

COHOMOLOGY AND DIFFERENTIAL SCHEMES

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Dedicated to the memory of Jerrold Kovacic

ABSTRACT. Replace this text with your own abstract.

1. Δ -SCHEMES

This section assembles basic results on Δ -schemes that are needed to set up a Grothendieck topology and show that the resulting cohomology agrees with Kolchin's constrained cohomology for differential varieties. Most of these results appear in various papers of Kovacic [2], [3] and others. All rings contain \mathbb{Q} and are commutative. We fix a differential Δ ring A throughout this section.

1.1. The topological space. Let A be a commutative Δ ring. We define

$$\text{Spec}^\Delta(A) = \{\mathfrak{p}/\mathfrak{p} \text{ is a differential prime ideal in } A\}.$$

We introduce the Kolchin topology on $X = \text{Spec}^\Delta(A)$ by introducing, for $f \in A$,

$$U_f = \{\mathfrak{p} \in X / f \notin \mathfrak{p}\}.$$

as a basis for the open sets. This means that the closed sets are of the form $V(I) = \{\mathfrak{p}/\mathfrak{p} \supseteq I \text{ for } I \subseteq A\}$. The ideal of functions vanishing on a set $S \subseteq X$ is then a radical Δ -ideal that we denote $\mathcal{I}(S)$. $\mathcal{I}(V(I)) = [I]$ and $V(\mathcal{I}(S)) = \overline{S}$, the closure of S in X . Points in X will be written with roman letters x, \dots when considered as points of a topological space and as \mathfrak{p}_x if we need to identify the corresponding prime ideals.

1.2. The structure sheaf and sheaves of quasi-coherent modules. Recall that a presheaf of abelian groups \mathcal{F} on X is a contravariant functor on the category of open sets of X with inclusion maps as morphisms. A presheaf is a sheaf on X if for every open set U and all coverings $\{U_\alpha\}_{\alpha \in I}$ of U by open sets, two conditions are satisfied:

- S1: if $x \in \mathcal{F}(U)$ and $x|_{U_\alpha} = 0$ for all $\alpha \in I$, then $x = 0$ and
- S2: if there are $x_\alpha \in \mathcal{F}(U_\alpha)$ such that $x_\alpha|_{U_\alpha \cap U_\beta} = x_\beta|_{U_\alpha \cap U_\beta}$, then there is an $x \in \mathcal{F}(U)$ with $x|_{U_\alpha} = x_\alpha$ for all $\alpha \in I$

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We say that a presheaf \mathcal{F} is separated if S1 is satisfied for any U and any covering $\{U_\alpha\}$ of U . In general it suffices to define a sheaf only on a basis for the topology since then the two sheaf conditions determine $\mathcal{F}(U)$ if $U = \cup U_\alpha$ where $\{U_\alpha\}$ is contained in a basis for the topology.

We would like to define a presheaf \tilde{A} by setting $\tilde{A}(U_f) = A_f$ on basic open sets. However the existence of differential units that are not units poses a problem as does the existence of differential zeros that are not zero. So we begin by defining them.

Definition 1. *Let M be a (differential) A module. $m \in M$ is a differential zero of M if $\frac{m}{1} = 0 \in A_x$ for all $x \in \text{Spec}^\Delta(A)$. $\mathcal{Z}(M)$ denotes the set of all differential zeros of M .*

Definition 2. *$a \in A$ is a differential unit if $a \notin \mathfrak{p}_x$ for any $x \in \text{Spec}^\Delta(A)$. $\Delta(A^\times)$ is then the set of all differential units of A and is multiplicatively closed.*

Observe that if we want a structure sheaf whose stalks at differential primes x are A_x , then differential units will be units in the sheaf and differential zeros will be zero in the sheaf since these statements are local statements aside from a patching condition. In the differential unit case, the multiplicative inverse is unique if it exists and so the patching condition will always be satisfied for the inverse of a differential unit. Note also that a differential semi-local ring A has no non zero differential zeros and no differential units that aren't units by considering the embedding $A \rightarrow \prod A_{\mathfrak{m}_i}$ where the product is over its localizations at all maximal ideals \mathfrak{m}_i .

