DIFFERENTIAL SCHEMES

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Dedicated to the memory of Jerry Kovacic

1. Δ SCHEMES

All rings contain $\mathbb{Q}$ and are commutative. We fix a differential Δ ring A throughout this section.

1.1. The topological space.

Let A be a commutative Δ ring. We define

$$Spec^{\Delta}(A) = \{\mathfrak{p}/\mathfrak{p} \text{ is a differential prime ideal in } A\}.$$

We introduce the Kolchin topology on $X = Spec^{\Delta}(A)$ by introducing

$$U_f = \{\mathfrak{p} \in X/f \notin \mathfrak{p}\}.$$

as a basis for the open sets. This means that the closed sets are of the form $V(I) = \{\mathfrak{p}/\mathfrak{p} \supseteq I \text{ for } I \subseteq A\}$. The ideal of functions vanishing on a set $S \subseteq X$ is a then a radical $\Delta-$ideal that we denote $\mathcal{I}(S)$. $\mathcal{I}(V(I)) = \sqrt{I}$ and $V(\mathcal{I}(S)) = \overline{S}$, the closure of S in X.

1.2. The structure sheaf and sheaves of quasi-coherent modules.

Recall that a presheaf of abelian groups $\mathcal{F}$ on X is a contravariant functor on the category of open sets of X with inclusion maps as morphisms. A presheaf is a sheaf on X if for every open set U and all coverings $\{U_\alpha\}_{\alpha\in I}$ of U by open sets, two conditions are satisfied:

S1: if $x \in \mathcal{F}(U)$ and $x|_{U_\alpha} = 0$ for all $\alpha \in I$, then $x = 0$ and
S2: if there are $x_\alpha \in \mathcal{F}(U_\alpha)$ such that $x_\alpha| = x_\beta| \in \mathcal{F}(U_\alpha \cap U_\beta)$, then there is an $x \in \mathcal{F}(U)$ with $x|_{U_\alpha} = x_\alpha$ for all $\alpha \in I$

We say that a presheaf $\mathcal{F}$ is separated if S1 is satisfied for any U and any covering $\{U_\alpha\}$ of U. In general it suffices to define a sheaf only on a basis for the topology since then the two sheaf conditions determine $\mathcal{F}(U)$ if $U = \cup U_\alpha$ where $\{U_\alpha\}$ is a basis for the topology.

We would like to define a presheaf $\widetilde{A}$ by setting $\widetilde{A}(U_f) = A_f$ on basic open sets. Similarly if M is an A module, we would like to define $\widetilde{M}(U_f) = M_f$. However the existence of differential units that are not units poses a problem as does the existence of differential zeros that are not zero. So we begin by defining them.

Definition 1. *$m \in M$ is a differential zero of M if $\frac{m}{1} = 0 \in A_x$ for all $x \in Spec^{\Delta}(A)$. $\mathcal{Z}(M)$ denotes the set of all differential zeros of M.*

Definition 2. *$a \in A$ is a differential unit if $a \notin \mathfrak{p}_x$ for any $x \in Spec^{\Delta}(A)$. $\Delta(A^{\times})$ is then the set of all differential units of A and is multiplicatively closed.*

Observe that if we want a structure sheaf whose stalks at differential primes x are A_x, then differential units will be units in the sheaf and differential zeros will be zero in the sheaf since these statements are local statements aside from a patching condition. In the differential unit case, the multiplicative inverse is unique if it exists and so the patching condition will always be satisfied for the inverse of a differential unit.

The differential units that are not units cause major difficulties. For instance let k be your favorite Δ field and consider the Δ ring $A := k\{X\}$ in one differential indeterminate. Then $U_{\delta X} \subset U_X$ and so there should be a restriction map, that is a differential homomorphism $A[X]^{-1} \to A[\delta X]^{-1}$. But X is not a unit in $A[\delta X]^{-1}$, only a differential unit, and so there is no canonical homomorphism. On the other hand any differential unit is a unit in A_x for any $x \in Spec^{\Delta}(A)$.

There is an alternate characterization of a Δ ring in which every differential unit is a unit that is more convenient.

