THE DIFFERENTIAL BRAUER GROUP

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All rings are noetherian, all rings are commutative with 1 unless specified at the beginning of the section; ring extensions are of finite type unless otherwise specified; and the notation $((-))$ stands for the category described by $-$.

0.1. **Review.** We will only consider 'local' topologies. Let $\mathcal{C}/X$ be a category with finite products, i.e. a terminal object X and products of pairs of objects.

Definition 1 (following Artin)**.** *A pretopology $\mathcal{T}$ on $\mathcal{C}/X$ is a set $Cov\,(\mathcal{T})$ consisting of families $(\pi_i : U_i \to U)_{i\in I}$ such that*

1. *If $\pi : V \to U$ is an isomorphism, then $(\pi) \in Cov\,(\mathcal{T})$.*
2. *If $(U_i \to U) \in Cov\,(\mathcal{T})$ and $\left(V_j^i \to U_i\right) \in Cov\,(\mathcal{T})$, then $\left(V_j^i \to U\right) \in Cov\,(\mathcal{T})$.*
3. *If $(U_i \to U) \in Cov\,(\mathcal{T})$ and $V \to U \in Mor\,(\mathcal{C}/X)$, then $U_i \times_U V$ exists and $(U_i \times_U V \to V) \in Cov\,(\mathcal{T})$.*

Such a category with its coverings is called a site. Here are some examples where X is a scheme.

Example 1. *X_{Zar}, the Zariski site on X, has $\mathcal{C}/X$ equal to the category of open subschemes $U \subset X$.*

$$Cov\,\{X_{Zar}\} = \left\{(U_i \subset U)\,/\,^{1)}U_i \text{ is open and } ^{2)}U = \bigcup U_i\right\}$$

Example 2. *X_{et}, the étale site on X, has $\mathcal{C}/X$ equal to the category of all étale schemes $U \to X$.*

$$Cov\,\{X_{et}\} = \left\{(p_i : U_i \to U)\,/\,^{1)}p_i \text{ is étale and } ^{2)}U = \bigcup p_i\,(U_i)\right\}$$

Example 3. *X_{pl}, the flat site on X, has $\mathcal{C}/X$ equal to the category of all schemes $U \to X$ which are locally of finite type and flat over X.*

$$Cov\,\{X_{pl}\} = \left\{(p_i : U_i \to U)\,/\,^{1)}p_i \text{ is flat, locally of finite type and } ^{2)}U = \bigcup p_i\,(U_i)\right\}$$

Let $D = Spec\,(R)$ where R is a differential ring containing $\mathbb{Q}$ with derivation δ. We introduce two new topologies, $D_{\delta-et}$ and $D_{\delta-pl}$ as follows. Let $\mathcal{C}/D_\delta$ be the category of schemes $\pi : X \to D$ such that X is a scheme with a derivation δ_X and π is a differentiation preserving morphism.

Example 4. *$D_{\delta-et}$, the $\delta-$étale site on D, has $\mathcal{C}/D$ equal to the full sub-category of $\mathcal{C}/D_\delta$ consisting of all étale schemes $\pi : U \to D$.*

$$Cov\,\{U_{\delta-et}\} = \left\{(p_i : U_i \to U)\,/\,^{1)}p_i \text{ is étale, } ^{2)}U = \bigcup p_i\,(U_i)\,, \text{ and } ^{3)}p_i \text{ is a differential morphism}\right\}$$

Example 5. *$D_{\delta-pl}$, the $\delta-$flat site on D, has $\mathcal{C}/D$ equal to the full sub-category of $\mathcal{C}/D_\delta$ consisting of all schemes $U \to D$ which are locally of finite type and flat over D.*

$$Cov\left\{U_{\delta-pl}\right\} = \left\{ (p_i : U_i \to U) / \begin{array}{c} {}^{1)}p_i \text{ is flat and locally of finite type, } {}^{2)}U = \bigcup p_i(U_i), \\ \text{and } {}^{3)}p_i \text{ is a differential morphism} \end{array} \right\}$$

Definition 2. *Let $\mathcal{C}/X$ be a site.*

(1) *A presheaf is a contravariant functor $\mathcal{F} : (\mathcal{C}/X)^{op} \to Sets$.*
(2) *A presheaf $\mathcal{F}$ is a sheaf of sets if, for all $\left(U_i \xrightarrow{p_i} U\right) \in Cov(\mathcal{T})$, the following conditions are satisfied:*

S1 *$\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is one-to-one*
S2 *If $(s_i) \in \prod \mathcal{F}(U_i)$ has $pr_1^*(s_i) = pr_2^*(s_j) \in \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \times_U U_j)$, then there is $s \in \mathcal{F}(U)$ such that $p_i^*(s) = s_i$ for all $i \in I$.*

An equivalent formulation of the two sheaf conditions is that $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is the difference kernel of

$$\prod \mathcal{F}(U_i) \begin{array}{c} \xrightarrow{pr_1^*} \\ \xrightarrow{pr_2^*} \end{array} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \times_U U_j)$$

where the two maps pr_i^* are projections onto the i^{th} factor. Consequently we can (and will) use this definition to define sheaves with values in any category $\mathcal{D}$ that has difference kernels. A presheaf satisfying $S1$ is said to be separated.