The differential units that are not units cause major difficulties. For instance let k be your favorite Δ field and consider the Δ ring $A := k\{X\}$ in one differential indeterminate. Then $U_{\delta X} \subset U_X$ and so there should be a restriction map, that is a differential homomorphism

$$A[X^{-1}] \rightarrow A[(\delta X)^{-1}].$$

But X is not a unit in $A[(\delta X)^{-1}]$, only a differential unit, and so there is no canonical homomorphism. Thus, to define a presheaf, we must first formally invert the differential units. So $\tilde{A}_X(U_f) := \left(\Delta(A_f^\times)\right)^{-1} A_f$ defines a presheaf of differential rings on a basis for the Kolchin topology on X .

Traditionally the Δ -structure sheaf \tilde{A}^Δ has been defined by requiring that $\tilde{A}_x^\Delta = A_x$ for all $x \in X$. Then a section $s \in \tilde{A}^\Delta(X)$ is specified by its value $s(x) \in A_x$ for all $x \in X$ subject to the condition that there is a Kolchin covering $\{U_{f_i}\}$ and elements $\frac{a_i}{b_i} \in A[f_i^{-1}]$ with $\frac{a_i}{b_i}(x) = s(x)$ for all $x \in U_{f_i}$. Alternatively we can proceed by introducing a sheafification functor as follows. Let $U \subset X$ be a Kolchin open set in X . Choose a Kolchin cover $\mathcal{U} = \{U_i\}_{i \in I}$ of U where $U_i = U_{f_i}$. Then, for each $(i, j) \in I^2$, choose a covering $\mathcal{V}_{i,j} = \left\{V_k^{ij}\right\}_{k \in I_{i,j}}$ of $U_i \cap U_j$ where $V_k^{ij} = U_{g_k}$. Set

$$\tilde{A}^\Delta(\mathcal{U}, \mathcal{V}) = \text{Ker} \left(\prod_I \tilde{A}_X(U_{f_i}) \rightarrow \prod_{I^2} \prod_{I_{i,j}} \tilde{A}_X(U_{g_k}) \right)$$

where the map sends $\prod s_i$ to $\prod \left(s_i |_{V_k^{ij}} - s_j |_{V_k^{ij}} \right)$ and define

$$\tilde{A}^\Delta(U) := \varinjlim \tilde{A}^\Delta(\mathcal{U}, \mathcal{V}).$$

Here the limit is taken in the usual Čech sense over maps $(\mathcal{U}, \mathcal{V}) \rightarrow (\mathcal{U}', \mathcal{V}')$ consisting of tuples of maps $\Phi : I \rightarrow I'$ and, for each $i, j \in I^2$, $\phi_{i,j} : I_{i,j} \rightarrow I'_{\Phi(i)\Phi(j)}$ such that $U_i \supseteq U_{\Phi(i)}$ and $V_k^{ij} \supseteq V_{\phi_{i,j}(k)}^{\Phi(i)\Phi(j)}$. This should be viewed as choosing a covering \mathcal{U} of U and representing a section s of the sheaf $\tilde{A}^\Delta(U)$ by sections $s_i \in \tilde{A}_X(U_{f_i})$ such that $s_i |_{U_{f_i f_j}} - s_j |_{U_{f_i f_j}} \in \tilde{A}_X(U_{f_i f_j})$ is a differential zero and so becomes 0 on a covering \mathcal{V}_{ij} of $U_{f_i f_j}$. Passing to the limit then makes the representation unique. A moment's thought shows that \tilde{A}^Δ agrees with the definition in Hartshorne.

Clearly the same prescription describes the sheaf \tilde{M}^Δ of \tilde{A}^Δ modules with stalks $\tilde{M}_x^\Delta = M_x$ for any (differential) A module M .