Lemma 1. *Let A be a Δ ring. Then the following are equivalent:*

(1) *Every differential unit in A is a unit.*
(2) *Every maximal ideal in A is a Δ ideal.*

Proof. Suppose every differential unit is a unit. Let $\mathfrak{m}$ be a maximal ideal in A and suppose that $\mathfrak{m}$ is not a Δ ideal but $\mathfrak{m} \subseteq \cup_{x\in X}\mathfrak{p}_x$.Then there is a finite set $\{a_1, \dots, a_r\} \subseteq \mathfrak{m}$ such that the Δ ideal $[a_i] = A$. On the other hand there are a finite number of maximal Δ ideals $\mathfrak{m}_1^{\Delta}, \dots, \mathfrak{m}_n^{\Delta}$ such that the ideal

$$(a_1, \dots, a_r) \subseteq \cup_{i=1}^{n}\mathfrak{m}_i^{\Delta}.$$

But then $(a_1, \dots, a_r) \subseteq \mathfrak{m}_i^{\Delta}$ for some i which is impossible. Consequently $\mathfrak{m}$ must contain some element b that is not in any Δ ideal and so is a unit. Again this is impossible so that $\mathfrak{m}m''t$ be a Δ ideal. The other direction is obvious. □

The differential zeros also should cause problems also but, once the differential unit issue has been dealt with, the differential zeros have also been handled. Accordingly we first turn our attention to differential units. $\Delta(A^{\times})$ consists of $a \in A$ such that a is in no Δ prime ideal of A. Thus if we define $A^{\Delta^{-1}} := A[\Delta(A^{\times})]^{-1}$, then every maximal Δ ideal in $A^{\Delta^{-1}}$ is actually a maximal ideal. Similarly if M is an A module, define $M^{\Delta^{-1}} := M[\Delta(A^{\times})]^{-1}$. Now if $m \in \mathcal{Z}(M)$ and $x \in Spec^{\Delta}(A)$, there is an $a \in Ann(m) - \mathfrak{p}_x$ such that $am = 0$. Thus $Ann(m) \cap \Delta(A^{\times}) \neq \emptyset$. Conversely if $Ann(m) \cap \Delta(A^{\times}) \neq \emptyset$, then $m \in \mathcal{Z}(M)$. Thus

$$\mathcal{Z}(M) = Ker\left(M \to M^{\Delta^{-1}}\right).$$

Of course the same argument applied to A shows that

$$0 \to \mathcal{Z}(A) \to A \to A^{\Delta^{-1}}$$

is exact and so inverting $\Delta(A^{\times})$ will also eliminate non zero differential zeros.

As we have seen above if A is a Δ ring, then $U_f \mapsto A_f$ is not a functor and so is not a presheaf. This has now been remedied by inverting all differential units for all basic opens. The result is a presheaf $\widetilde{A}^{\Delta}$ on $Spec^{\Delta}(A)$ defined by setting

$$\widetilde{A}^{\Delta}(U_f) = (A_f)^{\Delta^{-1}}.$$

Following Jerry, we set $\widehat{A} = \Gamma\left(X, \widetilde{A}^{\Delta}\right)$ and let $i_A : A \to \widehat{A}$ denote the natural map from A to $\widehat{A}$ that we get by inverting all differential units of A.

Lemma 2. *Let A be a Δ ring in which all differential units are units and having no differential zeros . Let $X = Spec^{\Delta}(A)$. Suppose $X = \cup U_{f_i}$, $0 \le i \le r$, defines a covering of X. Then*

(1) *if $g \in A$ has image 0 in all the rings A_{f_α}, then $g = 0$, and*
(2) *if $g_\alpha \in A_{f_\alpha}$ are elements such that g_α and g_β have the same image in $A_{f_\alpha f_\beta}$,. then there is an element $g \in A$ which has image g_α in A_{f_α} for all α.*

Proof. Since $\{U_{f_\alpha}\}$ is a covering of X, $\{U_{f_i}\}$ is also a covering. Now there is an element $f = \sum a_i f_i \notin x$ for any $x \in Spec^{\Delta}(X)$, and this element, being a differential unit, must be a unit in A. Consequently $1 \in [f_\alpha]_{\alpha \in A}$. We will use this fact repeatedly.

Now let $g \in A$. Since $\frac{g}{1} = 0 \in A_{f_i}$ there is an integer N, which we may assume is independent of i, such that $f_i^N g = 0$. But $\sqrt{\langle f_1^N, \dots, f_r^N \rangle} = A$ and so we can write 1 as $1 = \sum a_i f_i^N \in A$. But $f_i \in \sqrt{Ann(g)}$ for all i and so $1 \in \sqrt{Ann(g)}$. Since A has no differential zeros, g must be 0.

