The Differential Brauer Group

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Outline

1. Review of Brauer groups of fields, rings, and Δ-rings
2. Cohomology
3. Cohomological interpretation of $\Delta-$Brauer groups (with connections to Hodge theory)

Brauer Groups of Fields

Finite dimensional division algebras Λ *over a field* K *are classified by* $Br(K)$, *the Brauer group of* K.

- If Λ is a central, simple algebra over K, then it is isomorphic to $M_n(D)$ for some division algebra D.
- Given two such algebras Λ and Γ, $\Lambda \otimes_K \Gamma$ is again a central simple K algebra.
- They are said to be Brauer equivalent if there are vector spaces V, W and a $K-$algebra isomorphism $\Lambda \otimes_K End(V) \approxeq \Gamma \otimes_K End(W)$. This is an equivalence relation, and $Br(K)$ is then defined to be the group formed from the equivalence classes with $\otimes$ as product. $Br(K)$ classifies division algebras over K.
- For any such algebra Λ over K, there is a Galois extension L/K such that $\Lambda \otimes_K L \approxeq End_L(V)$.
- Galois cohomology is then used to classify all such equivalence classes using an isomorphism $Br(K) \approxeq H^2\left(G_{\overline{K}/K}, \overline{K}^*\right)$.

Brauer Groups of Rings

- Azumaya algebras over a commutative ring R
  - A finitely generated central R algebra Λ is an Azumaya algebra algebra if $\Lambda \otimes_R L$ is a central simple algebra over L for any homomorphism from R to a field L.
  - An Azumaya algebra Λ is a central, finitely generated R algebra which is a projective $\Lambda \otimes_R \Lambda^{op}$ algebra.
- Two such Azumaya algebras Λ and Γ are Brauer equivalent if there are faithful, projective R modules P, Q and an $R-$algebra isomorphism $\Lambda \otimes_R End\left(P\right) \approxeq \Gamma \otimes_R End\left(Q\right)$.
- If R is a local ring, there is an etale extension S/R such that $\Lambda \otimes_R S \approxeq End_S\left(P\right)$ for some projective S module P.
- Etale cohomology is then used to classify all such equivalence classes using an isomorphism $\partial : Br\left(R\right) \underset{\approxeq}{\rightarrow} H^2\left(R_{et}, \mathbb{G}_m\right)$.

Brauer Groups of Δ-rings

Let $\Delta = \{\delta_1, ..., \delta_n\}$ be a set of n commuting derivations on R, a ring containing $\mathbb{Q}$.

- A Δ Azumaya algebra over R is an Azumaya algebra Λ over R equipped with derivations extending the action of Δ on R.
- Two such Δ Azumaya algebras Λ and Γ are Δ Brauer equivalent if there are faithful, projective $\Delta - R$ modules P, Q and a $\Delta - R$ algebra isomorphism $\Lambda \otimes_R End\,(P) \approxeq \Gamma \otimes_R End\,(Q)$. This is an equivalence relation, and $Br_\Delta(R)$ is the resulting group on the set of equivalence classes with $\otimes_R$ as the product.
- If R is local, there is an etale extension S and a $\Delta - S$ isomorphism $\Lambda \otimes_R S \approxeq End_S\,(P)$ for some $\Delta - S$ projective module P.

Cohomology

Let C be a category with fibred products. A pretopology on C consists of specifying for all $X \in ob(C)$, a set $Cov(X)$ whose members are collections $\{f_\alpha : U_\alpha \to X | \alpha \in A\} \in Cov(X)$ satisfying

1. If $f : X \to X$ is an isomorphism, $\{f\} \in Cov(X)$.
2. If $\{f_\alpha : U_\alpha \to X\} \in Cov(X)$ and $\{g_i^\alpha : V_i^\alpha \to U_\alpha\} \in Cov(U_i)$ for all i , then $\{f_\alpha g_i^\alpha : V_i^\alpha \to X\} \in Cov(X)$.
3. If $\{f_\alpha : U_\alpha \to X\} \in Cov(X)$ and $Y \to X \in \mathcal{C}$, then $\{f_\alpha \times_X Y : U_\alpha \times_X Y \to Y\} \in Cov(Y)$.