Following [2], let $\hat{A} = \Gamma(X, \tilde{A}^\Delta)$ and $i_A : A \rightarrow \hat{A}$ be the corresponding Δ homomorphism with similar notation for a (differential) module M . Recall that i_M is said to be almost surjective if for any $x \in X$ and $m \in \hat{M}$, there are $a \in A - \mathfrak{p}_x$ and $m' \in M$ such that

$$i_A(a)m = i_M(m').$$

Proposition 1. *Let A be a Δ ring and M an A module, possibly differential. Then i_M is almost surjective.*

Proof. Let $x_0 \in X$ and $m \in \hat{M}$. Note that $U_{v f_i} \supseteq U_{f_i}$ if v is a differential unit in A_{f_i} . Then, for any $x \in X$, m can be represented by a fraction $\frac{m_i}{f_x}$ in the semi-local ring A_{x, x_0} with $f_x \in A - \mathfrak{p}_{x_0} \cup \mathfrak{p}_x$. Hence we may represent m as $\left\{ \frac{m_i}{f_i} \right\}$ relative to a covering $\mathcal{U} = \{U_{f_i}\}$ of X where $x_0 \in U_{f_i}$ for all $i \in I$. We may also assume the index set I is finite since $1 \in \{f_i\}$ requires only a finite number of f_i . Fix one of these opens, say U_0 , containing x_0 . Then, on $U_{f_i} \cap U_{f_j} = U_{f_i f_j}$,

$$m_i f_j - m_j f_i \in \mathcal{Z}(A_{f_i f_j}).$$

Since $\mathcal{Z}(A_{x, x_0}) = 0$, we can find, for each $i, j \in I^2$, a finite covering $\mathcal{V}_{i,j} = \left\{ V_k^{ij} \right\}_{k \in I_{i,j}}$ of $U_i \cap U_j$ with $V_k^{ij} = U_{g_k^{ij}}$ for some $g_k^{ij} \in A - \mathfrak{p}_{x_0}$ such that

$$\left(g_k^{i,j} \right)^N m_i f_j = \left(g_k^{i,j} \right)^N m_j f_i \in A_{f_i f_j}$$

for some fixed N and all i, j, k . Let $g = \prod_{I^2} \prod_{I_{i,j}} \left(g_k^{i,j} \right)^N$ so that $g m_i f_j = g m_j f_i$ in

$A_{f_i f_j}$ for all i, j . Let $a = g \prod_I f_i$. Then $a \notin \mathfrak{p}_{x_0}$ and $i_A(a)m = i_M \left(\left(g \prod_{k \in I - \{0\}} f_k \right) m_0 \right) \in$

\hat{M} since

$$\left(g \prod_I f_i \right) \left(\frac{m_0}{f_0} \right) = \left(g \prod_{k \in I - \{0\}} f_k \right) m_0 = \left(g \prod_{k \in I - \{i\}} f_k \right) m_i \in A_{f_0 f_i}.$$

□

We define a Δ ring A (or module M) to be entire if $i_A : A \rightarrow \widehat{A}$ ($i_M : M \rightarrow \widehat{M}$) is an isomorphism. The almost surjectivity condition is sufficient to guarantee that \widehat{A} (resp. \widehat{M}) is entire [1]. We outline a proof here since the manuscript is no longer available.

Proposition 2. *Let A be a Δ ring. Then i_A (resp. i_M) is almost surjective if and only if \widehat{A} (resp. \widehat{M}) is entire.*

Proof. It suffices to show that $i_A : A \rightarrow \widehat{A}$ induces a homeomorphism $i_A^\# : \widehat{X} := \text{Spec}^\Delta(\widehat{A}) \rightarrow X$ and that for all $\widehat{x} \in \widehat{X}$, $i_x : A_x \rightarrow \widehat{A}_{\widehat{x}}$ is an isomorphism if $x = i_A^\#(\widehat{x})$.