The second sheaf condition requires only a little more work. In view of the first sheaf condition we need only work with the finite cover $\{U_{f_i}\}_{1 \le i \le n}$. Let $g_i = \frac{b_i}{f_i^n} \in A_{f_i}$ with $\frac{g_i}{1} = \frac{g_j}{1} \in A_{f_i f_j}$ where, since the cover is finite, we may use a common n for all i. Then there is a common power N such that

$$(f_i f_j)^N \left(b_i f_j^n - b_j f_i^n\right) = 0 \in A.$$

Observe that $g_j = \frac{b_j f_j^N}{f_j^{N+n}}$. Now the ideal $I = \left\langle f_i^{N+n} \right\rangle = A$ since $X = \cup U_{f_i}$ and so

$$1 = \sum_{i=1}^{s} c_i f_i^{N+n}.$$

for some $c_i \in A$. Hence if we set $g = \sum c_i b_i f_i^N$, we have

$$f_j^{N+n} g = \sum_{i=1}^{s} f_j^{N+n} c_i b_i f_i^N = \sum_{i=1}^{s} f_i^{N+n} c_i b_j f_j^N = b_j f_j^N.$$

□

It is now straightforward to show that $\widetilde{A}^{\Delta}$ is a sheaf.

Proposition 1. *Let A be a Δ ring. Then $\widetilde{A}^{\Delta}$ is a sheaf.*

Proof. $\widetilde{A}^{\Delta}$ is clearly a separated presheaf and satisfies the patching condition for finite covers by the Lemma. If $\{U_{f_\alpha}\}_{\alpha \in I}$ is an infinite cover of U_f, and $g_\alpha \in A_{f_\alpha}^{\Delta^{-1}}$ with $g_\alpha| = g_\beta| \in A_{g_\alpha g_\beta}^{\Delta^{-1}}$, then $[f_\alpha] = A_f^{\Delta^{-1}}$. Since A_f is a Ritt ring, there are $\{f_i, 1 \le i \le r\} \subset [f_\alpha]_{\alpha \in I}$, such that $[f_i] = [f_\alpha]$. According to the lemma there is then a $g \in \widetilde{A}^{\Delta}(U_f)$ with

$$g|_{U_{f_i}} = g_i.$$

Since $\widetilde{A}^{\Delta}$ is separated and both $g|$ and g_α in $\widetilde{A}^{\Delta}(U_{f_\alpha})$ have the same image in $\widetilde{A}^{\Delta}(U_{f_\alpha} \cap U_{f_i})$ for $1 \le i \le r$, $g| = g_\alpha \in \widetilde{A}^{\Delta}(U_{f_\alpha})$. □

Corollary 1. *$Spec^{\Delta}(A) = Spec^{\Delta}\left(\widehat{A}\right)$ as topological spaces, and $i_{\widehat{A}} : \widehat{A} \to \widehat{\widehat{A}}$ is an isomorphism.*

Proof. $A^{\Delta^{-1}}$ has the same set of differential prime ideals since inverting differential units never removes a differential ideal and localization never adds ideals. The topology clearly also doesn't change. Moreover $\widehat{\widehat{A}}$ is constructed from $\widehat{A}$ by inverting any differential units that are not units. But all differential units in $\widehat{A}$ come from differential units in A by localization and so are already units in $\widehat{A}$. Similarly $\mathcal{Z}\left(\widehat{A}\right) = 0$ since $\widehat{A}$ is the global sections of a sheaf. □

The same argument applies to modules. Thus, given a Δ-A module M, we define the sheaf $\widetilde{M}^{\Delta}$ by first defining the presheaf $\widetilde{M}^{\Delta}(U_f) = M_f^{\Delta^{-1}}$ and noting that this is a sheaf on $Spec^{\Delta}(A)$ by the same argument. Of course the corollary also holds that $\widehat{M} = \widehat{\widehat{M}}$.

Suppose that A and B are Δ rings. If $\phi : A \to B$ is a Δ homomorphism, then differential units and differential zeros of A have images in B that are differential units and differential zeros respectively. Consequently ϕ induces a map of local ringed spaces

$$(\psi, \phi) : \left(Spec^{\Delta}(B), \widetilde{B}^{\Delta}\right) \to \left(Spec^{\Delta}(A), \widetilde{A}^{\Delta}\right)$$

where $\psi(x) = y \in Spec^{\Delta}(A)$ if the differential prime ideal $y = \phi^{-1}(x)$ and ϕ is extended to the sheaves by localizing with respect to the differential units and then factoring out the differential zeros.