Examples of sheaves with notation include:

- $W(\mathcal{O}_X) \in X_{pl}$ and $W_\delta(\mathcal{O}_X) \in X_{d-pl}$
- $\mathbb{G}_m \in X_{pl}$ and $\mathbb{G}_{m,\delta} \in X_{d-pl}$ where $\Gamma(U, \mathcal{G}_{m,\delta}) =$ units in $\Gamma(U, \mathcal{O}_U)$
- If O is a constant group, $\Gamma(U, O) := O^{\pi_0(U)}$ defines a sheaf where $\pi_0(U)$ is the set of connected components of the scheme U.
- $Gl_n \in X_{pl}$ and $Gl_{n,\delta} \in X_{\delta-pl}$ where we observe that $End(\mathcal{O}_x^n) \in X_{pl}$ is a sheaf with the usual values on U and the exactness of (**??**) shows that an endomorphism on A that is an automorphism on B must have been an automorphism to begin.
- Presheaves defined from any commutative group scheme ([Milne, 1980, II, Corollary 1.7]) with a subscript δ if they are regarded as sheaves on the site $X_{\delta-et}$ of $X_{\delta-pl}$.
- $S(X_{pl}) :=$ ((sheaves of abelian groups on X_{pl})) is an abelian category with enough injectives.

Definition 3. *Let M be a B module. Descent data for M consists of a $B \otimes B$ isomorphism $\phi : M \otimes B \to B \otimes M$.*

Descent data (M, ϕ) satisfies the cocycle condition if the diagram of isomorphisms

$$\begin{array}{ccc} & & B \otimes M \otimes B \\ & \nearrow \phi_{12} & \\ M \otimes B \otimes B & & \downarrow \phi_{23} \\ & \searrow \phi_{13} & \\ & & B \otimes B \otimes M \end{array}$$

commutes, i.e. $\phi_{23}\phi_{12} = \phi_{13} : M \otimes B \otimes B \to B \otimes B \otimes M$.

Given $A \to B$, the category of descent data for B over A consists of pairs (M, ϕ) where M is a finitely generated B module and ϕ is descent data for M. $Hom\left((M,\phi),(M',\phi')\right)$ consists of B module homomorphisms $f : M \to M'$ such that the diagram

$$\begin{array}{ccc} M \otimes B & \stackrel{\phi}{\to} & B \otimes M \\ \downarrow f \otimes B & & \downarrow B \otimes f \\ M' \otimes B & \stackrel{\phi'}{\to} & B \otimes M' \end{array}$$

commutes.

Theorem 1. *Let $A \to B$ be a faithfully flat ring homomorphism. Then the functor*

$-\otimes B : ((\textit{Finitely generated } A\textit{-modules})) \to ((\textit{Descent data for } B\textit{-modules + cocycle}))$

is an equivalence of categories.

Definition 4. *Given a faithfully flat ring extension $A \to B$ and an A module F, define*

$$Z^1(B/A, Aut(F)) := \{\phi \in Aut(F \otimes B \otimes B) \mid \ \phi_{23}\phi_{12} = \phi_{13}\}$$

Let $\phi, \sigma \in Z^1(B/A, Aut(F))$. Then $\phi \sim \sigma$ if there is $f : F \otimes B \to F \otimes B \in Aut(F \otimes B)$ such that

$$\sigma = f_2^{-1}\phi f_1.$$

Definition 5.

$$H^1(B/A, Aut(F)) := Z^1(B/A, Aut(F)) / \sim .$$

The following result is then a direct consequence of 2.

Theorem 2. *Let $A \to B$ be a faithfully flat ring homomorphism. Then the functor*

$-\otimes B : ((\textit{Finitely generated } A\textit{-modules})) \to ((\textit{Descent data for } B\textit{-modules + cocycle}))$

is an equivalence of categories.

Theorem 3. *Let B be a faithfully flat A algebra, F an A module. Then there is a natural isomorphism of pointed sets*

$$H^1(B/A, Aut(F)) \approxeq \left\{ N \mid \begin{array}{c} N \textit{ is an } A \textit{ module and there is} \\ a\ B \textit{ module isomorphism } F \otimes B \approxeq N \otimes B \end{array} \right\}.$$

If F happens to have additional structure given by a tensor such as multiplication, then $N \in H^1(B/A, Aut(F))$ will have such a structure also since it is described as a kernel.

We get rid of the dependence on a single covering in the site X_* by observing that for a given $U = Spec(B) \in \mathcal{C}/X$, $Cov(U)$ becomes a directed set and so we define

$$H^1(B_*, Aut(F)) = \lim_{(U_i \to U) \in Cov(U)} H^1(B_i/B, Aut(F))$$

where $U_i = Spec(B_i)$. (The general definition applies, of course, to arbitrary schemes in X_* and coverings of $U \in X_*$.) This results in a better version of Theorem 3 as follows.