If \mathfrak{p} is a prime Δ ideal in A , let

$$\widehat{\mathfrak{p}} = \left\{ s \in \widehat{A} \mid s(\mathfrak{p}) \in \mathfrak{p}A_{\mathfrak{p}} \right\}.$$

$\widehat{\mathfrak{p}}$ is clearly a prime Δ ideal in \widehat{A} with $i_A^{-1}(\widehat{\mathfrak{p}}) = \mathfrak{p}$. If there are two prime Δ ideals, $\widehat{\mathfrak{p}}$ and $\widehat{\mathfrak{p}'}$, in \widehat{A} both lying over \mathfrak{p} , there is an $\widehat{f} \in \widehat{\mathfrak{p}} - \widehat{\mathfrak{p}'}$ and a pair $a \in A - \mathfrak{p}$ and $f \in A$ with $i_A(a)\widehat{f} = i_A(f)$. But then \widehat{f} must be in both \mathfrak{p} and \mathfrak{p}' which is impossible. Consequently $i_A^\#$ is a continuous, one-to-one and onto map. Finally we show $i_A^\#$ is open. Suppose $U_{\widehat{f}}$ is a basic open in \widehat{X} . If $\widehat{x} \in U_{\widehat{f}}$ and $\mathfrak{p} = i_A^{-1}(\mathfrak{p}_{\widehat{x}}) = i_A^\#(\widehat{x})$, we know that $\widehat{f} \notin \widehat{\mathfrak{p}}$. Locating $a \in A - \mathfrak{p}$ and $f \in A$ with $i_A(a)\widehat{f} = i_A(f)$ means that $f \notin \mathfrak{p}$ and so the neighborhood $U_f \cap U_a$ of \mathfrak{p} is contained in $i_A^\#(U_{\widehat{f}})$.

We claim i_A is an isomorphism on stalks. Let $\widehat{f} \in \widehat{A}$. Then there are $a, f \in A$ with $a \notin \mathfrak{p}_x$ such that $i_x\left(\frac{f}{a}\right) = \frac{\widehat{f}}{1} \in \widehat{A}_{\widehat{x}}$ and the surjectivity of i_x now follows immediately. Next suppose $i_x\left(\frac{g}{1}\right) = 0 \in \widehat{A}_{\widehat{x}}$, then there is a $\widehat{f} \in \widehat{A} - \mathfrak{p}_{\widehat{x}}$ such that $\widehat{f}i_A(g) = 0$. But then $0 = i_A(ag)\widehat{f} = i_A(gf)$ which means that $gf \in \mathcal{Z}(A)$. Since $\mathcal{Z}(A_x) = 0$, $\frac{g}{1} = 0 \in A_x$.

A similar argument establishes the result for (differential) modules. \square

Corollary 1. *$i_A^\# : \widehat{X} \rightarrow X$ is a homeomorphism, and $i_{\widehat{A}} : \widehat{A} \rightarrow \widehat{\widehat{A}}$ and $i_{\widehat{M}} : \widehat{M} \rightarrow \widehat{\widehat{M}}$ are Δ isomorphisms.*

Be careful. The fact that $\{U_{f_i}\}$ is a covering of X means that the differential radical ideal $I = [f_i]$ contains 1, but this does not allow a representation of an element $s \in \widehat{A}$ as $\frac{a}{b}$ with a common denominator b . In fact Kovacic [3] has given the details of an example, due to Kolchin, of a Δ integral domain A and such an element $s \in \widehat{A}$. This is caused by having to differentiate generators of I to express 1 in terms of these generators.

This corollary allows us to define affine and differential schemes in the following manner.

