Symbolic Computation for Overdetermined Systems of Nonlinear Differential Equations.

Evelyne Hubert
INRIA Sophia Antipolis

Kolchin Seminar in Differential Algebra - graduate center series

Symbolic Computation for Nonlinear Differential Systems

Algebraic structures and nice theorems on those

Algorithm

In: a differential system and a ranking

Out: a finite set of *nice* differential systems
(membership, elimination, completion)

s.t. the set of solutions of the input system is the union of the non singular solutions of the output systems

Implementation publically available

Notes on triangular sets and triangulation decomposition algorithms. I Polynomial systems. II Differential Systems. LNCS 2630 (2003)

http://www.inria.fr/cafe/Evelyne.Hubert/diffalg

Contents

1 Sample problems **2**

2 Software **4**

3 Ring of differential polynomials **5**

4 Differential ideals **8**

5 Representation of radical differential ideals **9**

1 Sample problems

Envelope

Consider a family of curves

$$y = c\,x + c^2$$

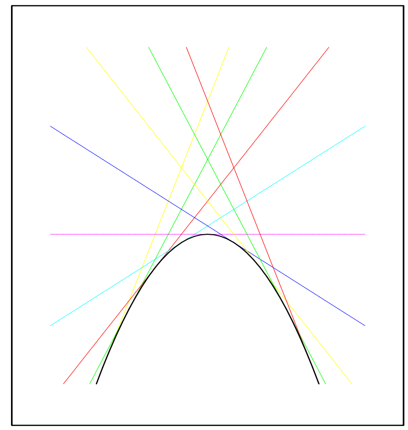
What is their envelope?

The differential equation satisfied by this family is:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + x\,\frac{\mathrm{d}y}{\mathrm{d}x} = y$$

General solution: $y = c\,x + c^2$

Singular solution: $y = -1/4x^2$



Kepler $\Rightarrow$ Newton

Kepler's observational laws:

K_1 the planets move along ellipses with the sun as focus
$r = p + e\,x$
where $r^2 = x^2 + y^2$
and $\dot{p}, \dot{e} = 0$

K_2 the vector from the sun to the planet sweeps equal area in equal times
$x\,\dot{y} - \dot{x}\,y = s$
with $\dot{s} = 0$

Newton's gravitational laws:

N_1 the acceleration is inversely proportional to the square of the distance to the sun.
$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\,a\right) = 0$
where $a^2 = (\ddot{x}^2 + \ddot{y}^2)$

N_2 the acceleration vector is directed towards the sun
$x\,\ddot{y} - \ddot{x}\,y = 0$

$K_2 \;\Rightarrow\; N_2, \qquad K_1, K_2 \;\Rightarrow\; N_1$?

Membership [Wu 91]

Orthogonal waves

[G. Metivier]

$$\begin{aligned} s\,(\phi_{xx} + \phi_{yy}) + s_x\,\phi_x + s_y\,\phi_y + \phi &= 0 \\ s\,(\psi_{xx} + \psi_{yy}) + s_x\,\psi_x + s_y\,\psi_y + \psi &= 0 \\ \psi_x\,\phi_x + \psi_y\,\phi_y &= 0 \end{aligned}$$

- Is there a solution? consistency
- What are the *degrees of freedom* completion

$$
\begin{aligned}
s &= f_1(y) + f_2(y)\,x + s_{20}\frac{x^2}{2} + s_{30}\,\frac{x^3}{6}\dots \\
\psi &= f_3(y) + c_1\,x + \psi_{20}\frac{x^2}{2} + \psi_{11}\,x\,y + \psi_{20}\,\frac{x^2}{2}\dots \\
\phi &= f_4(y) + \phi_{10}\,x + \phi_{01}y + \phi_{20}\frac{x^2}{2} + \phi_{11}\,x\,y + \dots
\end{aligned}
$$

4 functions of 1 variable, 1 constant.

- Conditions on s elimination

Equivalence

[Neut 03]

$$
y'' = f(x,y,y') \quad \underset{X=\xi(x,y),\,Y=\eta(x,y)}{\overset{\exists?\xi,\eta}{\longrightarrow}} \quad Y''=0
$$

$$
y_2 = f(x,y_0,y_1) \quad \underset{X=\xi(x,y_0),\,Y_0=\eta(x,y_0)}{\overset{\exists?\xi,\eta}{\longrightarrow}} \quad Y_2=0
$$

$$
\left\{
\begin{array}{l}
\xi_{y_1} \;=\; 0, \quad \eta_{y_1} \;=\; 0, \\
(\eta_{y_0}\xi_x - \xi_{y_0}\eta_x)\,f + (\eta_{y_0y_0}\xi_{y_0} - \eta_{y_0}\xi_{y_0y_0})\,{y_1}^3 \\
+\left(2\,\eta_{xy_0}\xi_{y_0} - 2\,\eta_{y_0\xi_{xy_0}-\eta_x}\xi_{y_0y_0} + \eta_{y_0y_0}\xi_x\right){y_1}^2 \\
+\,(2\,\eta_{xy_0}\xi_x - 2\,\eta_x\xi_{xy_0} - \eta_{y_0}\xi_{xx} + \eta_{x,x}\xi_{y_0})\,y_1 - \eta_x\xi_{xx} + \eta_{xx}\xi_x \;=\; 0
\end{array}
\right.
$$