Theorem 4. *Let $Spec(A)_*$ be in a site for one of our Grothendieck topologies, and let F be an A module. Then there is a natural isomorpism of pointed sets*

$$H^1(A_*, Aut(F)) \approxeq \left\{ \begin{array}{c} N \mid N \textit{ is an } A \textit{ module and there is a covering} \\ Spec(B) \to Spec(A) \in Cov(Spec(A)) \textit{ and a } B \textit{ module isomorphism } F \otimes B \approxeq N \otimes B \end{array} \right\}$$

Thus attention is focuissed on the set of objects that are 'locally' isomorphic to F in the $* = Zar$, et, pl, $\delta - et$, or $\delta - pl$ topology.

Example 6. (1) $F = A$, $Aut(F) = G_m$, *and*

$$H^1(A, Aut(F)) = H^1(A_*, G_m) = \{L \mid L \otimes B \approxeq B \text{ for some faithfully flat } A \text{ algebra } B \text{ of finite type over } A\} =$$

(2) $F = (A, \delta)$, $Aut(F) = G_m^\delta$, *and*

$$H^1(A_{\delta-*}, Aut(F)) = H^1\left(A_{\delta-*}, G_m^\delta\right) = \{L \mid L \otimes B \approxeq B \text{ as differential modules where } B \text{ is a covering in the } \delta$$

(3) $F = A^{\oplus n}$, $Aut(F) = Gl_n$, *and*

$$H^1(A, Aut(F)) = H^1(A_*, Gl_n) = \{P \mid P \otimes B \text{ is free of rank } n\}$$

(4) $F = (A^{\oplus n}, \delta)$, $Aut(F) = Gl_n^\delta$, *where* δ *is differentiation from* A *in coefficients relative to the given basis of* F *and*

$$H^1(A, Aut(F)) = H^1\left(A_{\delta-*}, Gl_n^\delta\right) = \{P \mid P \otimes B \text{ is free of rank } n \text{ as differential modules where } B \text{ is a covering}$$

Here we note that automorphisms of a free differential module of rank n *are elements of* Gl_n *that commute with the derivation and so are denoted* Gl_n^δ.

(5) $F = M_n(A)$, $Aut(F) := PGl_n$, *and*

$$H^1(A, Aut(F)) = H^1(A_*, PGl_n) = \{\Lambda \mid \Lambda \otimes B \approxeq M_n(B) \text{ for some covering } A \to B \text{ in the } * \text{ topology}\}$$

In this case, the isomorphism is a B *algebra isomorphism since it is easy to see that the descended module, which is* Λ, *is closed under multiplication.*

(6) $F = (M_n(A), ')$, $Aut(F) := Aut(M_n, ')$ *consists of differential automorphisms of the ring of matrices with coordinatewise differentiation and*

$$\begin{aligned} H^1(A_{\delta-pl}, Aut(M_n))(&= H^1(A_{\delta-pl}, PGl_n)) \\ &= \left\{\Lambda \mid \begin{array}{c} \Lambda \otimes B \approxeq M_n(B) \text{ as differential algebras where} \\ B \text{ is } \delta\text{-flat and } M_n(B) \text{ has coordinatewise differentiation} \end{array}\right\} \\ &= \left\{\Lambda \mid \Lambda \text{ is a differential Azumaya algebra of rank } n^2 \text{ over } A\right\} \end{aligned}$$

0.2. **The Differential Brauer Group.** Fix a commutative differential ring A with derivation δ. All tensor products and endomorphism rings will be taken over A unless otherwise specified. For simplicity we assume that $Spec(A)$ is connected, i.e. A has no idempotents other than 1. Let $Az_\delta(A)$ denote the set of differential Azumaya algebras over A, and let $\mathcal{P}_\delta(A)$ denote the set of all locally free A modules, i.e. all projective A modules. We introduce the usual differential structures on $P, Q \in \mathcal{P}_\delta(A)$ by defining a derivation $\delta^\vee : P^\vee \to P^\vee$ with the rule

$$\begin{aligned} \left(\delta^\vee f\right)(p) &= \delta_A(f(p)) - f(\delta_P p), \\ \delta_\otimes(p \otimes q) &= \delta_P(p) \otimes q + p \otimes \delta_Q(q), \text{ and} \\ \delta_{End(P)}(\alpha(p \otimes f))(p') &= \alpha\left(p \otimes \delta^\vee f + \delta p \otimes f\right)(p') \\ &= (\delta(f(p')) - f(\delta p'))\, p + f(p')(\delta p) \end{aligned}$$

where, in the last definition, $\alpha : P \otimes_A P^\vee \to End_A(P)$ is the isomorphism given by $\alpha(p \otimes f)(p') = pf(p')$ $(= f(p')\,p$ since A is commutative) and the derivation is defined from the $\delta_\otimes$ and $\delta^\vee$.

Definition 6. *For any pair* $\Lambda, \Gamma \in Az_\delta(A)$, $\Lambda \sim \Gamma$ *if there are* $P, Q \in \mathcal{P}_\delta(A)$ *and a differential* A *alghebra isomorphism*

$$\Lambda \otimes End(P) \approxeq \Gamma \otimes End(Q).$$

It is straightforward to check that this is an equivalence relation on $Az_\delta(A)$ which exactly mimics the definition used for non-differential commutative rings. We define $Br_\delta(A)$ to be the set of equivalence classes and indicate the equivalence class of Λ with $[\Lambda]$. The following Lemma shows that $Br_\delta(A)$ is a group with respect to the operation given by $\otimes$.