Definition 3. *An affine Δ scheme is a local Δ ringed space (X, \mathcal{O}_X) such that $(X, \mathcal{O}_X) \rightarrow \left(\text{Spec}^\Delta(\Gamma(X, \mathcal{O}_X)), \widetilde{\Gamma(X, \mathcal{O}_X)}^\Delta \right)$ is an isomorphism.*

Definition 4. *A Δ scheme is a local Δ ringed space (X, \mathcal{O}_X) which has an open covering by affine Δ schemes. Morphisms of (affine) Δ schemes consist of morphisms of local ringed spaces that preserve the action of Δ .*

Note that in view of the corollary above, an affine Δ scheme is, up to isomorphism, a local Δ ringed space $(\text{Spec}^\Delta(A), \tilde{A}^\Delta)$ for some Δ ring A . Structure sheaves of Δ schemes will be written as $\mathcal{O}_{X,\Delta}$ to distinguish them from structure sheaves of schemes which will be written as \mathcal{O}_X if there is a possibility of confusion.

1.3. Basic properties. Benoist (see also ()) now goes on to establish the fundamental adjointness, affine product, and product theorems for differential schemes. For the convenience of the reader and to set notation, we outline these results here letting $((\Delta \text{ rings}))$, $((\text{Entire } \Delta \text{ rings}))$, and $((\Delta \text{ schemes}))$ denote the categories of commutative Δ rings, entire commutative Δ rings, and Δ schemes respectively. We begin though with the result, observed many times, that identifies a procedure for passing from $((\text{Rings}))$ to $((\Delta \text{ rings}))$ and its properties.

Proposition 3. *There is an adjoint pair of functors*

$$\Delta : ((\text{Rings})) \rightarrow ((\Delta \text{ rings})) : F$$

where F is the forgetful functor and $\Delta(R)$ is the ring R with formally added derivatives so that $\text{Hom}_{\text{Ring}}(R, FA) \cong \text{Hom}_{\Delta \text{ ring}}(\Delta(R), A)$.

Proof. $\Delta(R) = R\{X_r\}_{r \in R} / [r - X_r]$. Then a ring homomorphism $\phi : R \rightarrow A$ extends to a Δ homomorphism $\Delta(\phi) : \Delta(R) \rightarrow A$ by sending the differential indeterminate X_r to $\phi(r) \in A$ and the various derivatives of X_r to the corresponding derivatives of $\phi(r)$. \square

Next we use a formally similar procedure to pass to entire Δ rings.

Proposition 4. *There is an adjoint pair of functors*

$$\hat{} : ((\Delta \text{ rings})) \rightarrow ((\text{Entire } \Delta \text{ rings})) : F$$

where F is the forgetful functor. In particular the category $((\text{Entire } \Delta \text{ rings}))$ has pushout diagrams given by

$$\begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ B' & \rightarrow & \widehat{B \otimes_A B'} \end{array}$$

and $B \widehat{\otimes}_A B' \rightarrow \widehat{B \otimes_A B'}$ is an isomorphism.

Proof. Since \hat{A} is entire, $i_A : A \rightarrow \hat{A}$ is universal for Δ homomorphisms from A to entire Δ rings. Alternatively this is the statement

$$\text{Hom}_{\Delta \text{ ring}}(A, FB) \cong \text{Hom}_{\text{Entire } \Delta \text{ ring}}(\hat{A}, B)$$

or that $\hat{}$ is a left adjoint to the forgetful functor. The last statement follows immediately since left adjoints preserve pushouts. \square

Define the functor $\Gamma_\Delta : ((\Delta \text{ schemes})) \rightarrow ((\text{Entire } \Delta \text{ rings}))$ to be the global sections functor on the category of Δ schemes. This, of course, identifies the category of affine Δ schemes with $((\text{Entire } \Delta \text{ rings}))^{op}$ and guarantees that the category of affine Δ schemes has pullback diagrams (= fibred products).