Differential indeterminates : ξ, η, f, functions of x, y_0, y_1

Derivations : $\partial_x = \frac{\partial}{\partial x}$, $\partial_{y_0} = \frac{\partial}{\partial y_0}$, $\partial_{y_1} = \frac{\partial}{\partial y_1}$

Differential indeterminates : ξ, η, f

Consider ∂_{y_0}, ∂_{y_1}, and $D_x = \partial_x + y_1\,\partial_{y_0} + f\,\partial_{y_1}$

$$
\partial_{y_0}\,D_x - D_x\,\partial_{y_0} = f_{y_0}\,\partial_{y_1}, \quad \partial_{y_1}\,D_x - D_x\,\partial_{y_1} = \partial_{y_0} + f_{y_1}\,\partial_{y_1}
$$

$$
\left\{
\begin{array}{l}
\xi_{y_1} \;=\; 0, \quad \eta_{y_1} \;=\; 0, \\
\xi_x\eta_{xx} - \xi_{xx}\eta_x \;=\; 0
\end{array}
\right.
$$

General case :

$$
f_{y_1y_1y_1y_1} = 0, f_{xxy_1y_1} = 4f_{xy_0y_1} + f_{y_1}f_{xy_1y_1} - 4f_{y_1}f_{y_0y_1} - 6f_{y_0y_0} + 3f_{y_1y_1}f_{y_0}
$$

Fiber preserving case ($\xi_{y_0} = 0$)

$$
f_{y_1y_1y_1} \;=\; 0, \quad f_{xy_1y_1} \;=\; f_{y_0y_1}, \quad f_{xy_0y_1} \;=\; 2\,f_{y_0y_0} - f_{y_1y_1}f_{y_0} + f_{y_1}f_{y_0y_1}
$$

2 Software

Software for nonlinear differential systems

diffalg: Differential Algebra — MapleV

1996 Rosenfeld-Gröbner algorithm [BLOP 1997]
by F. Boulier (1996) then at SCG, U. of Waterloo. — Maple V.5

1998 Singular solutions [H 99]; Efficiency improvement [H 00]
by E. Hubert then at SCG, UWaterloo. — Maple 6

1999 Redesign of help pages. — Maple 7

2004 Non-commuting derivations [H 2005]
`http://www.inria.fr/cafe/Evelyne.Hubert/diffalg`

BLAD: Bibliothèques Lilloises d'Algèbre Différentielle — GLGPL

C libraries distributed under Gnu Lesser General Public License

`http://www2.lifl.fr/~boulier/BLAD` by F. Boulier

RIF: Reduced Involutive Forms — Maple6

A. Wittkopf, G. Reid — Maple 6, DEtools

diffgrob: Differential Gröbner bases — Maple

E. Mansfield (`http://www.kent.ac.uk/ims/personal/elm2`)

CRACK: PDE solver — Reduce

T. Wolf (`http://lie.math.brocku.ca/crack`)

For linear functional systems

kan/sm1 by N. Takayama (`http://www.math.kobe-u.ac.jp/KAN`)

D-Macaulay by A. Leykin & H. Tsai (`http://www.ima.umn.edu/~leykin/Dmodules`)

Cocoa, `http://cocoa.dima.unige.it/`

Plural:Singular by V. Levandovskyy & H. Schönemann (`http://www.singular.uni-kl.de/plural`)

Groebner/OreModule by F. Chyzak, A. Quadrat & D. Robertz. (`http://wwwb.math.rwth-aachen.de/OreModules`)

Calix:Aldor by Ralf Hemmecke (`http://www.hemmecke.de/ralf`)

...