Lemma 1. *If Λ is a differential Azumaya algebra, the natural map $\phi : \Lambda \otimes \Lambda^{op} \to End(\Lambda)$ given by $\phi(\lambda' \otimes \lambda'')(\lambda) = \lambda'\lambda\lambda''$ is a differential A algebra isomorphism.*

Proof. It is well known that this is an A algebra isomorphism. We need only check that it is differential. Unfortunately it is not easy to compare δ on $\Lambda \otimes \Lambda^{op}$ and on $End(\Lambda)$ since the latter differentiation comes from the isomorphism $\alpha : \Lambda \otimes \Lambda^\vee \to End(\Lambda)$. However there is a faithfully flat ring extension $A \to B$ and a differential B algebra isomorphism $\Lambda \otimes B \approxeq M_n(B)$. Consequently it is enough to check the isomorphism on $\phi : M_n(B) \otimes M_n(B)^{op} \to End(M_n(B))$. Here it is easier to make the necessary identifications. (All tensor products are now over B.) Recall that e_{ij} stands for the matrix entry with a 1 in the i^{th} row and j^{th} column and $)'s$ elsewhere. e_{ij} and $e_{ij}^\vee$ form a basis for $M_n(B)$ and $(M_n(B))^\vee$ respectively. So if $e_{(ij)(kl)} \in End(M_n(B))$ is the B module generator defined by

$$e_{(ij)(kl)}(e_{ab}) = \begin{cases} 0 \text{ if } i \neq a \text{ or } j \neq b \\ e_{kl} \text{ if } i = a \text{ and } j = b \end{cases},$$

then

$$\left(\alpha_{ki}e_{ki} \otimes a_{lj}e_{lj}^{tr}\right)(a_{ij}e_{ij}) = a_{ki}e_{ki}a_{ij}a_{lj}e_{jl} = a_{ki}a_{ij}a_{lj}e_{kl}$$

where Λ^{op} is represented by the transpose of the matrix in case $\Lambda = M_n(B)$. But representing $End(M_n(B))$ as $M_n(B) \otimes M_n(B)^\vee$ gives a representation for this map as

$$\left(a_{ki}e_{kl} \otimes a_{lj}e_{ij}^\vee\right)(a_{ij}e_{ij}) = a_{ki}a_{lj}a_{ij}e_{kl}.$$

clearly the value of either map at e_{mn} is 0 unless $m = i$ and $n = j$. Thus we have our representation of $a_{ki}a_{lj}a_{ij}e_{(ij)(kl)}$ and we need only check that coordinatewise differentiation of the first representation matches δ_{End} for the second representation. But

$$\begin{aligned} \alpha\left(\delta(\alpha_{ki}e_{ki}) \otimes a_{lj}e_{lj}^{tr} + \alpha_{ki}e_{ki} \otimes \delta\left(a_{lj}e_{lj}^{tr}\right)\right)(a_{ij}e_{ij}) &= (\delta a_{ki})\, e_{ki}a_{ij}e_{ij}a_{lj}e_{jl} + a_{ki}e_{ki}a_{ij}e_{ij}(\delta a_{lj})\, e_{jl} \\ &= ((\delta a_{ki})\, a_{ij}a_{lj} + a_{ki}a_{ij}(\delta a_{lj}))\, e_{kl} \end{aligned}$$

and

$$\begin{aligned} \delta\left(\phi\left(\alpha_{ki}e_{ki} \otimes a_{lj}e_{lj}^{tr}\right)\right)(a_{ij}e_{ij}) &= \delta\left(a_{ki}e_{kl} \otimes a_{lj}e_{ij}^\vee\right)(a_{ij}e_{ij}) \\ &= (\delta(a_{lj}a_{ij}) - a_{lj}(\delta a_{ij}))\, a_{ki}e_{kl} + a_{lj}a_{ij}(\delta a_{ki})\, e_{kl} \\ &= (\delta a_{lj})\, a_{ij}a_{ki} + a_{lj}a_{ij}(\delta a_{ki}) \end{aligned}$$

and so the map is a differential isomorphism. Since B is A faithfully flat, $\phi : \Lambda \otimes \Lambda^{op} \to End(\Lambda)$is also a differential isomorphism as claimed. □

Now it is well known that $Br(A) \approxeq H^2(A_{pl}, 3G_m)$ if A is regular. (Otherwise $Br(A) \approxeq_{tors} H^2(A_{et}, 3G_m)$.) Our main result is that in this case $Br_\delta(A) = Br(A)$. Since we already know that any Azumaya algebra Λ over A has a derivation compatible with that of A, this assertion amounts to showing that the forget the derivation functor produces a monomorphism $Br_\delta(A) \to Br(A)$. We do this by constructing an injection $Br_\delta(A) \to H^2(A_{\delta-pl}, \mathbb{G}_m)$ and then using cohomology to finish the argument. As a corollary we show that $Br(A) = Br\left(A^\delta\right)$.

