{array}$$

Now, using 1 and 2 and 3, we see that

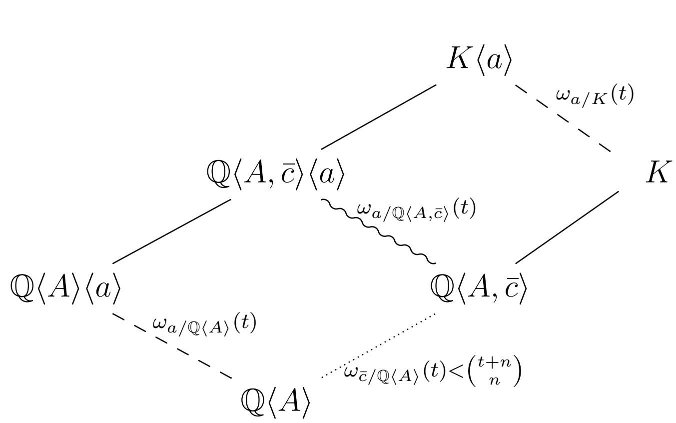
$$kf(t) + \omega_{\bar{c}/A}(t) \leq \omega_{\bar{c}\bar{b}/A}(t) = \omega_{\bar{b}/A}(t) \leq kf(t) + kh(t)$$

$$\omega_{\bar{c}/A}(t) \leq kh(t) < \binom{t+n}{n}.$$

□

Remark 1.6. Let us discuss the previous result briefly. Following suggestions from the Kolchin seminar members, I have provided a disgram of the pertinent field extensions. The theorem says that any forking which only affects the terms of the Kolchin polynomial of degree below n can be achieved by adding a tuple $\bar{c}$ to the canonical base of the restricted type which itself has Kolchin polynomial of degree less than n.

Let $a \models p$. Then we have the following diagram:



Our hypotheses are on the Kolchin polynomials of the field extensions with dashed lines, while the conclusion of the theorem is the above bound on the Kolchin polynomial of the field extension with the dotted line along with the equality of the Kolchin polynomials of the field extensions of the curly line and the upper dashed line.

2. Consequences of a conjecture

In my slides, I mention the following conjecture:

Conjecture. *For any type, $RU(p) \geq \omega^{\tau(p)}$.*

The above theorem would be a natural consequence of the conjecture along with some model-theoretic developments from superstability, which I do not explain here. Here, I want to explain what the conjecture means from the differential algebraic standpoint. Let us suppose, for simplicity that $\tau(p) = 1$. Then the conjecture says that the Lascar rank of p should be at least ω. By considering the generic types of generalizations of the heat equation, one can see that this is best one can hope for - that typical differential dimension can not be related to Lascar rank for general types.

To say that p is a of rank at least ω means that p has a forking extension of Lascar rank n for each $n \in \mathbb{N}$. Recall that Lascar rank zero means that p is the type if an element which is algebraic over the differential field generated by the base set being considered. So, when considering types over algebraically closed differential fields, Lascar rank zero types are simply the types of the elements in the field.

Taking $a \models p \in S(K)$, this means that for each n, there is a tuple $\bar{c}$ in a differential field extension of K such that $tp(a/K\langle \bar{c} \rangle)$ has Lascar rank n. Let us explain this in differential algebro-geometric terms.

Consider the variety $V = loc(a/K\langle \bar{c} \rangle)$ which is the zero set of finite many differential polynomials in $K\langle \bar{c} \rangle\{x\}$ for some tuple of differential indeterminants, x. Now, suppose that there is a parameterized family of disjoint subvarieties $V_d \subset loc(a/K\langle \bar{c} \rangle)$, that is we have a finite set of differential polynomials $f_1(x, y), \ldots f_n(x, y)$ such that for some differential algebraic variety, D,

- For $d \in D$, a generic point $a_1 \in Z(f_1(x, d), \ldots f_n(x, d)) \cap V$ has $RU(tp(a_1/K\langle c, d \rangle) \geq n - 1$.
- For $d_1 \neq d_2 \in D$, we have $Z(f_1(x, d_1), \ldots f_n(x, d_1)) \cap Z(f_1(x, d_2), \ldots f_n(x, d_2)) \cap V = \emptyset$.

The above two conditions are equivalent to $tp(a/K\langle \bar{c} \rangle)$ having Lascar rank n. So, specifying the Lascar rank of the generic type of a variety is equivalent to finding chains of uniform families of subvarieties of the original variety. The conjecture might thus be described as a uniform version of the Kolchin catenary conjecture for intermediate levels of the Kolchin polynomial.

References

[1] Saharon Shelah. *Classification theory and the number of non-isomorphic models. Studies in Logic and the Foundations of Mathematics.* Volume 92, North-Holland Publishing Company, New York, 1978.

[2] William Sit. On the well-ordering of certain numerical polynomials. *Transactions of the American Mathematical Society*, 212:37–45, 1975.