Differential and Difference Chow Form, Sparse Resultant, and Toric Variety

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Outline

- Background
- Sparse Differential Resultant
- Differential Chow Form
- Difference Binomial and Toric Variety

Sparse Differential Resultant for Laurent Differential Polynomials

Sylvester Resultant

Two polynomials:
$$f = a_l x^l + a_{l-1} x^{l-1} + \cdots + a_1 x + a_0$$
$$g = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0.$$

$$\mathrm{Res}(f,g) = \begin{vmatrix} a_l & a_{l-1} & a_{l-2} & \cdots & a_0 & & \\ & a_l & a_{l-1} & a_{l-2} & \cdots & a_0 & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & a_l & a_{l-1} & a_{l-2} & \cdots & a_0 \\ b_m & b_{m-1} & b_{m-2} & \cdots & b_0 & & \\ & b_m & b_{m-1} & b_{m-2} & \cdots & b_0 & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & b_m & b_{m-1} & b_{m-2} & \cdots & b_0 \end{vmatrix}.$$

Property: $\mathrm{Res}(f,g) = 0 \iff f(x) = g(x) = 0$ has common solutions

J.J. Sylvester, Phil Trans of Royal Soc of London, 407-548, 1883.

A Brief History of Resultant

Algebraic Resultant

- Sylvester (1883) resultant for two polynomials ($n = 1$)
- Macaulay (1902) multivariate resultant
- Gelfand & Sturmfels (1994) sparse resultant

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Differential Resultant

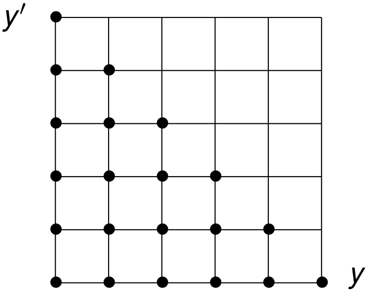
- Ritt (1932): Differential resultant for $n = 1$.
- Ferro (1997): Diff-Res as Macaulay resultant. **Not complete.**
- Zwillinger (1998): *Handbook of Differential Equations*.

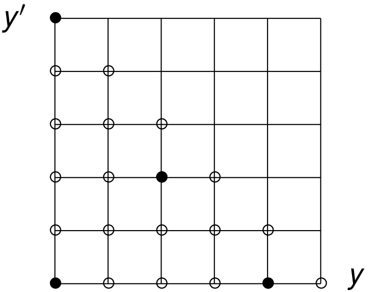
No rigorous definition for differential multi-variate resultant
No study of differential sparse resultant

Sparse Differential Polynomials

- **Sparse Differential Polynomials**: with fixed monomials

  Most differential polynomials in practice are sparse





Dense Diff Polynomials
$f = \sum_{i+j\leq 5} * y^i (y')^j$

Sparse Diff Polynomials
$f = * + * y^4 + * y'^5 + * y^2 y'^2$

Notations

Ordinary differential field: $(\mathcal{F}, \delta)$, e.g. $(\mathbf{Q}(x), \frac{d}{dx})$

Diff Indeterminates: $\mathbb{Y} = \{y_1, \ldots, y_n\}$.

Notation: $y_i^{(k)} = \delta^k y_i$.

Laurent Diff Monomial: $M = \prod_{k=1}^{n} \prod_{l=0}^{o} (y_k^{(l)})^{d_{kl}}$ with $d_{kl} \in \mathbb{Z}$;

Laurent Diff Poly: $f = \sum_{k=1}^{m} a_k M_k$, M_k Laurent diff monomials.

Support of f: $\mathcal{A} = \{M_1, \ldots, M_m\}$.

Laurent Diff Poly Ring: $\mathcal{F}\{\mathbb{Y}^{\pm}\}$.

Example. Laurent Differential Polynomial

$$\mathbb{P} = y_1 + y_1' y_2 \quad \Leftrightarrow \quad \mathbb{P} = 1 + y_1^{-1} y_1' y_2$$

Differential Dimension Conjecture in Generic Case

Intersection Theorem is not true in diff case:

$$\mathbf{dim}(V \cap W) \geq \mathbf{dim}(V) + \mathbf{dim}(W) - n$$

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Theorem

$\mathcal{I} \subset \mathcal{F}\{\mathbb{Y}\}$: a prime diff ideal with dimension $d > 0$ and order h.
f: a generic diff poly of order s with $\mathbf{u}_f$ the set of its coefficients.

Then $\mathcal{I}_1 = [\mathcal{I}, f]$ is a prime diff ideal in $\mathcal{F}\langle\mathbf{u}_f\rangle\{\mathbb{Y}\}$ with dimension $d - 1$ and order $h + s$.

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Theorem (Generic Dimension Theorem)

$f_1, \ldots, f_r (r \leq n)$: generic diff polynomials. Then

$[f_1, \ldots, f_r]$: a prime diff ideal of dimension $n - r$ and order $\sum_i \mathbf{ord}(f_i)$.

