# Free Commutative Integro-differential Algebras

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# Free differential algebras

- Differential algebra

$$d(xy) = d(x)y + xd(y) + \lambda d(x)d(y).$$

$$d(uv) \mapsto_\phi d(u)v + ud(v) + \lambda d(u)d(v), \forall u, v \in R.$$

This leads to normal forms $w$ with no products in $d$.

- Thus free commutative differential algebra (of weight $\lambda$) on a set $X$ is of the form

$$\mathbf{k}\{X\} := \mathbf{k}[\Delta X], \quad \Delta X := \{x^{(n)} \mid x \in X, n \geq 0\}$$

with concatenation product. Define $d_X : \mathbf{k}\{X\} \to \mathbf{k}\{X\}$ as follows. Let $w = u_1 \cdots u_k, u_i \in \Delta(X)$, $1 \leq i \leq k$, be a commutative word from the alphabet set $\Delta(X)$. If $k = 1$, so that $w = x^{(n)} \in \Delta(X)$, define $d_X(w) = x^{(n+1)}$. If $k > 1$, recursively define

$$d_X(w) = d_X(u_1)u_2 \cdots u_k + u_1 d_X(u_2 \cdots u_k) + \lambda d_X(u_1) d_X(u_2 \cdots u_k).$$

Further define $d_X(1) = 0$. Then $(\mathbf{k}\{X\}, d_X)$ is the free commutative differential algebra of weight $\lambda$ on the set $X$.

# Integral algebra

- What about integral algebra?
- What is an integral algebra? It is a special case of Rota-Baxter algebra.
- Let **k** be a commutative ring. Let $\lambda \in \mathbf{k}$ be fixed. A Rota-Baxter operator of weight $\lambda$ on a **k**-algebra $R$ is a linear map $P : R \to R$ such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \ \forall x, y \in R.$$

- References:
1. L. Guo, WHAT IS a Rota-Baxter Algebra, *Notice of Amer. Math. Soc.* **56** (2009), 1436-1437.
2. L. Guo, An Introduction to Rota-Baxter Algebra, International Press, 2012.

# The integration operator $I$

- For continuous functions $f(x)$ and $g(x)$, define

$$F(x) := I[f](x) := \int_0^x f(s)ds, G(x) := I[g](x) := \int_0^x g(s)ds. \qquad (1)$$

  Then $F'(x) = f(x), G'(x) = g(x)$.
- The integration by parts formula

$$\int_0^x F'(t)G(t)dt = F(t)G(t)\big|_0^x - \int_0^x F(t)G'(t)\,dt$$

  can be "rewritten" as

$$\int_0^x f(t)G(t)\,dt = F(x)G(x) - \int_0^x F(t)g(t)\,dt.$$

- Using Eq. (1), get $I[f\,I[g]](x) = I[f]I[g](x) - I[I[f]\,g](x)$.

$$I[f\,I[g]] = I[f]I[g] - I[I[f]\,g], \quad I[f]I[g] = I[f\,I[g]] + I[I[f]\,g].$$

- An integral algebra is an algebra $R$ together with a linear operator $I : R \to R$ that satisfies

$$I[f]I[g] = I[f\,I[g]] + I[g\,I[f]], \quad \forall f, g \in R.$$

# Free commutative Rota-Baxter algebras

- Rota-Baxter algebra

$$P(x)P(y) = P(P(x)y) + P(xP(y)) + \lambda P(xy).$$

$$P(x)P(y) \mapsto_P P(P(x)y) + P(xP(y)) + \lambda P(xy).$$

This leads to normal forms $w$ with no subwords of the form $P(x)P(y)$. In a commutative Rota-Baxter algebra, this means

$$\mathfrak{a} = a_0 P(a_1 P(a_2 P(\cdots P(a_n)\cdots))), a_i \in A.$$

$$\mathfrak{a} = a_0 \otimes a_1 \otimes \cdots \otimes a_n \in A^{\otimes(n+1)}.$$

The product is given by

$$\mathfrak{a}\mathfrak{b} = (a_0 b_0) \otimes \big((a_1 \otimes \cdots a_n) \diamond (b_1 \otimes \cdots \otimes b_m)\big).$$

$\diamond$ is a shuffle like product, called mixable shuffle product.