0.3. **Sheaves in the $\delta - pl$ site.** In order to use the power of sheaf theory and fpqc descent we need to explicitly identify some sheaves and introduce some notation.

In this section we work with $X_{\delta-pl}$ where $X = Spec\,(A)$ and A has a derivation denoted δ. We are interested in six sheaves for this site, most of which have alredy been identified above, but let's make them explicit. Note that the subscript δ indicates that we are dealing with δ quantities and the superscript δ indicates that we are dealing with δ invariant quantities, i.e constants for δ.

(1) $Aut\,(W\,(\mathcal{O})) = \mathbb{G}_m$ is the sheaf of automorphisms of the structure sheaf viewed as a rank 1 free module on the site $Spec\,(A)_{\delta-pl}$.

$Aut_\delta\,(W\,(\mathcal{O})) = \mathbb{G}_{m,\delta} = \mathbb{G}_m^\delta$ is the sheaf of differential automorphisms of the structure sheaf viewed as a rank 1 free module on the site $Spec\,(A)_{\delta-pl}$. Note that any differential automorphism of the rank 1 free δ module must be given by multiplication by a constant unit. Consequently $\mathbb{G}_{m,\delta} = \mathbb{G}_m^\delta$.

(2) $Aut\,(W\,(\mathcal{O}^{\oplus n})) = Gl_n$ is the sheaf of automorphisms of the free module of rank n with basis $\mathcal{B} = \{e_1, \ldots, e_n\}$

$Aut_\delta\,(W\,(\mathcal{O}^{\oplus n})) = Gl_{n,\delta} = Gl_n^\delta$ is the sheaf of differential automorphisms of the free module of rank n with basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ with coordinate differentiation denoted $'$. Note that any differential automorphism of $W\,(A^{\oplus n})$, the free δ module with coordinate wise differentiation, must send the basis $\mathcal{B}$ which consists of constants to a basis consisting of independant constant vectors. Consequently it can be described by its action on constant vectors and so $Gl_{n,\delta} = Gl_n^\delta$.

(3) $Aut\,(M_n) = PGl_n$ is the sheaf of algebra automorphisms of M_n where

$$0 \to \mathbb{G}_m \xrightarrow{i} Gl_n \xrightarrow{j} PGl_n \to 1.$$

Here $i\,(u) = \begin{pmatrix} u & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u \end{pmatrix}$ is the diagonal embedding of a unit into an $n \times n$ matrix and the sequence is a central extension of sheaves on $X_{\delta-pl}$. Note that I use a 1 to indicate a non-abelian sheaf and a 0 for an abelian sheaf in describing a short exact sequence of sheaves of groups. Thus PGl_n is defined to be the quotient sheaf of groups Gl_n/G_m, and we must observe that the Skolem-Noether theorem says that over a local ring, any automorphism of M_n is given by conjugation by a unit. Since sheaf exactness is a local property, this establishes our claim.

$Aut_\delta\,(M_n) = PGl_{n,\delta} = PGl_n^\delta$ is the sheaf of differential algebra automorphisms of M_n where

$$0 \to \mathbb{G}_{m,\delta} \xrightarrow{i} Gl_{n,\delta} \xrightarrow{j} PGl_{n,\delta} \to 1$$

and the map i is defined as above. The fact that a differential automorphism of $M_n\,(B)$ is given by conjugation by a unit $u \in M_n\,(B)$ if B is a semi-local ring follows in exactly the same way. However a corollary of the Morita Theorems given last time shows that after taking a $\delta - pl$ refinement of $B \to B'$, $u^{-1}\delta u \in B'$. Consequently the quotient sheaf $PGl_{n,\delta} = Aut_\delta\,(M_n)$, and, even more, $Aut_\delta\,(M_n) = Aut\left((M_n)^\delta\right) = PGl_n^\delta$.

Note that in each of these examples there is a sheaf consisting of automorphisms that are not required to preserve the differentiation. These three sheaves are really sheaves on the site X_{pl} but are being regarded as sheaves on $X_{\delta-pl}$.

Our identifications of these automorphisms sheaves allow us to use the basic fpqc descent result and non-abelian cohomology to understand the situation better. The connection with the Brauer group is through the following result which we quote.

Theorem 5. *Let X_* be a site. Suppose that*

$$0 \to A \to H \to G \to 1$$

is a central extension of sheaves of groups for X_*. Then, for any $U \in \mathcal{C}/X$, there is a 7 term exact sequence of groups and pointed sets

$$\begin{aligned} 0 \quad &\to \quad H^0(U,A) \to H^0(U,H) \to H^0(U,G) \to \\ H^1(U,A) \quad &\to \quad H^1(U,H) \to H^1(U,G) \xrightarrow{\partial} H^2(U,A). \end{aligned}$$

We apply this to the central extensions 3 and 3 to get boundary maps

$$H^1(A_{\delta-pl}, PGl_{n,\delta}) \xrightarrow{\partial} H^2(A_{\delta-pl},, \mathbb{G}_{m,\delta})$$

and

$$H^1(A_{\delta-pl}, PGl_n) \xrightarrow{\partial} H^2(A_{\delta-pl},, \mathbb{G}_m).$$

This defines maps

$$H^1(A_{\delta-pl}, PGl_{n,\delta}) = \left\{\Lambda \mid \Lambda \text{ is a differential Azumaya } A \text{ algebra of rank } n^2\right\} \xrightarrow{\partial_n} H^2(A_{\delta-pl}, \mathbb{G}_{m,\delta})$$

and

$$H^1(A_{\delta-pl}, PGl_n) = \left\{\Lambda \mid \Lambda \text{ is an Azumaya } A \text{ algebra of rank } n^2\right\} \xrightarrow{\partial_n} H^2(A_{\delta-pl}, \mathbb{G}_m).$$

The key to our result is to repeat what is well known for the usual Brauer group.