Sparse Differential Resultant

- **Generic Sparse Differential Polynomials:**

  $\mathcal{A}_i = \{M_{i0}, M_{i1}, \ldots, M_{il_i}\}$ $(i = 0, \ldots, n)$: Monomial sets

  $\mathbb{P}_i = \sum_{j=0}^{l_i} u_{ij} M_{ij}$ and $\mathbf{u}_i = \{u_{i1}, \ldots, u_{il_i}\}$.

  $[\mathbb{P}_0, \mathbb{P}_1, \ldots, \mathbb{P}_n] \subset \mathbf{Q}\{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbb{Y}, \mathbb{Y}^{-1}\}$

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- **Sparse Differential Resultant Exists, if the eliminant ideal:**

  $$[\mathbb{P}_0, \ldots, \mathbb{P}_n] \cap \mathbf{Q}\{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n\} = \mathbf{sat}(\mathbf{R}(\mathbf{u}_0, \ldots, \mathbf{u}_n))$$

  is of codimension 1

Definition

R: **Sparse Differential Resultant** of $\mathbb{P}_0, \ldots, \mathbb{P}_n$ or $\mathcal{A}_0, \ldots, \mathcal{A}_n$.

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$\Leftrightarrow$ $\mathbb{P}_i$ **are Laurent differentially essential:**

There exist k_i $(i = 0, \ldots, n)$ with $1 \le k_i \le l_i$ such that

$$\text{d.tr.deg}\, \mathbf{Q}\langle \frac{M_{0k_0}}{M_{00}}, \frac{M_{1k_1}}{M_{10}}, \ldots, \frac{M_{nk_n}}{M_{n0}} \rangle / \mathbf{Q} = n.$$

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Examples

Example

$n = 2,$

$\mathbb{P}_i = u_{i0} y_1'' + u_{i1} y_1''' + u_{i2} y_2''' \ (i = 0, 1, 2).$

$\mathrm{d.tr.deg}\, \mathbf{Q}\langle \frac{y_1'''}{y_1''}, \frac{y_2'''}{y_1''} \rangle / \mathbf{Q} = 2 \implies \mathbb{P}_i$ form a diff essential system.

The sparse differential resultant is

$$\mathbf{R} = \begin{vmatrix} u_{00} & u_{01} & u_{02} \\ u_{10} & u_{11} & u_{12} \\ u_{20} & u_{21} & u_{22} \end{vmatrix}.$$

Criterion for Existence of Sparse Resultant

$\mathbb{P}_i = \sum_{j=0}^{l_i} u_{ij} M_{ij}\ (i = 0, \ldots, n)$.

- $M_{ij}/M_{i0} = \prod_{k=1}^{n} \prod_{l=0}^{s_i} (y_k^{(l)})^{d_{ijkl}}$. $\quad d_{ijk} = \sum_{l=0}^{s_i} d_{ijkl} x_k^l \in \mathbf{Q}[x_k]$.

  Symbolic Support Vector of M_{ij}/M_{i0}: $\beta_{ij} = (d_{ij1}, \ldots, d_{ijn})$

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- **Symbolic Support Vector of** $\mathbb{P}_i$: $\beta_i = \sum_{j=0}^{l_i} u_{ij} \beta_{ij} = (d_{i1}, \ldots, d_{in})$.
- **Symbolic Support Matrix of** $\mathbb{P}_0, \ldots, \mathbb{P}_n$:

$$\mathbf{M}_{\mathbb{P}} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} d_{01} & d_{02} & \ldots & d_{0n} \\ d_{11} & d_{12} & \ldots & d_{1n} \\ & & \ddots & \\ d_{n1} & d_{n2} & \ldots & d_{nn} \end{pmatrix}$$

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Theorem (Like Linear Algebra!)

Sparse resultant exists for $\mathbb{P}_i \Longleftrightarrow \mathrm{rk}(\mathbf{M}_{\mathbb{P}}) = n$.

Properties of Sparse Differential Resultant

Lemma

$(\mathbb{P}_i, \mathbf{u}_i)$ *specializes to* $(\overline{\mathbb{P}}_i, \mathbf{v}_i)$ *by setting* $\mathbf{u}_i = \mathbf{v}_i \in \mathcal{F}$.

If $\overline{\mathbb{P}}_0 = \cdots = \overline{\mathbb{P}}_n = 0$ *has a* **non-poly solution**,

then $\mathbf{R}(\mathbf{v}_0, \ldots, \mathbf{v}_n) = 0$.

Necessary Condition for ∃ of Non-poly Solutions

Lemma

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Example (Why Non-Polynomial solution?)

$\mathcal{F} = \mathbf{Q}(x)$, differential operator: $\frac{\partial}{\partial x}$
$\mathbb{P}_i = u_{i0}y_1'' + u_{i1}y_1''' + u_{i2}y_2''' \ (i = 0, 1, 2)$.