A presheaf $F : \mathcal{C}^{op} \to ((Sets))$ is a sheaf if for all $X \in \mathcal{C}$ and $\{U_\alpha \to X\} \in Cov(X)$,

$$F(X) \hookrightarrow \prod F(U_\alpha) \rightrightarrows \prod F(U_\alpha \times_X U_\beta)$$

is exact.

Example

1. X_{et} has $\{\{f_\alpha : V_a \rightarrow U \mid f_\alpha$ is an etale map and $U = \bigcup f_\alpha(V_a)\}\} = Cov_{et}(U)$.
2. $X_{\Delta-fl}$ has $\{\{g_\alpha : V_a \rightarrow U \mid g_\alpha$ is a flat Δ map of finite type and $U = \bigcup g_\alpha(V_a)\}\}$ $Cov_{\Delta-fl}(U)$.

If G is a scheme, then its functor of points defines a sheaf in either of these topologies. Moreover there is a map of sites $\tau : X_{\Delta-fl} \rightarrow X_{et}$ since any etale map is a flat Δ map. Thus $\tau^{-1}(\{f_\alpha\}) \in Cov_{\Delta-fl}(U)$. Moreover $H^*(X_{et}, G) \underset{\approx}{\rightarrow} H^*(X_{\Delta-fl}, G)$ for sheaves G defined by smooth, quasi-projective group schemes over X like the sheaf of units, $\mathbb{G}_m$.

In particular on $X_{\Delta-fl}$ we have the exact sequence

$$0 \to \mathbb{G}_m^{\Delta} \to \mathbb{G}_m \xrightarrow{d\ln} Z_X^1 \to 0$$

whose cohomology sequence contains

$$\begin{aligned} H^0\left(X_{\Delta-fl}, Z_X^1\right) &\to H^1\left(X_{\Delta-fl}, \mathbb{G}_m^{\Delta}\right) \to Pic\left(X\right) \xrightarrow{c_1} H^1\left(X_{\Delta-fl}, Z_X^1\right) \\ &\to H^2\left(X_{\Delta-fl}, \mathbb{G}_m^{\Delta}\right) \to Br\left(X\right) \to 0 \end{aligned}$$

if X is smooth since then $H^2\left(X_{et}, \mathbb{G}_m\right)$ is torsion unlike the vector space $H^2\left(X_{\Delta-fl}, Z_X^1\right)$!
How do we interpret this??

Cohomological Interpretation

Theorem

Let X be a quasi-projective variety of finite type over a field K of characteristic 0. If $x \in H^2(X, \mu_N)$, then there is an Azumaya algebra Λ equipped with an integrable connection constructed from x such that $\partial([\Lambda]) = i_N(x) \in H^2(X_{et}, \mathbb{G}_m)$ where $i_N : \mu_N \to \mathbb{G}_m$ is inclusion.

For simplicity, let's consider the case where $X = Spec(R)$ is a local ring and R contains a primitive N^{th} root of unity. Then there is an etale extension $R \to S \in Cov(Spec(R))$, i.e. $U = Spec(S) \to Spec(R)$, and a Cech 2 cocycle $\zeta \in \mu_N(S^3)$ such that $[\zeta] = x \in \check{H}^2((R \to S), \mu_N)$. Now by refining S we may assume that it is in the standard form $S = (R[T] / (p(T)))_{g(t)}$ where $p(T)$ is a monic polynomial of degree D. So we approximate S by $R[t] := R[T] / (p(T)) = \oplus_1^D R$.