Theorem 6. *Let (A,δ) be a differential ring. Then the boundary maps ∂_n as n varies over the set $\mathbb{N}$ ordered by factorization defines an embedding $Br_\delta(A) \overset{\partial}{\hookrightarrow} H^2(A_{\delta-pl}, \mathbb{G}_{m,\delta})$.*

Sketch of proof. We first calculate ∂_n explitly. We start with the differential sequence 3 and a cocycle $\sigma \in Z^1(B/A, PGl_n)$ for some $\delta - pl$ covering $A \to B$. $\sigma : M_n(B \otimes B) \to M_n(B \otimes B)$ is a differential isomorphism. Then there is a $g \in Gl_n(B \otimes B)$ such that $\sigma(x) = gxg^{-1}$. (Actually this is a lie since j, being surjective, is not necessarily onto until we take a $\delta - pl$ covering of $B \otimes B$. However we can use hypercoverings to get around this, and then the argument proceeds exactly as follows.) Then $j\left(g_{12}g_{23}g_{13}^{-1}\right)$ is the identity automorphism since σ is a cocycle. Consequently there is a unit $u_\sigma \in B \otimes B \otimes B$ such that

$$i(u_\sigma) = \begin{pmatrix} u_\sigma & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_\sigma \end{pmatrix} = g_{23}g_{12}g_{13}^{-1} \in Gl_n(B \otimes B \otimes B).$$

It is easy to check that u is a 2 cocycle and its cohomology class depends only on the 1 cohomology class of σ. So $\partial_n([\sigma]) = [u] \in H^2(B/A, \mathbb{G}_m)$. This construction makes it clear that if $\sigma \in Z^1(B/A, PGl_n)$ with lifting g_σ and $\tau \in Z^1(B/A, PGl_m)$ with lifting g_τ, then $\sigma \otimes \tau$ is an automorphism of $M_n(B \otimes B) \otimes_{B \otimes B} M_m(B \otimes B) = M_{mn}(B \otimes B)$ and $\sigma \otimes \tau$ is a i cocycle for $M_{mn}(B \otimes B)$. Again it is immediate to observe that if $\sigma = f_2^{-1}\sigma' f_1$ and $\tau = g_2^{-1}\tau' g_1$, i.e. σ, τ are cohomologous to σ', τ' respectively, then

$$\sigma \otimes \tau = \left(f_2^{-1} \otimes g_2^{-1}\right) \sigma' \otimes \tau' \left(f_1 \otimes g_1\right).$$

Consequently this defines a monoid operation on

$$\lim_{n|N} H^1\left(A_{\delta-pl}, PGl_n\right)$$

where $H^1\left(A_{\delta-pl}, PGl_n\right) \to H^1\left(A_{\delta-pl}, PGl_N\right)$ by tensoring with the identity $I_{N/n} : M_{N/n}\left(B \otimes B\right) \to M_{N/n}\left(B \otimes B\right)$. The above definition of ∂_n then shows that there is a monoid map $\partial : \lim_{n|N} H^1\left(A_{\delta-pl}, PGl_n\right) \to H^2\left(A_{\delta-pl}, \mathbb{G}_{m,\delta}\right)$ since the cocycle condition requires $\partial_N\left([\sigma \otimes 1]\right) = [u_\sigma]$. If $\partial_n\left([\sigma]\right) = 0 \in H^2\left(B/A, \mathbb{G}_m\right)$, then there is a unit $v_\sigma \in B \otimes B$ such that replacing g_σ with $v_\sigma g_\sigma$, we find that $v_\sigma g_\sigma \in Z^1\left(B/A, Gl_n\right)$ and remains a lifting of $\sigma \in Z^1\left(B/A, PGl_n\right)$. This means that the differential Azumaya algebra Λ_σ defined by the cocycle σ is actually differentially isomorphic to $End_\delta\left(P\right)$ where P is the rank n differential module defined by the cocyle $v_\sigma g_\sigma$. These facts together with the above result that $\Lambda \otimes \Lambda^{op} \to End_\delta\left(\Lambda\right)$ finishes the outline. □

There is a stronger theorem for a regular commutative ring A. It says that there is an isomorphism $Br\left(A\right) \to H^2\left(A_{et}, \mathbb{G}_m\right)$. In the case of fields this is the statement that the second Galois cohomology group with coefficients in K_s^*, $H^2\left(G\left(K_s/K\right), K_s^*\right)$, is the Brauer group of the field K where K_s is the separable closure of K. Thus etale cohomology replaces Galois cohomology but the result remains the same. We will see that it is also true in the differential case, but this requires changing the site.