The sparse differential resultant $\mathbf{R} = \begin{vmatrix} u_{00} & u_{01} & u_{02} \\ u_{10} & u_{11} & u_{12} \\ u_{20} & u_{21} & u_{22} \end{vmatrix} \neq 0.$

Let $a_1 = x + 1, a_2 = x^2 + x + 1$.
Then $a_1'' = a_2''' = 0$. (a_1, a_2): a solution of $\mathbb{P}_0 = \mathbb{P}_1 = \mathbb{P}_2 = 0$

Conditions for Existence of Non-poly Solutions

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- $\mathcal{Z}_0(\mathcal{A}_0, \ldots, \mathcal{A}_n)$: set of F_i having a common **non-poly solution**.
- $\overline{\mathcal{Z}_0(\mathcal{A}_0, \ldots, \mathcal{A}_n)}$: Kolchin diff closure of $\mathcal{Z}_0(\mathcal{A}_0, \ldots, \mathcal{A}_n)$.

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Theorem

$$\overline{\mathcal{Z}_0(\mathcal{A}_0, \ldots, \mathcal{A}_n)} = \mathbb{V}\big(\mathbf{sat}(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n})\big).$$

On a Kolchin open set of $\mathbb{V}\big(\mathbf{sat}(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n})\big)$,

$F_0 = \cdots = F_n = 0$ **have non-poly solutions** $\Leftrightarrow \mathrm{Res}_{F_0,\ldots,F_n} = 0$.

Order and Differential homogeneity

$\mathbb{G} = \{g_1, \ldots, g_n\}$: differential polynomials.

Jacobi Number: $\mathrm{Jac}(\mathbb{G}) = \max_\sigma \sum_{i=1}^{n} \mathbf{ord}(g_i, y_\sigma(i))$,

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Order and Differential homogeneity

- $\delta\mathbf{Res}(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ *is* **differentially homogeneous** *in each* $\mathbf{u}_i$ *and is of order* $h_i = s - s_i$ *in* $\mathbf{u}_i$ $(i = 0, \ldots, n)$ *where* $s = \sum_{l=0}^n s_l$.
- $\mathbf{S}\text{-}\delta\mathbf{Res}(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ *is* **differentially homogeneous** *in each* $\mathbf{u}_i$ *and is of order* $h_i \le J_i = \mathrm{Jac}(\mathbb{P}_{\hat{i}})$ *in* $\mathbf{u}_i$, *where* $\mathbb{P}_{\hat{i}} = \{\mathbb{P}_0, \ldots, \mathbb{P}_n\} \backslash \{\mathbb{P}_i\}$.

Poisson-Type Product Formula

- Algebraic Resultant: $\mathrm{Res}(A(x), B(x)) = c \prod_{\eta, B(\eta)=0} A(\eta)$.

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  When 1) Any n of the $\mathcal{A}_i$ diff independent and

  2) $\mathbf{e}_j \in \mathrm{Span}_{\mathbb{Z}}\{\alpha_{ij} - \alpha_{i0}\}$,

  the result can be strengthened:

  $\mathbf{S}\text{-}\delta\mathbf{Res}(\mathbf{u}_0, \ldots, \mathbf{u}_n) = A\prod_{\tau=1}^{t_0}\left(\frac{\mathbb{P}_{\mathbf{0}}(\eta_{\tau\mathbf{1}}, \ldots, \eta_{\tau\mathbf{n}})}{\mathbf{M_{00}}(\eta_{\tau\mathbf{1}}, \ldots, \eta_{\tau\mathbf{n}})}\right)^{(\mathbf{h_0})}$.

  And $\eta_\tau = (\eta_{\tau 1}, \ldots, \eta_{\tau n})$ are generic points of $[\mathbb{P}_0^N, \ldots, \mathbb{P}_n^N] : \mathbb{m}$.

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  $$\delta\mathbf{Res}(\mathbf{u}_0, \ldots, \mathbf{u}_n) = \sum_{i=0}^{n} \sum_{j=0}^{s-s_i} h_{ij}\mathbb{P}_i^{(j)}$$
  where $s_i = \mathbf{ord}(\mathbb{P}_i)$ and $s = s_0 + \cdots + s_n$, and
  $\mathbf{deg}(G_{ij}\mathbb{P}_i^{(j)}) \leq (m+1)\mathbf{deg}(R) \leq (m+1)^{ns+n+2}$.

Degree Bound of Sparse Differential Resultant

Laurent Diff Essential System: $\mathbb{P}_i$, **ord**$(\mathbb{P}_i) = s_i$ and **deg**$(\mathbb{P}_i) = m_i$.

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Theorem (Degree Bounds)

1. $\mathbf{deg}(\mathbf{R}) \leq \prod_{i=0}^{n}(m_i + 1)^{h_i+1} \leq (m + 1)^{ns+n+1}$, *where* $m = \max_i\{m_i\}$.