3 Ring of differential polynomials

Classical construction [Ritt 51, Kolchin 73]

$\mathbb{F} = \mathbb{Q}$ or $\mathbb{Q}(x,y)$

$$\begin{cases} s\,(\phi_{xx}+\phi_{yy})+s_x\,\phi_x+s_y\,\phi_y+\phi = 0 \\ s\,(\psi_{xx}+\psi_{yy})+s_x\,\psi_x+s_y\,\psi_y+\psi = 0 \\ \psi_x\,\phi_x+\psi_y\,\phi_y = 0 \end{cases}$$

$\mathbb{F}$ a field

$\delta_1 = \frac{\partial}{\partial x},\ \delta_2 = \frac{\partial}{\partial y}$

$\Delta = \{\delta_1, \ldots, \delta_m\}$ derivations on $\mathbb{F}$

$\delta_i(a+b) = \delta_i(a) + \delta_i(b) \quad \delta_i(a\,b) = a\delta_i(b) + \delta_i(a)\,b$

$\mathcal{Y} = \{s, \phi, \psi\}$

$\mathcal{Y} = \{y_1, \ldots, y_n\}$

$\mathbb{F}[s, s_x, s_y, s_{xx}\ldots] = \mathbb{F}\,[\![s,\phi,\psi]\!]$

$s_{xxy} \rightsquigarrow s_{x^2y} \rightsquigarrow s_{(2,1)}$

$\mathbb{F}\,[\![\mathcal{Y}]\!] = \mathbb{F}[\,y_\alpha \mid \alpha \in \mathbb{N}^m,\ y \in \mathcal{Y}\,]$

$\bar{\delta}_1\,(s_{xy}) = s_{xxy}$

$\rightsquigarrow \bar{\delta}_1\,\left(s_{(1,1)}\right) = s_{(2,1)}$

$\bar{\delta}_i\,(y_\alpha) = y_{\alpha+\epsilon_i}$

$\epsilon_i = (0,\ldots,\underset{i^{\text{th}}}{1},\ldots,0)$

$\bar{\delta}_i|_{\mathbb{F}} = \delta_i$

$\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}$

$\delta_i\delta_j = \delta_j\delta_i$

Link with differential geometry

Independent variables $x_1, \ldots, x_n$

- $\mathbb{F} = \mathbb{Q}(x_1, \ldots, x_m)$
- $\delta_1 = \frac{\partial}{\partial x_1}, \ldots, \delta_m = \frac{\partial}{\partial x_m}$

Dependent variables $y_1, \ldots, y_n$

- $\mathcal{Y} = \{y_1, \ldots, y_n\}$ the differential indeterminates

$\mathbb{F}\,[\![\mathcal{Y}]\!]$ is the coordinate ring for the infinite jet space

Total derivatives:

$$\bar{\delta}_i = \frac{\partial}{\partial x_i} + \sum_{y\in\mathcal{Y},\,\alpha\in\mathbb{N}^m} y_{\alpha+\epsilon_i}\,\frac{\partial}{\partial y_\alpha}$$

Declare it in Maple

```
> with(diffalg);
```

[Rosenfeld_Groebner, characters, delta_polynomial, denote, derivatives, differential_ring, differentiate, equations, essential_components, field_extension, greater, inequations, initial, initial_conditions, is_prime, leader, power_series_solution, preparation_polynomial, print_ranking, rank, reduce, reduced_form, rewrite_rules, separant]

```
> R := differential_ring( ranking=[[s,phi,psi]], derivations=[x,y], notation=jet );
```

$R := differential\_polynomial\_ring$

```
> differentiate( s[x,y], x, R );
```

$s_{x,x,y}$

```
> denote( s[x,y], vjet, R );
```

$$s_{1,1}$$

```
> denote( s[x,y], diff, R );
```

$$\frac{\partial^2}{\partial x \partial y} s(x, y)$$

Rankings $\prec$ on $\mathbb{F}[\![\mathcal{Y}]\!]$

A total order on $\mathcal{D}\mathcal{Y} = \{y_\alpha \mid \alpha \in \mathbb{N}^m,\ y \in \mathcal{Y}\}$ s.t.

- $y_\alpha \prec y_{\alpha+\gamma}$ $\qquad \alpha, \beta, \gamma \in \mathbb{N}^m$
- $y_\alpha \prec z_\beta \ \Rightarrow\ y_{\alpha+\gamma} < z_{\beta+\gamma}$ $\qquad y, z \in \mathcal{Y}$

$$|\alpha| = \alpha_1 + \ldots + \alpha_m$$

Orderly ranking: $|\alpha| < |\beta| \ \Rightarrow\ y_\alpha \prec z_\beta,$ $\qquad \forall y, z \in \mathcal{Y}$

Semi-orderly ranking: $|\alpha| < |\beta| \ \Rightarrow\ y_\alpha \prec y_\beta$

Elimination ranking: $y_\alpha \prec z_\beta,$ $\qquad \forall \alpha, \beta \in \mathbb{N}^m$