# Mixable shuffle product

- Let $A$ be a commutative $\mathbf{k}$-algebra. Let $Ш^+(A) = \bigoplus_{n\geq 0} A^{\otimes n} (= T(A))$. Consider the following products on $Ш^+(A)$.
- **Mixable shuffle product:** Guo-Keigher (2000) on Rota-Baxter algebras, Goncharov (2002) on motivic shuffle relations and Hazewinckle on overlapping shuffle products.
- A shuffle of $\mathfrak{a} = a_1 \otimes \ldots \otimes a_m$ and $\mathfrak{b} = b_1 \otimes \ldots \otimes b_n$ is a tensor list of $a_i$ and $b_j$ without change the order of the $a_i$s and $b_j$s.
- A mixable shuffle is a shuffle in which some pairs $a_i \otimes b_j$ are merged into $a_i b_j$.

  Define $(a_1 \otimes \ldots \otimes a_m) \diamond (b_1 \otimes \ldots \otimes b_n)$ to be the sum of mixable shuffles of $a_1 \otimes \ldots \otimes a_m$ and $b_1 \otimes \ldots \otimes b_n$.
- **Example:**

$$\begin{aligned}
&a_1 \diamond (b_1 \otimes b_2)\\
&= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 \quad \text{(shuffles)}\\
&+ a_1 b_1 \otimes b_2 + b_1 \otimes a_1 b_2 \quad \text{(merged shuffles)}.
\end{aligned}$$

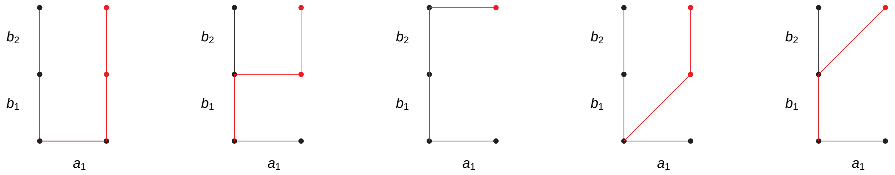
- **Quasi-shuffle product:** Hoffman (2000) on multiple zeta values.
- Let $Ш^+(A) = \bigoplus_{n\geq 0} A^{\otimes n} (= T(A))$.
- Define $\mathbf{1_k} \in \mathbf{k}$ to be the unit. Let $\mathfrak{a} = a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$. Write $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$, $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$. Recursively define

$$(a_1 \otimes \mathfrak{a}') * (b_1 \otimes \mathfrak{b}') = a_1 \otimes (\mathfrak{a}' * (b_1 \otimes \mathfrak{b}'))) + b_1 \otimes ((a_1 \otimes \mathfrak{a}') * \mathfrak{b}') + a_1 b_1 \otimes (\mathfrak{a}' * \mathfrak{b}'),$$

  with the convention that if $\mathfrak{a} = a_1$, then $\mathfrak{a}'$ multiples as the identity. It defines the shuffle product without the third term.
- **Example.**
  $a_1 * (b_1 \otimes b_2) = a_1 \otimes (\mathfrak{a}' * (b_1 \otimes b_2)) + b_1 \otimes (a_1 * b_2) + (a_1 b_1) \otimes (\mathfrak{a}' * b_2)$
  $= a_1 \otimes (b_1 \otimes b_2) + b_1 \otimes (a_1 * b_2) + (a_1 b_1) \otimes b_2.$
  $= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 + b_1 \otimes a_1 b_2 + a_1 b_1 \otimes b_2.$