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Theorem (Degree Bounds)

1. $\mathbf{deg}(\mathbf{R}) \leq \prod_{i=0}^{n}(m_i + 1)^{h_i+1} \leq (m+1)^{ns+n+1}$, *where* $m = \max_i\{m_i\}$.
2. $\mathbf{R} = \sum_{i=0}^{n} \sum_{j=0}^{s-s_i} h_{ij}\mathbb{P}_i^{(j)}$

   $\mathbf{deg}(G_{ij}\mathbb{P}_i^{(j)}) \leq (m+1)\mathbf{deg}(R) \leq (m+1)^{ns+n+2}$.

BKK Degree Bound for Differential Resultant

Theorem

$\mathbb{P}_i\,(i=0,\ldots,n)$: *generic diff polynomials in* $\mathbb{Y}$ *with order* s_i, *coefficient set* $\mathbf{u}_i$, *and* $s=\sum_{i=0}^{n} s_i$. *Then*

$$\mathbf{deg}(\mathbf{R},\mathbf{u}_i)\le\sum_{k=0}^{s-s_i}\mathcal{M}\big((\mathcal{Q}_{jl})_{j\ne i,0\le l\le s-s_j},\mathcal{Q}_{i0},\ldots,\mathcal{Q}_{i,k-1},\mathcal{Q}_{i,k+1},\ldots,\mathcal{Q}_{i,s-s_i}\big).$$

$\mathcal{Q}_{jl}$: *Newton polytope of* $\mathbb{P}_j^{(l)}$ *as a polynomial in* $y_1^{[s]},\ldots,y_n^{[s]}$.
$\mathcal{M}(\mathcal{Q}_1,\ldots,\mathcal{Q}_n)$: *Mixed volume of* $\mathcal{Q}_1,\ldots,\mathcal{Q}_n$.

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Example

$$\mathbb{P}_0 = u_{00} + u_{01}y + u_{02}y' + u_{03}y^2 + u_{04}yy' + u_{05}(y')^2$$
$$\mathbb{P}_1 = u_{10} + u_{11}y + u_{12}y' + u_{13}y^2 + u_{14}yy' + u_{15}(y')^2$$

Bézout-type degree bound: $\mathbf{deg}(\mathbf{R}) \leq (2+1)^4 = 81$.
BKK-type degree bound: $\mathbf{deg}(\mathbf{R}) \leq 20$.

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An Algorithm for Sparse Differential Resultant

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1. Search for $\mathbf{R}(\mathbf{u}_0, \ldots, \mathbf{u}_n)$
   with order $h_i = 0, \ldots, s - s_i$ and
   with degree from $D = 1, \ldots, \prod_{i=0}^{n}(m_i + 1)^{h_i+1}$.
2. With fixed h_i and D, computing coefficients of $\mathbf{R}$ and G_{ik} by solving linear equations raising from

$$\mathbf{R}(\mathbf{u}_0, \ldots, \mathbf{u}_n) = \sum_{i=0}^{n} \sum_{k=0}^{h_i} h_{ik} \mathbb{P}_i^{(k)}.$$

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Theorem (Computing Complexity)

$O(m^{O(nls^2)})$ $\mathbf{Q}$-arithmetic operations.

n: number of variables; s: order of system; l: **size of sparse system**

Difference Sparse Resultant

Comparison with differential sparse resultant:

	Difference Case	Differential Case
Definition	$\mathbf{sat}(\mathbf{R}, R_1, \ldots, R_m)$ Problem: $m = 0$?	$\mathbf{sat}(\mathbf{R})$
Criterion	$\mathbf{M}_{\mathbb{P}} \in \mathbb{Z}[x]^{(n+1)\times n}$	$\mathbf{M}_{\mathbb{P}} \in \mathbb{Z}[u_{ij}, x_1, \ldots, x_n]^{(n+1)\times n}$
Matrix	$\mathbf{R} = \det(M)/\det(M_0)$	**?**
$\exists$ solutions	Necessary non-zero sols $\overline{\mathcal{Z}_0} = \mathbb{V}(\mathbf{sat}(\mathbf{R}, R_1, \ldots, R_m))$	Nec and Suff non-poly sol $\overline{\mathcal{Z}_0} = \mathbb{V}(\mathbf{sat}(\mathbf{R}))$
Homogeneity	Transformally homogenous $f(\lambda\mathbb{Y}) = M(\lambda) f(\mathbb{Y})$	Differentially homogenous $f(\lambda\mathbb{Y}) = \lambda^m f(\mathbb{Y})$
Degree	Dense: "$=$" BKK number Sparse: Bezout Type bound	Dense: BKK bound Sparse: Bezout Type bound
Order	Sparse: Jacobi bound Dense: $s - s_i$	The same The same

Differential Chow Form

Example: Plücker Coordinates

Using coordinates to represent algebraic variety

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Using coordinates to represent algebraic variety

Lines in $\mathbf{P}(3)$:

- Line $\mathbf{L} := \begin{cases} a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0 \\ b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 = 0 \end{cases}$

  $\Leftrightarrow$ (one to one correspondence)

  Plücker Coordinates: $p^{ij} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}, i,j = 0,1,2,3$

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- Plücker coordinate $C = (p^{01}, p^{02}, p^{03}, p^{23}, p^{31}, p^{12}) \in \mathbf{P}(5)$

  C represents a line in $\mathbf{P}(3)$

  $\longleftrightarrow$

  C is on hypersurface $p^{23}p^{01} + p^{31}p^{02} + p^{12}p^{03} = 0.$

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Background: Chow Form

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- **Differential Analog?**

Definition of Differential Chow Form

$\mathcal{I} \subset \mathcal{F}\{\mathbb{Y}\}$: prime differential ideal of dimension d.