Rankings with *diffalg*

```
> DY := [seq(seq(seq(u[x$i,y$(j-i)],i=0..j),j=0..2),u=[s,phi,psi])];
```

$$DY := [s, s_y, s_x, s_{y,y}, s_{x,y}, s_{x,x}, \phi, \phi_y, \phi_x, \phi_{y,y}, \phi_{x,y}, \phi_{x,x}, \psi, \psi_y, \psi_x, \psi_{y,y}, \psi_{x,y}, \psi_{x,x}]$$

```
>R2:=differential_ring(ranking=[grlex[s],grlex[phi,psi]],derivations=[x,y]):
> sort(DY, (a,b) - > greater(b,a,R2));
```

$$[\psi, \phi, \psi_y, \psi_x, \phi_y, \phi_x, \psi_{y,y}, \psi_{x,y}, \psi_{x,x}, \phi_{y,y}, \phi_{x,y}, \phi_{x,x}, s, s_y, s_x, s_{y,y}, s_{x,y}, s_{x,x}]$$

```
> R1:=differential_ring(ranking=[[s,phi,psi]],derivations=[x,y]):

> sort(DY, (a,b) - > greater(b,a,R1));
```

$$[\psi, \phi, s, \psi_y, \phi_y, s_y, \psi_x, \phi_x, s_x, \psi_{y,y}, \phi_{y,y}, s_{y,y}, \psi_{x,y}, \phi_{x,y}, s_{x,y}, \psi_{x,x}, \phi_{x,x}, s_{x,x}]$$

Non-commuting derivations

$$y_2 = f(x, y_0, y_1) \quad \underset{Y_0 = \eta(x, y_0)}{\overset{X = \xi(x, y_0)}{\longrightarrow}} \quad Y_2 = 0$$

$$\begin{cases} \xi_{y_1} = 0, \quad \eta_{y_1} = 0, \\ \xi_x \eta_{xx} - \xi_{xx} \eta_x = 0 \end{cases}$$

Differential indeterminates : ξ, η, f

Derivations ∂_{y_0}, ∂_{y_1}, and D_x

$\partial_{y_0} D_x - D_x \partial_{y_0} = f_{y_0} \partial_{y_1},$

$\partial_{y_1} D_x - D_x \partial_{y_1} = \partial_{y_0} + f_{y_1} \partial_{y_1}$

$\partial_{y_1} \partial_{y_0} - \partial_{y_0} \partial_{y_1} = 0$

$\mathcal{Y} = \{y_1, \ldots, y_n\}$

$\mathcal{D} = \{\delta_1, \ldots, \delta_m\}$ with

$$\delta_i \delta_j - \delta_j \delta_i = \sum_{l=1}^{m} c_{ijl}\, \delta_l$$

Differential polynomial ring $\mathbb{K}\llbracket\mathcal{Y}\rrbracket$ with non commuting derivations [H05]

$\mathcal{Y} = \{y_1, \dots, y_n\}$

$\mathcal{D} = \{\delta_1, \dots, \delta_m\}$

$\mathbb{K}[y_\alpha \mid \alpha \in \mathbb{N}^m,\ y \in \mathcal{Y}]$

$$\delta_i\,(y_\alpha) = \begin{cases} y_{\alpha+\epsilon_i} \text{ if } \alpha_1 = \dots = \alpha_{i-1} = 0 \\ \delta_j \delta_i \left(y_{\alpha-\epsilon_j}\right) + \sum_{l=1}^{m} c_{ijl}\,\delta_l \left(y_{\alpha-\epsilon_j}\right) \\ \quad \text{where } j < i \text{ is s.t. } \alpha_j > 0 \\ \quad \text{while } \alpha_1 = \dots = \alpha_{j-1} = 0 \end{cases}$$

Example (m=2): $\delta_1 y_{(1,1)} = y_{(2,1)}$

$$\begin{aligned} \delta_2 y_{(1,1)} = \delta_2\delta_1 y_{(0,1)} = \ & \delta_1\delta_2 y_{(0,1)} + c_{121}\,\delta_1 y_{(0,1)} + c_{122}\,\delta_2 y_{(0,1)} \\ & \delta_1 y_{(0,2)} + c_{121}\,y_{(1,1)} + c_{122}\,y_{(0,2)} \\ & y_{(1,2)} + c_{121}\,y_{(1,1)} + c_{122}\,y_{(0,2)} \end{aligned}$$

If the c_{ijl} satisfy

- $c_{ijl} = -c_{jil}$
- $\delta_k(c_{ijl}) + \delta_i(c_{jkl}) + \delta_j(c_{kil}) = \sum_{\mu=1}^{m} c_{ij\mu}c_{\mu kl} + c_{jk\mu}c_{\mu il} + c_{ki\mu}c_{\mu jl}$

