RETHINKING PICARD-VESSIOT THEORY

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1. General Theory

We fix the following setting. A will be a Δ-ring, C the subring of constants, E a Δ module, locally free if necessary, $T_n = \oplus_{i=1}^n At_i$ where $\partial_j(t_i) = 0$ for all i, j. We let $((A - \Delta \text{ alg}))$ denote the category of differential A algebras. If M is any Δ module, let $\mathcal{M}$:$((A - \Delta \text{ alg})) \to ((Ab))$ be the associated covariant functor given by $\mathcal{M}(R) = M \otimes_A R$. Note that $\mathcal{HOM}(F, E)$ is the Hom functor consisting of homomorphisms that are not necessarily Δ maps. Moreover

$$\mathcal{HOM}(F, E)(R) = Hom_{A \text{ mod}}(F, E)_R \underset{\approx}{\to} Hom_{R \text{ mod}}(F_R, E_R)$$

where $M_R := M \otimes_A R$.

Lemma 1. *$\mathcal{M}$ is represented by the Δ algebra $\mathbb{S}(M^\vee)$ if M is locally free as an A module. $\mathbb{S}(M^\vee)$ is an $A - \Delta$ algebra.*

Proof. $\mathcal{M}(B) = M_B \xrightarrow{\alpha} Hom_{A \text{ mod}}(M^\vee, A)_B \underset{\approx}{\to} Hom_{A \text{ alg}}(\mathbb{S}(M^\vee), B)$ which is a Δ isomorphism since $\alpha(\delta m)(f) = f(\delta m)$ and $\delta(\alpha(m))(f) = \delta(f(m)) - (\delta f)(m) = \delta(f(m)) - [\delta(f(m)) - f(\delta m)]$. $Hom(M^\vee, A)_B \underset{\approx}{\to} Hom_{A \text{ alg}}(\mathbb{S}(M^\vee), B)$ is a Δ isomorphism which is easier to check as $Hom_{A \text{ mod}}(M, A)_B \underset{\approx}{\to} Hom_{A \text{ alg}}(\mathbb{S}(M), B)$. □

Note that representation does not mean with Δ maps, i.e. not Δ representable. However we view the various functors on $((A - \Delta \text{ alg}))$.

Corollary 1. *If M is a locally free $A - \Delta$ module, $\mathcal{M}^\Delta$ is represented by Δ homomorphisms from $\mathbb{S}(M^\vee)$. Thus $\mathcal{M}^\Delta(B) := (M_B)^\Delta = Hom_{A,\Delta \text{ alg}}(\mathbb{S}(M^\vee), B)$.*

Proof. $(M_B)^\Delta \xrightarrow{\alpha} Hom(M^\vee, A)_B^\Delta = Hom_\Delta(M_B^\vee, B) \underset{\approx}{\to} Hom_{A \text{ alg}}(\mathbb{S}(M^\vee), B)^\Delta = Hom_{A-\Delta \text{ alg}}(\mathbb{S}(M^\vee), B)$. □

Corollary 2. *$\mathcal{M} \oplus \mathcal{N}$ is represented by $\mathbb{S}(M^\vee) \otimes_A \mathbb{S}(N^\vee)$ if M, N are locally free A modules.*

Corollary 3. *If M is locally free and $A \to B$ is any Δ ring homomorphism, then $\mathbb{S}(M_B^\vee)$ represents the functor $R \mapsto M_B \otimes_B R$ on the category of $((B - \Delta \text{ alg}))$.*

This construction of the symmetric algebra of $M^\vee$ to represent the functor of the module M is clearly functorial.