$d+1$ **Generic Differential Primes**:

$\mathbb{P}_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n \, (i = 0, \ldots, d)$.

$\mathbf{u}_i = (u_{i0}, \ldots, u_{in})$: coefficient set of $\mathbb{P}_i$

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Theorem

By intersecting $\mathcal{I}$ with the $d+1$ primes, the **eliminant ideal**

$$[\mathcal{I}, \mathbb{P}_0, \ldots, \mathbb{P}_d] \cap \mathcal{F}\{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d\} = \mathbf{sat}(F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d))$$

is a prime ideal of **co-dimension one**.

Differential Chow form of $\mathcal{I}$ or $\mathbb{V}(\mathcal{I})$:

$$F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, \ldots, u_{d0})$$

Order of Differential Chow Form

Chow form of $\mathcal{I}$: $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, u_{10}, \ldots, u_{d0})$

Property of Chow form.

- $F(\ldots, \mathbf{u}_\sigma, \ldots, \mathbf{u}_\rho, \ldots) = (-1)^{r_{\sigma\rho}} F(\ldots, \mathbf{u}_\rho, \ldots, \mathbf{u}_\sigma, \ldots)$.
- $\mathbf{ord}(F, u_{00}) \neq 0$, $\mathbf{ord}(F, u_{00}) = \mathbf{ord}(F, u_{ij})$ if u_{ij} occurs in F

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Theorem (Order of Chow Form)

$\mathbf{ord}(F) = \mathbf{ord}(\mathcal{I})$.

Degree of Differential Chow Form

Differentially homogenous diff poly of degree m:

$$p(ty_0, ty_1 \ldots, ty_n) = t^m p(y_0, y_1, \ldots, y_n)$$

Theorem

$F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$: differential Chow form of V.

Then $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ is differentially homogenous of degree r in each set $\mathbf{u}_i$ and F is of total degree $(d+1)r$.

Definition (Differential degree)

r as above is defined to be the **differential degree** of $\mathcal{I}$, which is an invariant of $\mathcal{I}$ under invertible linear transformations.

Factorization of Differential Chow Form

V: a diff irreducible variety of dimension d and order h.
$F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$: the differential Chow form of V.

Theorem ($F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ can be uniquely factored)

$$\begin{aligned} F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) &= A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \prod_{\tau=1}^{g} (u_{00} + \sum_{\rho=1}^{\mathbf{n}} \mathbf{u}_{\mathbf{0}\rho} \xi_{\tau\rho})^{(h)} \\ &= A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \prod_{\tau=1}^{g} \mathbb{P}_0(\xi_{\tau 1}, \ldots, \xi_{\tau n})^{(h)} \end{aligned}$$

where $g = \mathbf{deg}(F, u_{00}^{(h)})$ and $\xi_{\tau\rho}$ are in an extension field of $\mathcal{F}$.

And the points $(\xi_{\tau 1}, \ldots, \xi_{\tau n})$ $(\tau = 1, \ldots, g)$ are ***generic points*** *of the variety V.*

Leading Differential Degree

Differential primes:

$\mathbb{P}_i := u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n \, (i = 1, \ldots, d),$

Algebraic primes:

${}^a\mathbb{P}_0 := u_{00} + u_{01}y_1 + \cdots + u_{0n}y_n,$

${}^a\mathbb{P}_0^{(s)} := u_{00}^{(s)} + \sum_{j=1}^{n} \sum_{k=0}^{s} \binom{s}{k} u_{0j}^{(k)} y_j^{(s-k)} \, (s = 1, 2, \ldots)$

Theorem

$(\xi_{\tau 1}, \ldots, \xi_{\tau n})\,(\tau = 1, \ldots, g)$ *are the* **only elements** *of V which lie on* $\mathbb{P}_1, \ldots, \mathbb{P}_d$ *as well as on* ${}^a\mathbb{P}_0, {}^a\mathbb{P}_0', \ldots, {}^a\mathbb{P}_0^{(h-1)}$.

Definition

Number g is defined to be the **leading diff degree** *of V or I.*

Differential Chow variety

A diff variety V has **index** (n, d, h, g, m) if $V \subset \mathcal{E}^n$ has **invariants**: dim d, order h, leading diff degree g, and diff degree m.

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Definition

A diff variety $\mathbb{V}$ is a **Chow Variety** if $(\bar{a}_i) \in \mathbb{V}$

$\Leftrightarrow$ $\bar{F}$ with coef $(\bar{a}_i)$: Chow form with index (n, d, h, g, m).