& there exists an *admissible ranking* $\prec$

- $|\alpha| < |\beta| \ \Rightarrow\ y_\alpha \prec y_\beta$,
- $y_\alpha \prec z_\beta \ \Rightarrow\ y_{\alpha+\gamma} \prec z_{\beta+\gamma}$,
- $\sum_{l\in\mathbb{N}_m} c_{ijl}\,\delta_l(y_\alpha) \prec y_{\alpha+\epsilon_i+\epsilon_j}$

then $\delta_i\delta_j(p) = \delta_j\delta_i(p) + \sum_{l=1}^{m} c_{ijl}\,\delta_l(p) \quad \forall p \in \mathbb{K}[\,y_\alpha \,|\, \alpha \in \mathbb{N}^m\,] = \mathbb{K}\llbracket\mathcal{Y}\rrbracket$

With *diffalg*

$$\partial_{y_0}\,D_x - D_x\,\partial_{y_0} = f_{y_0}\,\partial_{y_1}, \quad \partial_{y_1}\,D_x - D_x\,\partial_{y_1} = \partial_{y_0} + f_{y_1}\,\partial_{y_1}$$

```
> R := differential_ring(ranking=[[xi,eta],f], derivations=[x,y0,y1], commutations=[ [y0,x]=[0,0,f[y0]], [y1,x]=[0,1,f[y1]] ]):

> differentiate( differentiate( eta, y0, R ), x, R );
```

$$\eta_{x,y0}$$

```
> differentiate( eta, x, y0, R);
```

$$\eta_{x,y0}$$

```
> differentiate( eta, y0, x, R);
```

$$\eta_{x,y0} + f_{y0}\,\eta_{y1}$$

Leader, initial, separant

- $\mathcal{Y} = \{y_1, \dots, y_n\}$, $\quad \mathcal{D} = \{\delta_1, \dots, \delta_m\}$, $\quad \mathbb{F}\llbracket\mathcal{Y}\rrbracket$, $\quad \prec$
- Nota^o: $\delta^\alpha = \delta_1^{\alpha_1} \dots \delta_m^{\alpha_m}$

  $\delta^\alpha\delta^\beta \neq \delta^{\alpha+\beta}$ but $\delta^\alpha\delta^\beta = \delta^{\alpha+\beta} + \sum_{|\gamma|<|\alpha+\beta|} a_\gamma\,\delta^\gamma, \quad a_\gamma \in \mathbb{K}\llbracket\mathcal{Y}\rrbracket$
- $p \in \mathbb{K}\llbracket\mathcal{Y}\rrbracket \setminus \mathbb{K}$

$$p = i_p\,{y_\alpha}^d + c_1\,{y_\alpha}^{d-1} \dots + c_d \qquad c_i \prec y_\alpha$$

y_α leader $\mathsf{lead}(p)$ | i_p initial $\mathsf{init}(p)$ | s_p separant $\mathsf{sep}(p)$

- $s_p = \dfrac{\partial p}{\partial y_\alpha} = d\, i_p\, y_\alpha^{d-1} + (d-1)\, c_1\, y_\alpha^{d-2} + \ldots + c_{d-1}$

Prop:

$$\mathsf{lead}(p) = y_\alpha \quad \Rightarrow \quad \delta^\beta(p) = \mathsf{sep}(p)\; y_{\alpha+\beta} + \underbrace{\ldots\ldots}_{\prec\, y_{\alpha+\beta}}$$

4 Differential ideals

Differential Ideals

$\mathcal{Y} = \{y_1, \ldots, y_n\}$, $\mathcal{D} = \{\delta_1, \ldots, \delta_m\}$, $\mathbb{F}[\![\mathcal{Y}]\!]$

$\{p_1, \ldots, p_k\} \subset \mathbb{F}[\![\mathcal{Y}]\!]$

I, a differential ideal of $\mathbb{F}[\![\mathcal{Y}]\!]$:

- $a \in I \;\Rightarrow\; \delta a \in I, \forall \delta \in \Delta$
- I is an ideal

$[p_1, \ldots, p_k]$

$[p] = (\, \delta^\alpha p \,|\, \alpha \in \mathbb{N}^m)$

J, radical differential ideal of $\mathbb{F}[\![\mathcal{Y}]\!]$:

- $a^k \in J \;\Rightarrow\; a \in J$
- J is a differential ideal

$[\![p_1, \ldots, p_k]\!]$

$[\![p]\!] = \{q |\, q^r \in [p]\}$

P, prime differential ideal of $\mathbb{F}[\![\mathcal{Y}]\!]$:

- $a\, b \in P \;\Rightarrow\; a \in P$ or $b \in P$
- is a differential ideal

P radical and irreducible.