Corollary 4. *If $M \to N$ is one-to-one, resp. onto, then $\mathbb{S}(N^\vee) \to \mathbb{S}(M^\vee)$ is onto, resp. one-to-one.*

2. PV theory

2.1. **Classical case: C is a field.** We apply the lemma in the following setting where E is an $A-\Delta$ module of rank n:

$$\begin{array}{ccc} \mathcal{ISOM}\left(T_n,E\right)^{\Delta} & \hookrightarrow & \mathcal{ISOM}\left(T_n,E\right) \\ \cap & \square & \cap \\ \mathcal{HOM}\left(T_n,E\right)^{\Delta} & \hookrightarrow & \mathcal{HOM}\left(T_n,E\right) \end{array}$$

where $T_n=\oplus_{i=1}^n At_i$ as above with $\partial_j\left(t_i\right)=0$ for all i,j and $\mathcal{ISOM}\left(T_n,E\right)$ is the open subfunctor consisting of isomorphisms, i.e. det is a unit. Thus, in view of the corollary,

$$\begin{aligned} \mathcal{ISOM}\left(T_n,E\right)^{\Delta}(B) &= Isom_{\Delta}\left(T_{n,B},E_B\right) \\ &= \{\mathbb{B}/\mathbb{B} \text{ is an ordered basis of } E_B \text{ consisting of constant elements}\} \\ &\subseteq Hom_{A-\Delta\text{ alg}}\left(\mathbb{S}\left(Hom\left(E^{\vee},T_n^{\vee}\right)\right),B\right). \end{aligned}$$

Thus we see that there is an initial differential A algebra, which, for simplicity, we write as CB_E, such that for all differential A algebras B,

$$Hom_{A-\Delta\text{ alg}}\left(CB_E,B\right)=\left\{\left(e_1,\dots e_n\right)/\left\{e_i\right\} \text{ is a basis of } E_B \text{ and } \partial_j\left(e_i\right)=0 \text{ for all } i,j\right\}.$$

For simplicity, we let $\mathcal{CB}_E$ stand for the functor $\mathcal{ISOM}\left(T_n,E\right)^{\Delta}$. Stated slightly differently, CB_E is an initial object in the category of all pairs consisting of an $A-\Delta$ algebra B and a constant ordered basis $\mathbb{B}$ for E_B. Such a pair means, of course, that $\left(E_B\right)^{\Delta}$ is a free B^{Δ} module with an isomorphism $\left(E_B\right)^{\Delta}\otimes_B B\to E_B$. This gives us one of the conditions for a Picard-Vessiot extension.

The category $((A-\Delta\text{ alg}))$ possesses sums which means that

$$Hom_{A-\Delta\text{ alg}}\left(B,-\right)\oplus Hom_{A-\Delta\text{ alg}}\left(B',-\right)=Hom_{A-\Delta\text{ alg}}\left(B\otimes_A B',-\right).$$

This is well known for commutative A algebras and is immediately checked for $A-\Delta$ algebras. Thus $CB_E\otimes_A CB_F$ represents the functor that we can think of as

$$B\mapsto\left\{\left(e_1,\dots e_m\right)/\left\{e_i\right\} \text{ is a constant basis of } E_B\right\}\oplus\left\{\left(f_1,\dots f_m\right)/\left\{f_i\right\} \text{ is a constant basis of } F_B\right\}$$

But, when we consider the case $E=F$, we can also describe this as

$$B\mapsto\left\{\left(e_1,\dots e_m\right)/\left\{e_i\right\} \text{ is a constant basis of } E_B\right\}\oplus\left\{\sigma\in Aut_{B,\Delta}\left(E_B\right)\right\}$$

where $\sigma\left(e_i\right)=\sum b_{ij}f_j$ which is the Δ module homomorphism sending the ordered basis $\left(e_i\right)$ to the ordered basis $\left(f_i\right)$. Note that $Aut_{B-\Delta}\left(E_B\right)=Gl_E\left(B^{\Delta}\right)$. Of course if E is not decomposed by B, then there will be restrictions on possible automorphisms reflecting the differential structure on E_B. Thus the action