- By Stuffles: Cartier (1972) on free commutative Rota-Baxter algebras, Ehrenborg (1996) on monomial quasi-symmetric functions and Bradley (2004) on $q$-multiple zeta values.
- By Delannoy paths: Fares (1999) on coalgebras, Aguiar-Hsiao (2004) on quasi-symmetric functions and Loday (2005) on Zinbiel operads.
- Let $D(m,n)$ be the set of lattice paths from $(0,0)$ to $(m,n)$ consisting steps either to the right, to the above, or to the above-right. For $d \in D(m,n)$ define $d(\mathfrak{a},\mathfrak{b})$ to be the path $d$ with $\mathfrak{a}=(a_1,\cdots,a_m)$ (resp. $\mathfrak{b}=(b_1,\cdots,b_n)$) sequentially labeling the horizontal (resp. vertical) and diagonal segments of $d$. Define

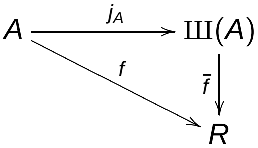
$$\mathfrak{a} \diamond_d \mathfrak{b} = \sum_{d \in D(m,n)} d(\mathfrak{a},\mathfrak{b}).$$




$a_1 \otimes b_1 \otimes b_2$ $\quad$ $b_1 \otimes a_1 \otimes b_2$ $\quad$ $b_1 \otimes b_2 \otimes a_1$ $\quad$ $a_1 b_1 \otimes b_2$ $\quad$ $b_1 \otimes a_1 b_2$

# Free commutative Rota-Baxter algebras

- **Theorem** All the above products define the same algebra on $\text{Ш}^+(A)$.
- A **free Rota-Baxter algebra over another algebra $A$** is a Rota-Baxter algebra $\text{Ш}(A)$ with an algebra homomorphism $j_A : A \to \text{Ш}(A)$ such that for any Rota-Baxter algebra $R$ and algebra homomorphism $f : A \to R$, there is a unique Rota-Baxter algebra homomorphism making the diagram commute




- When $A = \mathbf{k}[X]$, we have the free Rota-Baxter algebra over $X$.
- Recall $(\text{Ш}^+(A), \diamond)$ is an associative algebra. Then the tensor product algebra (scalar extension) $\text{Ш}(A) := A \otimes \text{Ш}^+(A)$ is an $A$-algebra.
  **Theorem** (Guo-Keigher) $\text{Ш}(A)$ with the shift operator $P(\mathfrak{a}) := 1 \otimes \mathfrak{a}$ is the free commutative RBA over $A$.

## Examples

- The free commutative Rota-Baxter algebra on **k** (i.e., on the empty set) is

$$Ш(\emptyset) = \oplus_{k\geq 1}\mathbf{k}a_k,$$

$$a_m a_n = \sum_{r=0}^{\min(m,n)} \binom{m+n-r}{m}\binom{m}{r}\lambda^r \mathbf{1}^{\otimes(m+n-r)}.$$

When $\lambda = 0$, we obtain the divided powers.

- The free commutative integral algebra (Rota-Baxter algebra of weight 0) on $\mathbf{k}[x]$ (i.e., on one generator $x$):

Let $\mathcal{I} := \coprod_{k\geq 1}\mathbb{N}^k_{\geq 0}$. For $I = (i_0, \cdots, i_k) \in \mathcal{I}$, denote

$$x^{\otimes I} := x^{i_0} \otimes \cdots \otimes x^{i_k}.$$

Then $\quad Ш(\mathbf{k}[x]) = \oplus_{I\in\mathcal{I}}\mathbf{k}x^{\otimes I}.$

For $x^{\otimes I} = x^{i_0} \otimes x^{\bar{I}}$ and $x^{\otimes J} = x^{j_0} \otimes x^{\bar{J}}$, we have

$$x^{\otimes I}x^{\otimes J} = x^{i_0+j_0} \otimes \left(x^{\bar{I}} Ш x^{\bar{J}}\right),$$

where Ш is the shuffle product.