Proposition 5. *There is an adjoint pair of functors*

$$\text{Spec}^\Delta : ((\text{Entire } \Delta \text{ rings}))^{op} \rightarrow ((\Delta \text{ schemes})) : \Gamma_\Delta$$

so that $\text{Hom}_{\text{Entire } \Delta \text{ ring}}(A, \Gamma_\Delta(S, \mathcal{O}_{S,\Delta})) \cong \text{Hom}_{\Delta \text{ scheme}}(\text{Spec}^\Delta(A), S)$.

Proof. We have an adjoint pair of functors $Spec : ((Rings))^{op} \rightarrow ((Schemes)) : \Gamma$ and $((Entire \Delta \text{ rings}))$ is a subcategory of $((Rings))$. \square

The usual argument then applies to give pullback diagrams (= fibred products) in the category of Δ schemes.

Proposition 6. *The pullback $T \times_S T'$ defined from the diagram*

$$\begin{array}{ccc} & T & \\ & \downarrow f & \\ T' & \xrightarrow{f'} & S \end{array} .$$

in $((\Delta \text{ schemes}))$ exists and is covered by affine Δ schemes of the form $Spec^\Delta(\widehat{B \otimes_A B'})$ if $(Spec^\Delta(A), \tilde{A}^\Delta)$ is an affine open Δ scheme in S and $(Spec^\Delta(B), B^\Delta)$ and $(Spec^\Delta(B'), B'^\Delta)$ are affine open Δ schemes in $f^{-1}((Spec^\Delta(A), \tilde{A}^\Delta))$ and $f'^{-1}((Spec^\Delta(A), \tilde{A}^\Delta))$ respectively.

Proof. Follow the proof in Hartshorne [?]. \square

We will adapt scheme language to our setting by saying that a property of a Δ homomorphism $g : B \rightarrow B'$ of entire Δ rings is Δ essential if there are Δ rings A and A' and a Δ ring homomorphism $f : A \rightarrow A'$ and isomorphisms h and h' and a commutative diagram

$$\begin{array}{ccc} \widehat{A} & \xrightarrow{\widehat{f}} & \widehat{A'} \\ \downarrow h & & \downarrow h' \\ B & \xrightarrow{g} & B' \end{array} .$$

For example,

Definition 5. *Let A and B be entire Δ rings. A Δ homomorphism $f : A \rightarrow B$ is said to be of Δ essentially finite type if there are Δ rings A' and B' and a Δ ring homomorphism $f' : A' \rightarrow B'$ which is of Δ finite type and Δ isomorphisms $\alpha : \widehat{A'} \rightarrow A, \beta : \widehat{B'} \rightarrow B$ such that*

$$\begin{array}{ccc} \widehat{A'} & \xrightarrow{\alpha} & A \\ \downarrow \widehat{f'} & & \downarrow f \\ \widehat{B'} & \xrightarrow{\beta} & B \end{array} .$$

commutes. A map of Δ schemes $\phi : X \rightarrow Y$ is said to be locally of Δ essentially finite type if for all $x \in X$ there are Δ affine neighborhoods U, V for $x, f(x)$ respectively such that $\phi^ : \Gamma(V, \mathcal{O}_{Y,\Delta}) \rightarrow \Gamma(U, \mathcal{O}_{X,\Delta})$ is of Δ essentially finite type. A map of Δ schemes is of Δ essentially finite type if there are finite coverings of X, Y by Δ affine Kolchin open sets $\mathcal{U} = \{U_i\}, \mathcal{V} = \{V_i\}$ such that $\phi(U_i) \subseteq V_i$ and $\phi^* : \Gamma(V_i, \mathcal{O}_{Y,\Delta}) \rightarrow \Gamma(U_i, \mathcal{O}_{X,\Delta})$ is of Δ essentially finite type for all i .*

1.4. Chevalley's theorem and applications. Our goal is to show that flat morphisms $f : X \rightarrow Y$ of Δ schemes are open maps. We follow the traditional algebraic geometry approach by first establishing that $f(C)$ is a constructible set in Y for any f and then showing that $f(X)$ is closed under generizations and so is open when f is flat. As before this is mostly definitions and references.