$$\mathcal{CB}_E\left(B\right)\times\mathcal{AUT}_{\Delta}\left(E\right)\left(B\right)\xrightarrow{\tau}\mathcal{CB}_E\left(B\right)\times\mathcal{CB}_E\left(B\right)$$

is an isomorphism for any B over which E_B has a constant basis. But this can always be achieved by a faithfully flat extension and so τ is an isomorphism of sheaves. We will need to use this action so we introduce the notation $\left(\mathbb{B},g\right)\in\mathcal{CB}_E\left(B\right)\times\mathcal{AUT}_{\Delta}\left(E\right)\left(B\right)\mapsto\left(\mathbb{B},g*\mathbb{B}\right)$. This establishes the next result.

Proposition 1. *The natural action of $\mathcal{AUT}_{\Delta}\left(E\right)$ on $\mathcal{CB}_E$ makes $\mathcal{CB}_E$ into a principal homogeneous space for $\mathcal{AUT}_{\Delta}\left(E\right)$. In particular CB_E is a principal homogeneous space for the algebraic group $Aut_{\Delta}\left(E\right)$.*

This has the following very nice Corollary.

Corollary 5. *Let* $\sigma_i : CB_E \to B$, $i = 1, 2$, *be two* $A - \Delta$ *homomorphisms to an arbitrary* $A - \Delta$ *algebra. Then there is* $\tau \in \mathcal{AUT}_\Delta(E)(B)$ *such that* $\sigma_2 = \tau * \sigma_1 : CB_E \to B$.

Here are some useful examples.

Example 1. *Suppose* $E = T_n$. *Then we have a given ordered basis of constants, namely* $(t_1, t_2, \ldots, t_n)$ *and so an ordered basis of constants for* $T_{n,B}$ *is determined by specifying* $\sigma \in Gl_n\left(B^\Delta\right)$. *Consequently* $CB_{T_n} = A\left[g_{ij}, \det^{-1}\right]$ *where* $1 \le i, j \le n$ *and* $\det(g_{ij})$ *is the determinant of the corresponding matrix.*

Example 2. *Suppose* $E = T_k \oplus E'$. *Then* $\mathcal{CB}_E \supseteq \mathcal{CB}_{T_k} \oplus \mathcal{CB}_{E'}$ *and so* $CB_E \twoheadrightarrow A\left[g_{ij}, \det^{-1}\right] \otimes_A CB_{E'}$ *where* g_{ij} *are constant,* $1 \le i, j \le k$.

There are other ways that constants can appear in CB_E. For instance, instead of a repeated T_1, E might contain a repeated rank r Δ submodule. Thus if $E = T_k \otimes_A E'$, CB_E would also contain new constants coming from $A\left[g_{ab}, \det^{-1}\right]$ where $1 \le a, b \le k$ that arise from letting $Gl_k\left(A^\Delta\right)$ act on the first tensor factor in E.

Next we turn to the question of constants and simplicity. So suppose A is a simple Δ domain with quotient field K and field of constants C. If B is a $A - \Delta$ algebra that is a domain, then its quotient field $Q(B)$ contains a transcendental new constant only if B contains a non-zero Δ ideal. Thus if B is simple, all new constants must be algebraic. So up to algebraic extensions, we need only deal with simplicity.

Now if $I \subseteq CB_E$ is any differential ideal, $I \cap A = (0)$. Thus if we wanted to, we could pass to $K = Q(A)$ to understand the differential ideal structure of CB_E, but let's not do this. Replacing CB_E with $PV_E := CB_E/\mathfrak{m}$ where $\mathfrak{m}$ is a maximal differential ideal in CB_E now gives a simple Δ ring such that E_{PV_E} has a constant basis and will represent the functor $\mathcal{CB}_E$ when restricted to the category of simple $A - \Delta$ algebras. Of course, we must first show that PV_E is independant of the choice of $\mathfrak{m}$.