Definition 3. *An affine Δ scheme is a local ringed space $(X, \mathcal{O}_X)$ that is isomorphic to $\left(Spec^{\Delta}(A), \widetilde{A}^{\Delta}\right)$ for some Δ ring A. A morphism of affine Δ schemes is a map of local ringed spaces $(\psi, \phi) : \left(Spec^{\Delta}(B), \widetilde{B}^{\Delta}\right) \to \left(Spec^{\Delta}(A), \widetilde{A}^{\Delta}\right)$. The category of affine Δ schemes is denoted $\left(\left(Affine^{\Delta}\right)\right)$.*

Definition 4. *A Δ ring A is said to be a max Δ ring if $\Delta(A^{\times})$ consists entirely of units in A. The category of max Δ rings and Δ homomorphisms will be called the max Δ ring category.*

Note that if A is a max Δ ring, then $\Delta\left(A_f^{\times}\right)/A_f^{\times}$ is generated by $\{a \in A/\emptyset \neq V(a) \subset V(f)\}$.

Proposition 2. *There is an adjoint pair $P : ((\Delta \text{ rings})) \to ((Max\ \Delta \text{ rings})) : i$ given by $P(A) = A[\Delta(A^{\times})]^{-1}$ and the inclusion functor $i(A) = A$. $P\left(B' \otimes_A B''\right)$ is canonically isomorphic to the coifibred coproduct $P\left(B'\right) \coprod_{P(A)} P\left(B''\right)$.*

Proof. Obvious. $Hom_{\text{ Prepared }\Delta}\left(A[\Delta(A^{\times})]^{-1}, B\right) \approx Hom_{\Delta}(A, i(B))$, and left adjoints preserve cofiibred coproducts. □

Actually we have an even stronger result.

Theorem 1. *Let A, B be max Δ algebras over a max Δ ring C. Then $A \otimes_C B$ is a max Δ ring.*

Proof. We first treat the case $C = \mathbb{Q}$.Let $\mathfrak{M} \subset A \otimes_{\mathbb{Q}} B$ be a maximal ideal, $F = A \otimes_{\mathbb{Q}} B/\mathfrak{M}, \mathfrak{m}_A = i_A(\mathfrak{M})^{-1}$, $k_A = A/\mathfrak{m}_A, \mathfrak{m}_B = i_B(\mathfrak{M})^{-1}$, and $k_B = B/\mathfrak{m}_B$ where $i_A : A \to A \otimes_{\mathbb{Q}} B$ and $i_B : B \to A \otimes_{\mathbb{Q}} B$ are given by $i_A(a) = a \otimes 1$ and $i_B(b) = 1 \otimes b$ respectively. Then $k_A \subseteq Q(k_A) \subseteq F$, $k_B \subseteq Q(k_B) \subseteq F$ and the ring compositum $k_A \cdot k_B = F$. Let T_B be a transcendence basis of $Q(k_B)$ over $\mathbb{Q}$. Then F is algebraic over $k_A \cdot \mathbb{Q}(T_B) = k_A(T_B)$. So if $a \in k_A - \{0\}$, then

$$\left(\frac{1}{a}\right)^n + \alpha_{n-1}\left(\frac{1}{a}\right)^{n-1} + \cdots + \alpha_0 = 0$$

for some $\alpha_i \in k_A(T_B)$, $0 \le i < n$. Multiplying by a^{n-1} produces the formula

$$\frac{1}{a} = -\alpha_0 a^{n-1} - \cdots - \alpha_{n-1} \in k_A(T_B) \cap Q(k_A).$$

Since T_B is transcendental over $\mathbb{Q}$, we conclude that a is a unit in k_A and $\mathfrak{m}_A$ is a maximal and so differential ideal in A. Similarly $\mathfrak{m}_B$ is a maximal and so differential ideal in B. Now look in the Δ ring $k_A \otimes_{\mathbb{Q}} k_B = (A \otimes_{\mathbb{Q}} B)/(\mathfrak{m}_A \otimes 1 + 1 \otimes \mathfrak{m}_B)$. Here $\mathfrak{M}/(\mathfrak{m}_A \otimes 1 + 1 \otimes \mathfrak{m}_B)$ is both a maximal and a minimal ideal in a Δ ring and so is a Δ ideal. (Alternatively we can use the fact that $k_A \cdot k_B = F$ and the Δ structure on k_A and k_B to conclude that $\mathfrak{M}$ is a Δ ideal.)

In the general case all we need to observe is that the kernel of the homomorphism $A \otimes_{\mathbb{Q}} B \to A \otimes_C B$ is a differential ideal since it is generated by elements of the form $a \otimes cb - ac \otimes b$. □

We are now in a position to define Δ schemes and extend the basic relation between affine schemes and schemes to Δ schemes as well as construct fibred products in Δ schemes.