$\Leftrightarrow$ V: diff cycle of index (n, d, h, g, m).

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Chow Coordinate of V: $(\bar{a}_i)$

In affine case, Chow Variety is a constructible set.

Differential Chow variety

Theorem (Gao-Li-Yuan, 2013)

In the case $g = 1$, the differential Chow variety exists.

Difficulty for the general case: Eliminating both differential and algebraic variables.

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Theorem (Freitag-Li-Scanlon, 2015)

The differential Chow variety exists.

Key Ideas: Use prolongation admissible varieties and prolongation sequences to reduce the construction to the algebraic case. Definability in ACF and DCF_0 is also used.

Differential Toric Variety

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- Consider the map

$$\begin{array}{rcl} \phi_{\mathcal{A}} : (\mathcal{E}^{\wedge})^n & \longrightarrow & \mathbf{P}(l) \\ \eta & & (M_0(\eta), M_1(\eta), \ldots, M_l(\eta)) \end{array}$$

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Theorem

$\mathrm{Res}_{\mathcal{A}}$: **Sparse differential resultant** *of* $\mathbb{P}_i = \sum_j u_{ij} M_j, i = 0, \ldots, n$ *is the differential Chow form of* $X_{\mathcal{A}}$.

Differential Toric Variety: An Example

Example

Let $n = 1$ and a set of monomials $\mathcal{A} = \{y_1, y_1', y_1^2\}$.

Toric Variety: all possible values of a set of monomials $\mathcal{A}$

$$X_{\mathcal{A}} = \{(y_1, y_1', y_1^2) \mid y_1 \in \mathcal{E}\}$$

Defining equations of the toric variety

$$X_{\mathcal{A}} = \text{Zero}(\textbf{sat}(z_1 z_2 - (z_0 z_2' - z_0' z_2))).$$

The sparse differential resultant $\mathrm{Res}_{\mathcal{A}}$ is equal to the differential Chow form of $X_{\mathcal{A}}$.

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The defining ideal of a diff toric variety is not **binomial**!

Binomial σ-ideal and Toric σ-variety

Notations

In this talk, **difference field** $(\mathcal{F}, \sigma)$: $\sigma : \mathcal{F} \Rightarrow \mathcal{F}$ is a field automorphism. $\mathcal{F}$ is also assumed to be algebraically closed.

Example: $\overline{\mathbf{Q}(x)} : \sigma(x) = x + 1$.

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σ**-Exponent**: For $p = \sum_{i=0}^{s} c_i x^i \in \mathbb{Z}[x]$, denote $a^p = \prod_{i=0}^{s} (\sigma^i a)^{c_i}$.

Example: $a^{3x^2-1} = (\sigma^2(a))^3 / a$

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$\mathbb{Y} = \{y_1, \ldots, y_n\}$: σ-indeterminates

σ**-monomial with support f**:
$\mathbb{Y}^{\mathbf{f}} = \prod_{i=1}^{n} y_i^{f_i}$ where $\mathbf{f} = (f_1, \ldots, f_n)^\tau \in \mathbb{N}[x]^n$
$\mathcal{F}\{\mathbb{Y}\}$: σ-polynomial ring

$\mathbb{Z}[x]$ Lattice

$\mathbb{Z}[x]$ **Lattice**: $\mathbb{Z}[x]$ module in $\mathbb{Z}[x]^n$

Two kinds of representations:

Generators: $L = \mathrm{Span}_{\mathbb{Z}[x]}\{\mathbf{f}_1, \ldots, \mathbf{f}_s\} = (\mathbf{f}_1, \ldots, \mathbf{f}_s)$, $\mathbf{f}_i \in \mathbb{Z}[x]^n$

Matrix representation: $F = [\mathbf{f}_1, \ldots, \mathbf{f}_s]_{n\times s}$

Rank of L: $\mathrm{rk}(F)$

Binomial σ-ideal

σ-binomial: $f = a\mathbb{Y}^{\mathbf{a}} + b\mathbb{Y}^{\mathbf{b}}$, $\mathbf{a}, \mathbf{b} \in \mathbb{N}[x]^n$, $a, b \in \mathcal{F}$.

Normal Form: $f = a\mathbb{Y}^{\mathbf{g}}(\mathbb{Y}^{\mathbf{f}^+} - c\mathbb{Y}^{\mathbf{f}^-})$,

$\mathbf{f} \in \mathbb{Z}[x]^n$ and $\mathbf{f} = \mathbf{a} - \mathbf{b} = \mathbf{f}^+ - \mathbf{f}^-$ for $\mathbf{f}^+, \mathbf{f}^- \in \mathbb{N}[x]^n$.

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Normal binomial σ-ideal $\mathcal{I}$:

- $\mathcal{I}$ is generated by σ-binomials
- $Mp \in \mathcal{I} \Rightarrow p \in \mathcal{I}$, M: σ-monomial

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Partial Character: A homomorphism from a $\mathbb{Z}[x]$ lattice L_ρ to the multiplicative group $\mathcal{F}^*$ satisfying $\rho(x\mathbf{f}) = \sigma(\rho(\mathbf{f}))$.