# Differential Rota-Baxter algebra

- A differential Rota-Baxter algebra (DRB) is a triple $(R, d, P)$ where $(R, d)$ is a differential algebra (of weight $\lambda$), $(R, P)$ is a Rota-Baxter algebra (of weight $\lambda$) such that $d \circ P = \mathrm{id}_R$.
- These give three rewriting rules that imply that a normal form for the DRB algebra is of the form $\mathfrak{x} := x_0 \otimes x_1 \otimes \cdots \otimes x_n, x_i \in \Delta X$.
- More generally, let $(A, d)$ be a differential algebra of weight $\lambda$. On the free commutative Rota-Baxter algebra $(Ш(A), P_A)$, define

$$d_A : Ш(A) \to Ш(A),$$

$$\begin{aligned} d_A(x_0 \otimes x_1 \otimes \cdots \otimes x_n) &= d_0(x_0) \otimes x_1 \otimes \ldots \otimes x_n \\ &+ x_0 x_1 \otimes x_2 \otimes \ldots \otimes x_n + \lambda d_0(x_0) x_1 \otimes x_2 \otimes \ldots \otimes x_n \end{aligned}$$

  Then $(Ш(A), d_A, P_A)$ is the free commutative differential Rota-Baxter algebra on $A$.
- Let $X$ be a set. The differential Rota-Baxter algebra $(Ш(\mathbf{k}\{X\}), d_{\mathbf{k}\{X\}}, P_{\mathbf{k}\{X\}})$ is the free differential Rota-Baxter algebra on $X$. It is the algebra of differential Rota-Baxter polynomials in $X$.

# Integro-differential algebras

- Note that the "integral by parts" formula in Rota-Baxter algebra is a purified version (so that only $I$ occurs)

$$I(f)I(g) = I(fI(g)) + I(I(f)g)$$

of the original formula

$$FG = I(F'G) + I(FG')$$

where the interaction of differentiation and integration is taken out of the picture. This needs to be put back in order to understand fully the algebraic structure in differential equations, in particular, to compute Green's operators of boundary problems for linear ordinary/partial differential equations.

# Definition of Integro-differential Algebras

- An integro-differential $\mathbf{k}$-algebra of weight $\lambda$ (also called a $\lambda$-integro-differential $\mathbf{k}$-algebra) is a differential $\mathbf{k}$-algebra $(R, D)$ of weight $\lambda$ with a linear operator $\Pi\colon R \to R$ such that

$$D \circ \Pi = \mathrm{id}_R$$

and the initialization

$$J := \Pi \circ D$$

satisfies

$$J(x)J(y) = J(x)y + xJ(y) - J(xy) \quad \text{for all } x, y \in R.$$

# Equivalent conditions

- Let $(R, D)$ be a differential algebra of weight $\lambda$ with a linear operator $\Pi$ on $R$ such that $D \circ \Pi = \mathrm{id}_R$. Denote $J = \Pi \circ D$, called the *initialization*, and $E = \mathrm{id}_R - J$, called the *evaluation*. Then the following statements are equivalent:
  1. $(R, D, \Pi)$ is an integro-differential algebra;
  2. $E(xy) = E(x)E(y) \quad$ for all $x, y \in R$;
  3. $\ker E = \mathrm{im} J$ is an ideal;
  4. $J(xJ(y)) = xJ(y)$ and $J(J(x)y) = J(x)y \quad$ for all $x, y \in R$;
  5. $J(x\Pi(y)) = x\Pi(y)$ and $J(\Pi(x)y) = \Pi(x)y \quad$ for all $x, y \in R$;
  6. $x\Pi(y) = \Pi(D(x)\Pi(y)) + \Pi(xy) + \lambda\Pi(D(x)y)$ and
     $\Pi(x)y = \Pi(\Pi(x)D(y)) + \Pi(xy) + \lambda\Pi(xD(y)) \quad$ for all $x, y \in R$;
  7. $(R, D, \Pi)$ is a differential Rota-Baxter algebra and
     $\Pi(E(x)y) = E(x)\Pi(y)$ and $\Pi(xE(y)) = \Pi(x)E(y) \quad$ for all $x, y \in R$;
  8. $(R, D, \Pi)$ is a differential Rota-Baxter algebra and
     $J(E(x)J(y)) = E(x)J(y)$ and
     $J(J(x)E(y)) = J(x)E(y) \quad$ for all $x, y \in R$.