Definition 5. *A Δ scheme is a local Δ ringed space $(X, \mathcal{O}_{X,\Delta})$ such that each point $x \in X$ has a neighborhood U such that $(U, \mathcal{O}_{X,\Delta}|_U)$ is Δ isomorphic to an affine Δ scheme $\left(Spec^{\Delta}(A), \widetilde{A}^{\Delta}\right)$ where A is a max Δ ring. A morphism of Δ schemes is then a morphism of local Δ ringed spaces.*

We will repeatedly use the fact that giving a morphism $(\psi, \phi) : (X, \mathcal{O}_{X,\Delta}) \to (Y, \mathcal{O}_{Y,\Delta})$ is equivalent to finding an open covering of $(X, \mathcal{O}_{X,\Delta}) = \cup (X_i, \mathcal{O}_{X_i,\Delta})$ and maps $(\psi_i, \phi_i) : (X_i, \mathcal{O}_{X_i,\delta}) \to (Y, \mathcal{O}_{Y,\Delta})$ such that

$$(\psi_i, \phi_i)|_{X_i \cap X_j} = (\psi_j, \phi_j)|_{X_i \cap X_j} : (X_i \cap X_j, \mathcal{O}_{X,\Delta}|_{X_i \cap X_j}) \to (Y, \mathcal{O}_{Y,\delta}).$$

We begin by observing that $\Phi : Mor_{\Delta}\left(\left(Spec^{\Delta}(B), \widetilde{B}^{\Delta}\right), \left(Spec^{\Delta}(A), \widetilde{A}^{\Delta}\right)\right) \to Mor_{\Delta}(A, B)$ defined by $\Phi(\psi, \phi) = \phi$ is an equivalence of categories between affine Δ schemes and max Δ rings. As is the case with schemes we build this up to Δ schemes.

Theorem 2. *Let A be a max Δ ring, and $X = \left(Spec^{\Delta}(A), \widetilde{A}^{\Delta}\right)$. Then $\Phi : Mor_{\Delta}(T, X) \to Mor_{\Delta}(A, \Gamma(T, \mathcal{O}_{T,\delta}))$, defined by*

$$\Phi((\psi, \phi)) = \phi(X) : A \to \Gamma(T, \mathcal{O}_{T,\delta}),$$

is an isomorphism.

Proof. Since A is a max Δ ring $\Gamma\left(X, \widetilde{A}^{\Delta}\right) = A$. If T is an affine Δ scheme, then the isomorphism Φ is the identification observed above. If T is not affine, then we write $T = \cup T_i$ where $T_i = \left(Spec^{\Delta}(B_i), \widetilde{B}_i^{\Delta}\right)_{i \in I}$ and observe that a morphism $(\psi, \phi) : T \to \left(X, \widetilde{A}^{\Delta}\right)$ is equivalent to giving a collection of maps (ψ_i, ϕ_i) : $\left(Spec^{\Delta}(B_i), \widetilde{B}_i^{\Delta}\right) \to \left(X, \widetilde{A}^{\Delta}\right)$ indexed by $i \in I$ which agree on pairwise intersections. But such a map means giving Δ homomorphisms $\phi(T_i) : \Gamma(T_i, \mathcal{O}_T) \to A$ which agree on $T_i \cap T_j$. □

Now the usual argument applies for the existence of fibred products in the category of Δ schemes since we already know that tensor product serves that purpose for max Δ rings.

Theorem 3. *Let X and Y be Δ schemes over the Δ scheme S. Then there is a Δ scheme $X \times_S Y$ and S morphisms $p_X : X \times_S Y \to X$ and $p_Y : X \times_S Y \to Y$ satisfying*

$$Mor_{\Delta}(T, X \times_S Y) = Mor_{\Delta}(T, X) \times_{Mor_{\Delta}(T,S)} Mor_{\Delta}(T, Y)$$

Proof. By covering S with Δ affine open schemes we first reduce to the case $S = \left(Spec^{\Delta}(A), \widetilde{A}^{\Delta}\right)$. Next we cover X with Δ affine open schemes $\left(Spec^{\Delta}(B_i'), \widetilde{B_i'}^{\Delta}\right)$ and Y with Δ affine open schemes $\left(Spec^{\Delta}(B_j''), \widetilde{B_j''}^{\Delta}\right)$. Again using the universal mapping property this reduces us to the same assertion for Δ affine schemes. Here we appeal to the existence of cofiibred coproducts in the category of prepared Δ rings. □

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