Lemma

$\mathcal{I}$ is a normal binomial σ-ideal $\Leftrightarrow$

$\mathcal{I} = \mathcal{I}(\rho) = \{\mathbb{Y}^{\mathbf{f}^+} - \rho(\mathbf{f})\mathbb{Y}^{\mathbf{f}^-} \mid \mathbf{f} \in L_\rho\}$ *for a partial character* ρ.

Criteria for Normal LBσ-ideal

Definition

A $\mathbb{Z}[x]$ lattice L in $\mathbb{Z}[x]^n$ is called

- **$\mathbb{Z}$-saturated** if, for $a \in \mathbb{Z}$ and $\mathbf{f} \in \mathbb{Z}[x]^n$, $a\mathbf{f} \in L$ implies $\mathbf{f} \in L$.
- **x-saturated** if, for $\mathbf{f} \in \mathbb{Z}[x]^n$, $x\mathbf{f} \in L$ implies $\mathbf{f} \in L$.
- **M-saturated** if, for $\mathbf{f} \in \mathbb{Z}[x]^n$ and $m \in \mathbb{N}$, $m\mathbf{f} \in L \Rightarrow (x - o_m)\mathbf{f} \in L$.

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Theorem

Let ρ be a partial character over $\mathbb{Z}[x]^n$.

- *L_ρ is $\mathbb{Z}$-saturated $\Leftrightarrow \mathcal{I}(\rho)$ is prime*
- *L_ρ is x-saturated $\Leftrightarrow \mathcal{I}(\rho)$ is reflexive*
- *If $\langle \mathcal{I}(\rho) \rangle : \mathbb{M} \neq [1]$, then L_ρ is M-saturated $\Leftrightarrow \mathcal{I}(\rho)$ is well-mixed*
- *If $\{\mathcal{I}(\rho)\} : \mathbb{M} \neq [1]$, then L_ρ is x-M-saturated $\Leftrightarrow \mathcal{I}(\rho)$ is perfect*

Toric σ-variety

For $\mathbb{a} = \{\alpha_1, \ldots, \alpha_n\}$, $\alpha_i \in \mathbb{Z}[x]^m, i = 1, \ldots, n$
Define a map $\phi_{\mathbb{a}} : (\mathbb{A}^*)^m \mapsto (\mathbb{A}^*)^n$:

$$\mathcal{T} = (t_1, \ldots, t_m) \mapsto \mathcal{T}^{\mathbb{a}} = (\mathcal{T}^{\alpha_1}, \ldots, \mathcal{T}^{\alpha_n}).$$

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Toric Variety $\mathbf{X}_{\mathbb{a}}$: the Cohn closure of $\phi_{\mathbb{a}}((\mathcal{C}^*)^m)$ in $(\mathbb{A})^n$.

- Toric Variety: σ-variety parameterized by σ-monomials.
- $\mathbf{X}_{\mathbb{a}}$ is an irreducible σ-variety of dim $\mathrm{rk}(A)$, where $A = [\alpha_1, \ldots, \alpha_n]_{m \times n}$

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Example

The support: $\mathbb{a} = \{[1,1]^\tau, [x,x]^\tau, [0,1]^\tau\}$.
The σ-monomial: $(t_1 t_2, t_1^x t_2^x, t_2)$.
The map: $y_1 = t_1 t_2, y_2 = t_1^x t_2^x, y_3 = t_2$
Toric σ-variety: $\mathbf{X}_{\mathbb{a}} : y_1^x - y_2 = 0.$ Note that y_3 is free.

Toric σ-ideal

Toric $\mathbb{Z}[x]$ Lattice L: $p\mathbf{f} \in L \Rightarrow \mathbf{f} \in L \quad (p \in \mathbb{Z}[x]$ and $\mathbf{f} \in \mathbb{Z}[x]^n)$

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Toric σ-ideal: $\mathcal{I}^+(L) = [\mathbb{Y}^{\mathbf{f}^+} - \mathbb{Y}^{\mathbf{f}^-} \mid \mathbf{f} \in L]$, where L is a toric $\mathbb{Z}[x]$ lattice.

- $\mathbf{I}^+(L)$ is reflexive and prime σ-ideal of dimension $\mathrm{rk}(L)$.

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Theorem

A σ-variety V is toric iff $\mathbb{I}(V)$ is a toric σ-ideal.

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Example (Reflexive prime but not toric)

Let $L = ([1-x, x-1]^\tau)$.

Since $[1-x, x-1] = (x-1)\cdot[1,-1]$, L is not $\mathbb{Z}[x]$ toric.

The σ-ideal $\mathbf{I}^+(L) = [y_1^{x^i} y_2^{x^j} - y_1^{x^j} y_2^{x^i}; 0 \le i \le j \in \mathbb{N}]$ is reflexive prime but not toric.