Theorem 1. *Let* A *be a* Δ *simple ring with* $C = A^\Delta$ *an algebraically closed field, and let* E *be an* $A - \Delta$ *module.*

(1) *If* $PV_E = CB_E/\mathfrak{m}$ *and* $PV'_E = CB_E/\mathfrak{m}'$ *are two simple extensions defined by maximal* Δ *ideals* $\mathfrak{m}$ *and* $\mathfrak{m}'$ *respectively, then there is a* $A - \Delta$ *algebra isomorphism* $PV_E \to PV'_E$.
(2) $Aut_{A-\Delta\ alg}(PV_E) = Stab_{Aut\left(E_{CB_E}\right)}(\mathfrak{m}) \subseteq Aut_{\Delta - CB_E}(E_{CB_E})$
(3) PV_E *is a principal homogeneous space for* $G(E) := Aut_{A-\Delta\ alg}(PV_E)$.

Proof. Choose a maximal Δ ideal $\mathcal{N}$ in $PV_E \otimes_A PV'_E$, and let $B = \left(PV_E \otimes_A PV'_E\right)/\mathcal{N}$. We first observe that PV_E and PV'_E, being simple, both embed into B. Moreover $B^\Delta = C$ since B is a simple Δ PV_E algebra and $(PV_E)^\Delta$ is also C. Thus B is a simple Δ extension containing both PV_E and PV'_E over which E has a basis of constants, one basis coming from PV_E while the other basis comes from PV'_E. Thus there is $\sigma \in Aut_\Delta(E_B) = Gl_n(C)$ such that $\sigma(PV_E) = PV'_E$ since we may apply the CB_E algebra automorphism defined from $\sigma \in Aut_{A-\Delta}(E)$ (which is $Gl_n(C)$ if E is free) to CB_E before passing to the residue rings.

Δ automorphisms of E_{CB_E} define $A - \Delta$ algebra automorphisms of CB_E and those that preserve $\mathfrak{m}$ clearly induce $A - \Delta$ algebra automorphisms of PV_E. This defines a map

$$Stab_{Aut\left(E_{CB_E}\right)}(\mathfrak{m}) \to Aut_{A-\Delta\ \mathrm{alg}}(PV_E).$$

Since any differential automorphism of PV_E will have to take a constant basis of E_{PV_E} to a different constant basis, it will come from a differential automorphism of E_{CB_E} and so this map is onto. Similarly if an automorphism of CB_E stabilizes $\mathfrak{m}$ and induces the identity algebra automorphism on PV_E, then it must come from the identity automorphism of E_{CB} and so be the identity algebra automorphism. Consequently

$$Aut_{A-\Delta \text{ alg}}(PV_E) = Stab_{Aut_\Delta(E_{CB_E})}(\mathfrak{m}).$$

$G(E)$ comes equipped with an embedding into $Aut_\Delta(E_{CB_E})$

The assertion that PV_E is a principal homogeneous space for$Aut_\Delta(E)$ will be established by restricting and evaluating the principal homogeneous space construction for $\mathcal{CB}_E$ (2). Let $\mathcal{PV}_E$ denote the functor defined by PV_E. Then

$$\begin{array}{ccc} \mathcal{PV}_E(B) = Hom_{A-\Delta \text{ alg}}(PV_E, B) & \hookrightarrow & \mathcal{CB}_E(B) = Hom_{A-\Delta \text{ alg}}(CB_E, B) \\ \| & & \| \\ \{\phi : CB_E \to B/\phi(\mathfrak{m}) = 0\} & & \{\phi : CB_E \to B\} \end{array}$$

and the functor $\mathcal{G}$ stabilizing $\mathcal{PV}_E$ is characterized as

$$\begin{array}{ccc} \mathcal{G}(B) & \hookrightarrow & Aut_\Delta(E_B) \\ \| & & \\ \{g \in Aut_\Delta(E_B)/g * \phi \in \mathcal{PV}_E(B) \text{ for all } \phi \in \mathcal{PV}_E(B)\} & & \end{array}.$$