Differential Nullstellensatz

$p_1, \ldots, p_r \in \mathbb{F}[\![\mathcal{Y}]\!]$

NOTE:
- if $q \in [\![p_1, \ldots, p_r]\!]$ then $q^e = \sum c_{i\alpha} \delta^\alpha p_i$ and q vanishes on all the common zeros of $p_1, \ldots, p_r$.
- if $1 \in [\![p_1, \ldots, p_r]\!]$ then $1 = \sum c_{i\alpha} \delta^\alpha p_i$ so that $p_1, \ldots, p_r$ have no common zero.

THEO:
- $p_1, \ldots, p_k$ admit a common zero iff $1 \notin [\![p_1, \ldots, p_r]\!]$
- $q \in \mathbb{F}[\![\mathcal{Y}]\!]$ vanishes on all the common zeros of $p_1, \ldots, p_k$ iff $q \in [\![p_1, \ldots, p_k]\!]$.

Ritt-Raudenbush Theorem

THEO: A radical differential ideal J of $\mathbb{F}[\![\mathcal{Y}]\!]$ is

- finitely generated: $J = [\![p_1, \ldots, p_r]\!]$ for some $p_1, \ldots, p_r \in \mathbb{F}[\![\mathcal{Y}]\!]$.

  Differential ideals need not be finitely generated.

- the intersection of finitely many prime differential ideals.

  The irredundant decomposition $J = \bigcap_{i=1}^{r} P_i$ is unique.

EX: $[\![y_x^2 + x\,y_x - y]\!] = [\![y_x^2 + x\,y_x - y, y_{xx}]\!] \cap [\![4\,y + x^2]\!]$.

Saturation ideals

- $h \in \mathbb{F}[\![\mathcal{Y}]\!]$ $\quad I : h^\infty = \{q \mid \exists k \text{ s.t. } h^k\, q \in I\}$
- $H = \{h_1, \ldots, h_s\}$ $\quad I : H^\infty = I : h^\infty$ where $h = h_1 \ldots h_s$

NOTE: The zeros of $[\![p]\!] : h^\infty$ are the zeros of p that don't vanish on h, except for some *adherent* piece.

EX: $[\![y_x^2 + x\,y_x - y]\!] = [\![y_x^2 + x\,y_x - y]\!] : (2y_x + x)^\infty \cap [\![4\,y + x^2]\!]$.

5 Representation of radical differential ideals

Purpose

Given $p_1, \ldots, p_r$ in $\mathbb{F}[\![\mathcal{Y}]\!]$ we want to compute a representation of $[\![p_1, \ldots, p_r]\!]$ that allows to

- test membership to $[\![p_1, \ldots, p_r]\!]$
- *measure* the zero set (completion)
- compute $[\![p_1, \ldots, p_r]\!] \cap \mathbb{F}[\![\mathcal{Z}]\!]$ (elimination)

There is no strict analogue of Gröbner bases.

Differential ideals do not admit in general

- a finite set
- that is generating for the differential ideal
- and *reduces* to zero the elements of the differential ideal

Characterisable differential ideals are defined by

- a finite set so called the characteristic set

– that *reduces* to zero the elements of the differential ideal
– generate the ideal *outside of some hypersurface*

Prime differential ideals are characterizable

Radical diff. ideal are $\cap^o$ of characterisable diff. ideals

Characteristic Decomposition

ALGO: (Rosenfeld-Groebner in *diffalg*)

In: $\{p_1, \dots, p_k\} \subset \mathbb{F}[\![\mathcal{Y}]\!]$, $\prec$
Out: $C_1, \dots C_r$ s.t.

$$[\![p_1, \dots, p_k]\!] = [C_1]:S_1^\infty \cap \dots \cap [C_r]:S_r^\infty$$

where $S_i = \mathsf{sep}(C_i)$

Membership: $p \in [C_i]:S_i \;\Leftrightarrow\; p \longrightarrow_{C_i} 0$

Completion: If $\prec$ orderly then $[C_i]:S_i^\infty \cap \mathbb{F}[\mathcal{D}_\kappa\mathcal{Y}] = (\mathcal{D}_\kappa C_i):S_i^\infty$

Elimination: If $\prec$ eliminates $\mathcal{Y} \setminus \mathcal{Z}$ and $C_i' = C_i \cap \mathbb{F}[\![\mathcal{Z}]\!]$ then

$$[C_i]:S_i^\infty \cap \mathbb{F}[\![\mathcal{Z}]\!] = [C_i']:S_i'^\infty$$

Output characteristic sets C

or pseudo-Gröbner bases

- C is a differential triangular set:

  $$a, b \in C, \quad \mathsf{lead}(a) = y_\alpha,\ \mathsf{lead}(b) = y_\beta \quad \Rightarrow \quad \beta \neq \alpha + \gamma$$