# Integral by parts revisited

- $(R, D, \Pi)$ is an integro-differential algebra if and only if $(R, D)$ is a differential algebra, $D \circ \Pi = \mathrm{id}_R$ and

$$\Pi(D(x)\Pi(y)) - x\Pi(y) + \Pi(xy) + \lambda\Pi(D(x)y) = 0, \quad \forall x, y \in R.$$

- Theorem Let $(A, D)$ be a differential algebra. Let $I_{ID}$ be the differential Rota-Baxter ideal of $\text{Ш}(A)$ generated by elements in the above equations. Then the quotient differential Rota-Baxter algebra $\text{Ш}(A)/I_{ID}$ is the free integro-differential algebra on $(A, D)$.
- The last equation suggests the rewriting rule

$$\Pi(D(x)\Pi(y)) \mapsto_{ID} x\Pi(y) - \Pi(xy) - \lambda\Pi(D(x)y).$$

Working in the free differential Rota-Baxter algebra $\text{Ш}(A)$ where $(A, d)$ is a differential algebra, this means that $d(x)$ should not appear in $\Pi$. More precisely, in $\mathfrak{a} = a_0 \otimes a_1 \otimes \cdots \otimes a_n$, $a_1, \cdots, a_{n-1} \in A$ should be "in complement of" $d(A)$, i.e., in $A_T$ such that $A = \mathrm{im} d \oplus A_T$. Such an $A$ is called regular.

# Example of regular differential algebras

- Let $A = \mathbf{k}\{u\} = \mathbf{k}[u_i, i \leq 0]$ be the ring of differential polynomials in one variable. Then a monomial of $A$ is of the form $u^{\vec{i}} := u_0^{i_0} \cdots u_k^{i_k}$ with $\vec{i} := (i_0, \cdots, i_k), i_j \geq 0, 0 \leq j \leq k, i_k \neq 0, k \geq 0$. Such a monomial is said to have order $k$. Also say $u^{()} := 1$ to have order $-1$. A monomial $u^{\vec{i}}$ is called functional if either $k \leq 0$ or $i_k > 1$. Then the linear span $A_T$ is a linear complement of $\ker d$.
- Let $X$ be a well ordered set. Let $u \in C(\Delta X)$ be in the form

  $$u = u_0^{j_0} \cdots u_k^{j_k}, \text{ where } u_0, \cdots, u_k \in \Delta X, u_0 > \cdots > u_k \text{ and } j_0, \cdots, j_k \in \mathbb{Z}_{\geq}$$

  Call $u$ functional if either $u \in \{1\}$, or $u_k \in X$, or $j_k > 1$. Let $A = \mathbf{k}[\Delta X]$ and $A_T$ be the linear span of the functional monomials. Then $\mathbf{k}[\Delta X] = A_T \oplus \mathrm{im} d$. Thus $\mathbf{k}\{X\}$ is a regular differential algebra.