$\mathcal{AUT}_\Delta(E)(B)$ acts transitively on $\mathcal{CB}_E(B)$ and so if $\phi, \phi' \in \mathcal{PV}_E(B)$, there is a unique $g \in Aut_\Delta(E_B)$ such that $g * \phi = \phi'$. But both $\phi(\mathfrak{m}) = \phi'(\mathfrak{m}) = 0$ and so $g \in \mathcal{G}(B)$. Now we restrict the isomorphism τ to get the diagram

$$\begin{array}{ccc} \mathcal{CB}_E(B) \times \mathcal{AUT}_\Delta(E)(B) & \xrightarrow{\tau} & \mathcal{CB}_E(B) \times \mathcal{CB}_E(B) \\ \cup & & \cup \\ \mathcal{PV}_E(B) \times \mathcal{G}(B) & \xrightarrow{\tau_1} & \mathcal{PV}_E(B) \times \mathcal{PV}_E(B) \end{array}$$

where the top map is an isomorphism. Clearly the bottom map is 1-to-1. On the other had $\mathcal{G}(B)$ acts transitively on $\mathcal{PV}_E(B)$ and so its restriction to the bottom is also an isomorphism.

Finally $G(E)$ is clearly $\mathcal{G}(PV_E)$ which finishes the argument. □

Note however that the constants have had to be algebraically extended to an algebraically closed field to realize this isomorphism. Consequently, in general, we need an algebraic extension B of A before realizing $PV_{E,B} \approxeq PV'_{E,B}$.

The key point in the proof is that Δ module automorphisms of E_{CB_E} are equivalent to Δ algebra automorphisms of CB_E and those that preserve $\mathfrak{m}$ clearly induce $A-\Delta$ algebra automorphisms of PV_E. Now once we know that PV_E is simple, then we know that $(PV_E)^\Delta$ is an algebraic extension of C. So if C is an algebraically closed field, we can conclude that PV_E is our usual Picard-Vessiot extension. Thus it is critical that $B-\Delta$ module automorphisms of E_B translate into $A-\Delta$ algebra automorphisms of CB_E when B is simple with algebraically closed constants.

Example 3. *Suppose $E = T_n$. Then $PV_E \approxeq A$.*

Example 4. *Suppose $E = T_k \oplus E'$. Then $PV_E \approxeq PV_{E'}$ This follows from the second example since PV_E is constructed by choosing a maximal Δ ideal in CB_E. But $CB_E \twoheadrightarrow CB_{E'} \otimes_A A[x_1, \ldots, x_k]$ is onto so we may choose a maximal ideal in $CB_{E'}$ to construct PV_E.*

The original question that gave rise to this sequence of ideas was how to calculate $PV_E \otimes_A PV_F$ for two $A - \Delta$ modules E and F. While we cannot completely answer the question we can observe that $PV_E \otimes_A PV_F \approxeq PV_E \otimes_{PV_E} PV_{F'} := PV$ where F' is a $\Delta - PV_E$ module satisfying $F_{PV_E} = T_{k,PV_E} \oplus F'$. If we could eliminate the constants from the PV_E algebra PV, we could complete the description, but this requires further analysis of the possible PV_E submodules of F'.

Note that we have not assumed that A is a Δ field, only that A^Δ is an algebraically closed field. This is of particular interest when A is the local ring of a closed point on a smooth variety X over a field k with derivations $\Delta = \{\partial/\partial x_i\}$ where $\mathfrak{m}_A = (x_i)$.

2.2. **The case C is a domain.** Here there should be a nascent theory comparable to ramification theory for a domain R with quotient field K and a Galois extension L/K. Following Kovacic we consider the case where $C = A^\Delta \to A$ is an almost constant extension or, equivalently, where the set of prime ideals of C equals the set of prime differential ideals of A ...