- C is coherent: [Rosenfeld 59]

  $$a, b \in C, \quad \mathsf{lead}(a) = y_\alpha,\ \mathsf{lead}(b) = y_\beta$$

  $$\gamma = \alpha + \bar{\alpha} = \beta + \bar{\beta} \quad \Rightarrow \quad s_a\, \delta^{\bar{\beta}} b - s_b\, \delta^{\bar{\alpha}} a \in (\mathcal{D}C_{<y_\gamma}):S^\infty$$

  Recall:

  $$\begin{array}{rcl} \delta^{\bar{\beta}} b & = & s_b\, y_\gamma + \dots \\ \delta^{\bar{\alpha}} a & = & s_a\, y_\gamma + \dots \end{array}$$

  A generalisation of

  $$\left.\begin{array}{l} u_x = f(x,y) \\ u_y = g(x,y) \end{array}\right\} \Rightarrow\ f_y - g_x = 0$$

- $p \longrightarrow_c q$ means: $s_c^e i_c^f\, p \equiv q \mod [c]$

  – q is free of proper derivatives of $\mathsf{lead}(c)$
  $s_c^e\, p = q_1 \mod (\delta^\beta c, \dots, \delta^\gamma c)$
  – the degree of q in $\mathsf{lead}(p)$ is lower than that of p
  $i_c^f\, q_1 = q \mod (p)$

Prop: $[C]:S^\infty$ is a radical differential ideal. Because $(C):S^\infty$ is.

Prop: If $[C]:S^\infty = \bigcap_i [C_i]:S_i$ then $\mathsf{lead}(C_i) = \mathsf{lead}(C)$

Envelope

> R:= differential_ring(ranking=[c,y],derivations=[x]):
> C := Rosenfeld_Groebner([$y - c\,x - c^2,\ c_x$], R);

$$C := [characterisable]$$

> equations(C);

$$[[-y_x + c,\ y_x^2 + x\,y_x - y]]$$

> p := equations(G[1])[-1];

$$p := y_x^2 + x\,y_x - y$$

> C:= Rosenfeld_Groebner([p],R);

$$C := [characterisable, characterisable]$$

> equations(C), inequations(C)

$$[[y_x^2 + x\,y_x - y],\ [4\,y + x^2]], \quad [[2y_x + x],\ [\,]]$$

That is: $[\![p]\!] = [p] : (2y_x + x)^\infty \cap [4\,y + x^2]$

Kepler $\Rightarrow$ Newton

> R:= differential_ring(ranking=[[x,y],[a,r],[p,e,s]],derivations=[t], parameters=[p,e,s]):
> K := $\{a^2 - x_{t,t}^2 - y_{t,t}^2, r^2 - x^2 - y^2), r - p + e\,x, x\,y_t - y\,x_t - s\}$:
> C := Rosenfeld_Groebner(K, $\{p, e, s\}$, R);

$$C := [characterisable]$$

> N := differentiate($r^2\,a$,t,R):

$$N := r\,(a_t r + 2\,r_t a)$$

> reduce(N, C[1]);

$$0$$

With $[\![K]\!] : (pes)^\infty = [e\,\mathbf{x} + r - p, es\,\mathbf{y} - prr_t,$
$r^2 p^2 \mathbf{r_t}^2 - s^2(e^2 r^2 + r^2 - 2\,rp + p^2), r^4 p^2 \mathbf{a}^2 - s^4] : (epsarr_t)^\infty$

Orthogonal Waves

$$\begin{aligned} s\,(\phi_{xx} + \phi_{yy}) + s_x\,\phi_x + s_y\,\phi_y + \phi &= 0 \\ s\,(\psi_{xx} + \psi_{yy}) + s_x\,\psi_x + s_y\,\psi_y + \psi &= 0 \\ \psi_x\,\phi_x + \psi_y\,\phi_y &= 0 \\ s_x, s_y, \phi_x, \phi_y, \psi_x, \psi_y &\neq 0 \end{aligned}$$

> with(diffalg)

> R:=differential_ring(ranking=[[s,phi,psi]],derivations=[x,y]):

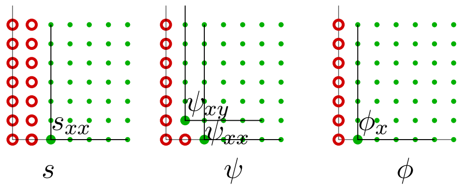
> S:=[s*(phi[x,x]+phi[y,y])+s[x]*phi[x]+. . .]