Cohomological Interpretation

Now our cocycle $\zeta \in \mu_N (S \otimes_R S \otimes_R S)$ is constant on each connected component of $S^{\otimes 3}$ but may vary from one component to another. So we must use an index set that accounts for this. We let

$$J = \left\{ \text{connected components of } S^3 \right\}$$

Of course $\mathcal{F} := \left(\prod_{\alpha \in J} R[t]_\alpha \right) = \left(\oplus_{\alpha \in J} \left(\oplus_1^D R \right) \right)$ is not usually connected but for each connected component α of $S^{\otimes 3}$ there is an $R[t]_\alpha$ which admits multiplication by the value ζ_α of ζ on that connected component and, as an R module, $\mathcal{F}$ is free of rank $M = D \cdot (\#(J))$. Then we define an R module isomorphism

$$\ell_\zeta = \bigoplus_J \zeta_\alpha : \mathcal{F} \otimes_R \mathcal{S} \otimes_R \mathcal{S} \to \mathcal{F} \otimes_R S \otimes_R S$$

by multiplying the α^{th} factor in $\mathcal{F}$ by ζ_α. Note that this amounts to a diagonal block matrix where the α^{th} block is $\zeta_\alpha I_D$.

Cohomological Interpretation

We let $c\left(\oplus_J \zeta_\alpha\right) : End_R\left(\mathcal{F}\right) \otimes_R S \otimes_R S \longrightarrow End_R\left(\mathcal{F}\right) \otimes_R S \otimes_R S$ be the algebra isomorphism given by conjugation by ℓ_ζ. Then we get descent data from the diagram

$$\begin{array}{cccc} & & & End\left(\mathcal{F}\right) \otimes_R \mathcal{S} \otimes_R \mathcal{S} \\ & & e_2 \nearrow & \\ \Lambda \longrightarrow & End\left(\mathcal{F}\right) \otimes_R \mathcal{S} & & \downarrow c\left(\oplus_J \zeta_\alpha\right) \\ & & e_1 \searrow & \\ & & & End\left(\mathcal{F}\right) \otimes_R \mathcal{S} \otimes_R \mathcal{S} \end{array}$$

where e_i means insert 1_S into the i^{th} copy of S. Note that $End\left(\mathcal{F}\right)$ is the algebra of $M \times M$ matrices with $\delta\left(e_{ij}\right) = 0$ for all $\delta \in \Delta$. Here $c\left(\oplus_J \zeta_\alpha\right) = c\left(\ell_\zeta\right)$ is the patching data used to define Λ and it preserves the action of Δ since $c\left(\ell_\zeta\right)$ is given by conjugation by an N^{th} root of unity on each block in $End\left(\mathcal{F}\right)$.

It satisfies the cocycle condition

$$
\begin{array}{ccc}
 & & End\left(\mathcal{F}\right) \otimes_R \mathcal{S}^{\otimes 3} \\
 & e_3\left(c\left(\ell_\zeta\right)\right) \nearrow & \\
End\left(\mathcal{F}\right) \otimes_R \mathcal{S}^{\otimes 3} & & \downarrow e_1\left(c\left(\ell_\zeta\right)\right) \\
 & e_2\left(c\left(\ell_\zeta\right)\right) \searrow & \\
 & & End\left(\mathcal{F}\right) \otimes_R \mathcal{S}^{\otimes 3}
\end{array}
$$

This commutes because $\left(e_2\left(c\left(\bigoplus_J \zeta_\alpha\right)\right)\right)^{-1}\left(e_1\left(c\left(\bigoplus_J \zeta_\alpha\right)\right)\right)\left(e_3\left(c\left(\bigoplus_J \zeta_\alpha\right)\right)\right)$ is conjugation on $End\left(\mathcal{F}\right) \otimes_R \mathcal{S} \otimes_R \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}$ by $1 \otimes \zeta$ since ζ is a 2 cocycle. But this is $1_{End(\mathcal{F}) \otimes S^{\otimes 3}}$ which is the cocycle condition for descent.

Cohomological Interpretation

Thus the Cech cocycle provides the needed descent data and we immediately see that

$$\partial\left(\left[c\left(\oplus_J \zeta_\alpha\right)\right]\right) = [\zeta] \in \check{H}^2\left(X, \mu_n\right)$$

where $\Lambda = \left[c\left(\oplus_J \zeta_\alpha\right)\right]$ is the desired Δ Azumaya algebra.