0.4. **Continuous maps or morphisms of sites and exact sequences of sheaves.** A map of topological spaces $\phi : X \to Y$ is continuous iff $\phi^{-1}\left(U\right)$ is open in X for all open sets U in Y. A morphism of sites is similar except that the main point is that coverings be preserved.

Definition 7. *Let* $\left(\mathcal{C}/X_1, \mathcal{T}_1\right)$ *and* $\left(\mathcal{C}/X_2, \mathcal{T}_2\right)$ *be two sites. A morphism of sites is a functor* $\Phi : \mathcal{C}/X_2 \to \mathcal{C}/X_1$ *such that if* $\left(p_i : U_i \to U\right) \in Cov\left(\mathcal{T}_2\right)$, *then* $\left(\Phi\left(p_i\right) : \Phi\left(U_i\right) \to \Phi\left(U\right)\right) \in Cov\left(\mathcal{T}_1\right)$.

We have numerous morphisms between the five sites we have already defined. Let's list them and label the functors. Note that the functors appear to go in the wrong direction. Milne indicates the morphism with a letter, say ϕ, and the functor with $\phi^{\cdot}$.

(1) $X_{pl} \xrightarrow{\tau} X_{et}$ where $\tau^{\cdot}\left(Y \to X\right) = Y \to X$ since any étale morphism is flat and locally of finite presentation. Clearly étale coverings are taken to flat coverings.
(2) $X_{et} \xrightarrow{\varepsilon} X_{Zar}$ where $\varepsilon^{\cdot}\left(U \subset X\right) = U \subset X$ since any open subscheme is étale. Clearly Zariski coverings are taken to étale coverings.
(3) $D_{et} \xrightarrow{=} D_{\delta-et}$ since any étale differential extension is an étale extension and differential étale coverings are étale coverings. Note that here we actually have an equivalence of categories and coverings since derivations extend uniquely to étale covers Consequently presheaves that are sheaves for one of these topologies will be a sheaf for the other. Moreover $H^1\left(X_{et}, \mathcal{F}\right) = H^1\left(X_{\delta-et}, F\right)$ for any sheaf $\mathcal{F}$.

(4) $D_{pl} \overset{\Delta}{\to} D_{\delta-pl}$ where $\Delta^{\cdot}(U \to D) = U \to D$ since any flat differential morphism is a flat morphism. Clearly flat differential coverings are flat coverings.

In addition to morphisms like these, we should also consider morphisms of sites caused by morphisms of schemes. Suppose $\phi : X' \to X$ is a morphism of schemes. Then there is a morphism of the Zariski, étale, and flat topologies since if $U \to X$ is an open embedding, an étale map, or a flat morphism locally of finite type, so is $U \times_X X' \to X^{'}$. Moreover coverings are obviously preserved. This leads to the continuous morphisms of sites $\phi_{Zar} : X'_{Zar} \to X_{Zar}$, $\phi_{et} : X'_{et} \to X_{et}$, and $\phi_{pl} : X'_{pl} \to X_{pl}$. The same reasoning for $\phi : D \to D'$ leads to morphisms of differential sites $\phi_{\delta-et} : X'_{\delta-et} \to X_{\delta-et}$ and $\phi_{\delta-pl} : X_{\delta-pl} \to X_{\delta-pl}$.

Clearly we have lots of opportunities for fun and games here. But let's make it even more interesting by producing some exact sequences of sheaves. Recall that a sheaf sequence $F \overset{\alpha}{\to} G \overset{b}{\to} H$ in $\mathcal{C}/X$ is exact means only that for each $U \in \mathcal{C}/X$ the homology of

$$\Gamma(U, F) \overset{\alpha(U)}{\to} \Gamma(U, G) \overset{b(U)}{\to} \Gamma(U, H)$$

vanishes 'locally', i.e there is a covering $(p : U_i \to U) \in Cov\{U\}$ such that applying p^* to the sequence produces a sequence

$$\Gamma(U_i, F) \overset{\alpha(U_i)}{\to} \Gamma(U_i, G) \overset{b(U_i)}{\to} \Gamma(U_i, H)$$

which is exact. There is a sheafification procedure satisfying the usual adjointness relations and this implies that sheaf maps are $1-1$ if and only if they are $1-1$ as presheaves. However a map of presheaves or sheaves may not be onto but still be surjective on the associated sheaves. All one needs is to show that 'locally' surjectivity is true, i.e. given a map of sheaves $f : F \to G$, f is onto if for all $U \in \mathcal{C}/X$, there is a covering $(U_i \to U)_{i \in I} \in Cov\{U\}$ such that $F(U_i) \to G(U_i)$ is onto for all $i \in I$.

Let's give some examples.

Example 7. (1) *The Kummer sequence*

$$0 \to \mu_n \to G_m \overset{\cdot n}{\to} G_m \to 0$$

is exact on X_{et} or X_{pl} but not on X_{Zar} where $\cdot n$ is the n^{th} power map and μ_n is the sheaf of n^{th} roots of unity. (Taking an n^{th} root of a unit defines an etale extension as long as n is a unit, and all our rings contain $\mathbb{Q}$.)