Conversion between $V = \mathbf{X}_{\mathbb{\alpha}}$ and $\mathbb{I}(V) = \mathbf{I}^{+}(\rho_L)$

(1) **Implicitization**:

Given $\mathbf{X}_{\mathbb{\alpha}}$ $(\mathbb{\alpha} = (\alpha_1, \ldots, \alpha_n)) \Rightarrow \mathbb{I}(V) \subset \mathcal{F}\{\mathbb{Y}\}$

$A = [\alpha_1, \ldots, \alpha_n]_{m \times n}$

$K_A = \ker(A) = (\mathbf{f}_1, \ldots \mathbf{f}_s)$: a toric $\mathbb{Z}[x]$ lattice; Gröbner basis

$\mathbb{I}(\mathbf{X}_\alpha) = \mathbf{I}^{+}(K_A) = \mathbf{sat}(\mathbb{Y}^{\mathbf{f}_1^{+}} - \mathbb{Y}^{\mathbf{f}_1^{-}}, \ldots, \mathbb{Y}^{\mathbf{f}_s^{+}} - \mathbb{Y}^{\mathbf{f}_s^{-}})$

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(2) **Parametrization**:

Given $\mathcal{I} = \mathbf{sat}(\mathbb{Y}^{\mathbf{f}_1^+} - \mathbb{Y}^{\mathbf{f}_1^-}, \ldots, \mathbb{Y}^{\mathbf{f}_s^+} - \mathbb{Y}^{\mathbf{f}_s^-}) \Rightarrow \mathbf{X}_{\mathbb{\alpha}} = \mathbb{V}(\mathbf{I})$

$L_\rho = (\mathbf{f}_1, \ldots, \mathbf{f}_s)$

$F = [\mathbf{f}_1, \ldots, \mathbf{f}_s]_{n\times s} \in \mathbb{Z}[x]^{n\times s}$

$K_F = \{X \in \mathbb{Z}[x]^n \mid \mathbf{F}^\tau X = 0\}$ is a free $\mathbb{Z}[x]$ module.

K_F has a basis $\{\mathbf{h}_1, \ldots, \mathbf{h}_{n-r}\}$

$H = [\mathbf{h}_1, \ldots, \mathbf{h}_{n-r}]_{n\times(n-r)}$

$\mathbb{\alpha} = \{\alpha_1, \ldots, \alpha_n\} \in \mathbb{Z}[x]^{n-r}$ the rows of H.

If L_ρ is a toric $\mathbb{Z}[x]$ lattice, then $\mathbf{X}_{\mathbb{\alpha}} = \mathbb{V}(\mathbf{I})$

Affine $\mathbb{N}[x]$ module:

$\mathbb{\beta} = \{\beta_1, \ldots, \beta_s\} \subset \mathbb{Z}[x]^m$

$M = \mathbb{N}[x](\mathbb{\beta}) = \{\sum_{i=1}^s a_i\beta_i \mid a_i \in \mathbb{N}[x]\} \subset \mathbb{Z}[x]^m$.

Affine σ-algebra

$$\mathcal{F}\{M\} = \{\sum_{\mathbf{f}\in M} a_f \mathcal{T}^{\mathbf{f}} \mid a_{\mathbf{f}} \in \mathcal{F}, a_\beta \neq 0 \text{ for finitely many } \beta\}.$$

Coordinate Ring of Toric σ-variety

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Theorem. X is a toric σ-variety
$\Leftrightarrow$ $X \cong \mathrm{Spec}^\sigma(\mathbf{Q}\{M\})$ for an affine $\mathbb{N}[x]$ module M.
$\Leftrightarrow$ the coordinate ring of X is $\mathbf{Q}\{M\}$.

Toric σ-variety in terms of group action

The map $\phi_{\mathbb{a}} : (\mathbb{A}^*)^m \longrightarrow (\mathbb{A}^*)^n$:

Quasi σ-torus: $T_{\mathbb{a}} = \phi_{\mathbb{a}}((\mathbb{A}^*)^m)$

In the algebraic case, $T_{\mathbb{a}}$ (the torus) is a variety: $T_{\mathbb{a}} = \mathbf{X}_{\mathbb{a}} \cap (\mathbb{C}^*)^m$
This is not valid in the difference case.

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σ-torus: a σ-variety isomorphic to the Cohn $*$-closure of $T_{\mathbb{\alpha}}$ in $(\mathbb{A}^*)^n$.

(1) T^* is a σ-variety which is open in $\mathbf{X}_{\mathbb{\alpha}}$.

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Theorem (Toric σ-variety in terms of group action)

A σ-variety X is toric iff X contains a σ-torus T^* as an open subset and with a group action of T^* on X extending the natural group action of T^* on itself.

Summary

- **Sparse differential/difference resultant** is defined and properties similar to that of the Sylvester resultant are given.

  A single exponential algorithm to compute the sparse differential resultant is given.

- **Differential/difference Chow Form** is defined and its basic properties are established.

- **Difference binomial ideals and difference toric varieties** are introduced, which connects the difference Chow form and difference sparse resultant.

Thanks !

Summary

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