# Construction of free integro-differential algebras

- Let $(A, d)$ be a regular differential algebra. Thus $A = \mathrm{im} d \oplus A_T$ and $A_T$ is a nonunitary subalgebra of $A$ when $\lambda \neq 0$.
- Let n
$$Ш_T(A) = \oplus_{n\geq 0} A \otimes A_T^{\otimes n} = A \oplus (A \otimes A_T) \oplus \cdots$$
be the **k**-submodule of $Ш(A)$. Since $A_T$ is assumed to be a nonunitary subalgebra if $\lambda \neq 0$, $Ш_T(A)$ is a subalgebra of $Ш(A)$.
- Let $A_\varepsilon := \{\varepsilon(a) \,|\, a \in A\}$ be another copy of the algebra $A$, but with the zero derivation. Then both $A$ and $A_\varepsilon$ are $K$-algebras for $K := \ker d$.
- Define $ID(A) := A_\varepsilon \otimes_K Ш_T(A) = A_\varepsilon \otimes_K (A \otimes Ш^+(A_T))$ to be the tensor product algebra.
- Define $d_A$ on $ID(A)$ by the product rule on the tensor product.
- For $a \in A$, define $P_A(a) = \phi(a) - \varepsilon(\phi(a)) + 1 \otimes \psi(a)$.
For $\mathfrak{a} := a_0 \otimes \cdots \otimes a_n \in A \otimes (A_\psi)^{\otimes n}$, write $\mathfrak{a} = a_0 \otimes \bar{\mathfrak{a}}, \bar{\mathfrak{a}} \in A_\psi^{\otimes n}$. Define
$$P_A(\mathfrak{a}) = \phi(a_0) \otimes \bar{\mathfrak{a}} - P_A(\phi(a_0)\bar{\mathfrak{a}}) + 1 \otimes \psi(a_0) \otimes \bar{\mathfrak{a}}.$$
- Theorem (Guo-Regensburger-Rosenkranz) The triple $(ID(A), d_u, P_u)$ is the free commutative integro-differential algebra on $(A, d)$.

# Normal forms of integro-differential algebras

- Back to the rewriting rule

$$\Pi(D(x)\Pi(y)) \mapsto_{ID} x\Pi(y) - \Pi(xy) - \lambda\Pi(D(x)y).$$

- Then in the free differential Rota-Baxter algebra $\text{Ш}(\mathbf{k}\{X\})$. Elements of the form $d(x)$ should not appear in $\Pi(-p\Pi(v))$. So for $\mathfrak{a} = a_0 \otimes a_1 \otimes \cdots \otimes a_n$ to be normal, should have $a_i \in A_T, 1 \leq i \leq n-1$. This is quite hard to verify directly.

# Free integro-differential algebras by normal forms

- By the method of Gröbner-Shirshov basis, we obtain.
  Theorem(Gao-Guo-Zheng) Let $X$ be a nonempty well-ordered set and $A := \mathbf{k}\{X\}$. Let $Ш(\mathbf{k}\{X\}) = Ш(\mathbf{k}[\Delta X])$, with the derivation $d$ and Rota-Baxter operator $P$, be the free commutative differential Rota-Baxter algebra of weight $\lambda$ on $X$. Let $I_{ID}$ be the differential Rota-Baxter ideal of $Ш(\mathbf{k}\{X\})$ generated by

  $$S := \{P(d(u)P(v)) - uP(v) + P(uv) + \lambda P(d(u)v) \mid u, v \in Ш(\mathbf{k}\{X\})\}.$$

  Let $A_f$ be the submodule of $A = \mathbf{k}\{X\}$ spanned by functional monomials. Then the composition

  $$Ш(A)_f := A \oplus \left( \bigoplus_{k \geq 0} A \otimes A_f^{\otimes k} \otimes A \right) \hookrightarrow Ш(A) \to Ш(A)/I_{ID}$$

  of the inclusion and the quotient map is a linear bijection. Thus $Ш(A)_f$ gives an explicit construction of the free integro-differential algebra $Ш(A)/I_{ID}$.

# Comparison

- Compare the two constructions:

$$ID(A) := A_\varepsilon \otimes_K Ш_T(A) = A_\varepsilon \otimes_K (A \otimes Ш^+(A_T))$$

$$= (A_\varepsilon \otimes_K A) \oplus (A_\varepsilon \otimes A \otimes A_T) \oplus (A_\varepsilon \otimes A \otimes A_T \otimes A_T) \oplus \cdots .$$

$$Ш(A)_f := A \oplus \left( \bigoplus_{k \geq 0} A \otimes A_f^{\otimes k} \otimes A \right).$$

$$= A \oplus (A \otimes A) \oplus (A \otimes A_T \otimes A) \oplus \cdots .$$

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- **Thank You!**