(2) *The log derivative sequence*

$$0 \to G_m^{\delta} \to G_m \overset{D\ln}{\to} W(\mathcal{O}) \to 0$$

is exact on $X_{\delta-pl}$ but not on $X_{\delta-et}$ where $D\ln(u) = \frac{u'}{u}$ is the logarithmic derivative of a unit. (The linear differential equation $Y' = aY$ can be solved by adjoining a variable X and declaring the derivative to be aX. Inverting X Gives an extension $U' \to U \in Cov(U)$ for the site $X_{\delta-pl}$ but not for $X_{\delta-et}$ since U' will be transcendental for some $a \in \Gamma(U, W(\mathcal{O}))$.)

(3) *Conjecturally the log derivative sequence*

$$0 \to Gl_n^{\delta} \to Gl_n \overset{D\ln}{\to} M_n \to 0$$

is exact on $X_{\delta-pl}$. Lourdes is investigating this. But even more intersting would be

$$0 \to PGl_n^{\delta} \to PGl_n \overset{j}{\to} ? \to 0.$$

Here I can't even guess what the cokernel should be nor what the map should be, but observe. PGl_n is also the automorphism group for $\mathbb{P}^{n-1}$. Consequently (and there are descent results that justify this) $H^1(A_{et}, PGl_n)$ describes schemes over $Spec(A)$ that locally for the etale topology are isomorphic to $\mathbb{P}^n$. These are called Severi-Brauer varieties. Suppose A is a field K. It is reasonable to ask if they admit coverings by open sets such that the patching isomorphisms are constant, i.e. patching isomorphisms are in $PGl_n(K)$ If $n = 2$, the Schwarzian derivative tells you when this occurs. Is there an extension to higher dimensions that would describe the above map j?

In our setting we are interested in the continuous map $\Delta : D_{pl} \to D_{\delta-pl}$ where $D = Spec(A)$ for some differential ring A. We know that any sheaf on D_{pl} is automatically a sheaf on $D_{\delta-pl}$ which simplifies life for us. In general given a sheaf $\mathcal{F}$ on D_{pl}, there is a spectral sequence

$$E_2^{p,q} = H^p(D_{\delta-pl}, R^q\Delta_*\mathcal{F}) \Longrightarrow H^n(D_{pl}, \mathcal{F})$$

where $R^q\Delta_*\mathcal{F}$ is the sheaf obtained by sheafifying the presheaf $U \mapsto H^q(U_{pl}, \Delta_*\mathcal{F})$. If we apply this to the sheaf $\mathbb{G}_m$ and use the fact that $H^1(U_{pl}, \Delta_*\mathbb{G}_m) = Pic(U)$ and $H^2(U_{pl}, \Delta_*\mathbb{G}_m) = Br(U)$, we conclude that the sheafifification in the $\delta - pl$ topology of these two presheaves produces sheaves that are identically 0. ($Pic(U)$ vanishes locally for the Zariski topology and $Br(U)$ vanishes locally for the etale topology.) Putting this into the spectral sequence and noting that $\Delta_*\mathbb{G}_m = \mathbb{G}_m$ on $D_{\delta-pl}$, we find the key fact

$$H^2(A_{pl}, \mathbb{G}_m) = H^2(A_{\delta-pl}, \mathbb{G}_m).$$

But if A is regular it is well known that $Br(A) \approxeq H^2(A_{pl}, \mathbb{G}_m) = H^2(A_{et}, \mathbb{G}_m)$ and so we conclude $Br(A) = H^2(A_{\delta-pl}, \mathbb{G}_m)$. This leads to our main result.

Theorem 7. *Let A be a regular differential ring. Then $Br_\delta(A) = Br(A)$.*

Proof. We use the exact sequence $0 \to G_m^{\delta} \to G_m \overset{D\ln}{\to} W(\mathcal{O}) \to 0$ to investigate matters together with the fact that if X is affine, then $H^q(X_{pl}, W(M)) = 0$. This follows from Milne ([Milne, 1980, III, Proposition 3.7]) and Serre's theorem on the vanishing of quasi-coherent sheaf cohomology on affine schemes. Thus we conclude that $H^2\left(A_{\delta-pl}, \mathbb{G}_m^{\delta}\right) \to H^2(A_{\delta-pl}, \mathbb{G}_m)$ is an isomorphism. Combining this with the embeddings coming from ∂ and the surjectivity of $Br_\delta(A) \to Br(A)$ which results from being able to extend derivations on Azumaya algebras over A yields the first equality.

Now look at the continuous map of sites $\phi : A_{\delta-pl} \to A_{\delta-et}$. Since $H^2\left(B_{\delta-pl}, \mathbb{G}_m^{\delta}\right) \to H^2(B_{\delta-pl}, \mathbb{G}_m)$ for any etale extension $A \to B$, we find again that $B \mapsto H^2\left(B_{\delta-pl}, \mathbb{G}_m^{\delta}\right)$ vanishes when sheafified on $A_{\delta-et}$ as does □

References

[Milne, 1980] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.