Definition 6. Let $f : X \rightarrow Y$ be a morphism of Δ schemes. f is said to be flat at $x \in X$ if $f^* : \mathcal{O}_{Y,f(x)\Delta} \rightarrow \mathcal{O}_{X,x\Delta}$ is a flat map of local Δ rings. f is said to be flat if f is flat at all $x \in X$. f is said to be Δ faithfully flat if f is flat and surjective.

Thus f^* induces an exact functor $((\mathcal{O}_{Y,\Delta} \text{ modules})) \rightarrow ((\mathcal{O}_{X,\Delta} \text{ modules}))$, also denoted f^* , if and only if f is flat while $f^*(\mathcal{F}) = 0$ forces $\mathcal{F} = 0$ if and only if f is, in addition, Δ faithfully flat. However, in the Δ affine case, this does not necessarily mean that the Δ ring extension is flat or faithfully flat unless the Δ rings involved are entire.

Recall that a Zariski topological space is a noetherian topological space S in which every irreducible closed set has a unique generic point. Constructible sets in S are then those in the smallest subset collection \mathcal{C} that contains all open sets and is closed under finite intersections and complements.

Theorem 1 (Chevalley). Let $f : X \rightarrow Y$ be a morphism of noetherian Δ schemes that is Δ essentially of finite type. If C is a constructible set in X , then $f(C)$ is a constructible set in Y .

Proof. Exercises in Hartshorne [?, Chapter II, Exercises 3.18, 3.19] outline a proof of Chevalley's theorem by topologically reducing it to the assertion that if X and Y are affine and irreducible and $C = X$, then $f(C)$ contains an open set $U \subseteq Y$. Then an algebraic extension theorem is proved to complete the argument. Noetherian Δ schemes are Zariski spaces so the topological reduction is the same, and we only need to establish the extension theorem for entire Δ integral domains. Topologically $\text{Spec}^\Delta(A)$ is homeomorphic to $\text{Spec}^\Delta(\widehat{A})$ and so Kolchin has established the necessary extension theorem that, formulated in our language, is the Lemma below. Here the semiuniversal condition replaces an algebraically closed field condition that is needed to guarantee that any Δ prime ideal in A can be realized as the kernel of a Δ homomorphism $A \rightarrow \mathcal{U}$.

Lemma 1 (Kolchin Basis Theorem). Let B be a Δ integral domain with a Δ subring A such that B is Δ finitely generated over A , and let $b \in B$ be a nonzero element. There exists a nonzero element $a \in A$ such that every Δ homomorphism, $f : A \rightarrow \mathcal{U}$, into a field \mathcal{U} which is a semiuniversal extension of F , the quotient field of $f(A)$, with $f(a) \neq 0 \in \mathcal{U}$ may be extended to a Δ homomorphism $g : B \rightarrow \mathcal{U}$ with $g(b) \neq 0$.

□

Proof. Theorem 3, Chapter III

□

This result is the basis of a variety of assertions since constructible sets that are closed under generization (resp. specialization) are open (resp. closed). The result we are immediately interested in is that Δ flat maps are open.

Proposition 7. Let $f : X \rightarrow Y$ be a Δ flat morphism of noetherian Δ schemes that is Δ essentially of finite type. Then f is open.

Proof. If $\mathcal{O}_{X,x\Delta}$ is flat as an $\mathcal{O}_{Y,f(x)\Delta}$ module then so is any generization of $f(x)$ since a map $f^*(M)_x \rightarrow f^*(N)_x$ which is monic remains so on all generizations of x .

□

2. Δ -FLAT TOPOLOGY

Connections with étale topology and flat topology.

2.0.1. *Principal homogeneous spaces.* Δ -flat descent à la Deligne

Constrainedly closed fields

PHS have sections over constrainedly closed field

Corollary 2. *Over a Δ -field Δ -flat phs are Kolchin phs*

3. KOLCHIN'S THEOREM

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