> H :={s[x],s[y],phi[x],. . . }

> C :=Rosenfeld_Groebner(S,H,R);

$$C := [characterisable]$$

$$C = \{s_{xx} = \ldots, \psi_{xx} = \ldots, \psi_{xy} = \ldots, \phi_x = \ldots\}$$



$$\begin{array}{rcl}
s & = & s_{00} + s_{10}\,x + s_{01}y + s_{20}\frac{x^2}{2} + s_{11}\,x\,y + s_{02}\,\frac{y^2}{2} + s_{30}\,\frac{x^3}{6}\ldots \\
\psi & = & \psi_{00} + \psi_{10}\,x + \psi_{01}y + \psi_{20}\frac{y^2}{2} + \psi_{11}\,x\,y + \psi_{02}\,\frac{x^2}{2}\ldots \\
\phi & = & \phi_{00} + \phi_{10}\,x + \phi_{01}y + \phi_{20}\frac{x^2}{2} + \phi_{11}\,x\,y + \phi_{02}\,\frac{y^2}{2}\ldots
\end{array}$$

$$\begin{array}{rcl}
s & = & f_1(y) + f_2(y)\,x + s_{20}\frac{x^2}{2} + s_{30}\,\frac{x^3}{6}\ldots \\
\psi & = & f_3(y) + c_1\,x + \psi_{20}\frac{x^2}{2} + \psi_{11}\,x\,y + \psi_{20}\,\frac{x^2}{2}\ldots \\
\phi & = & f_4(y) + \phi_{10}\,x + \phi_{01}y + \phi_{20}\frac{x^2}{2} + \phi_{11}\,x\,y + \ldots
\end{array}$$

Differential Dimension Polynomial

$\prec$ orderly

$$\begin{array}{rcl}
H(s) & = & \dim\left([C]\!:\!S \cap \mathbb{F}\,[\![\mathcal{D}_s\mathcal{Y}]\!]\right) = \dim(\mathcal{D}_sC)\!:\!S^\infty \\
& = & a_m\begin{pmatrix} m+s \\ s\end{pmatrix} + \ldots + a_d\begin{pmatrix} d+s \\ s\end{pmatrix} + \ldots + a_0
\end{array}$$

for $s >> 0$

Algorithm for computing the characteristic decomposition

1. $[\![\Sigma]\!] = [A_1]\!:\!H_1^\infty \cap \ldots \cap [A_k]\!:\!H_k^\infty$

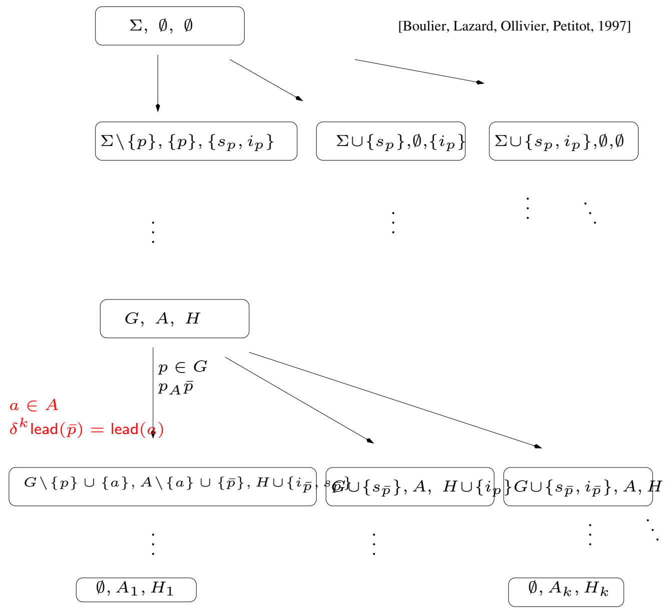
   (A_i, H_i) regular differential systems [BLOP 97]

2. $(A_i)\!:\!H_i^\infty = \bigcap_j (C_{ij})\!:\!S_{ij}^\infty$

3. $[\![\Sigma]\!] = \bigcap_{ij} [C_{ij}]\!:\!S_{ij}^\infty$

[H 00]

Algorithm for the decomposition in Regular Differential Systems - Case of a single derivation



Bibliography

BLOP97 Boulier, Lazard, Ollivier & Petitot., Computing representations for radicals of finitely generated differential ideals, LIFL:IT-306, 1997.

H00 : E. Hubert. Factorization free decomposition algorithms in differential algebra, Journal of Symbolic Computations, 2000, volume 29(4-5) pp641-662

HLR03: E. Hubert & N. Le Roux. Computing Power Series Solutions of Nonlinear PDE systems, ISSAC 2003.

H05 : E. Hubert. Differential Algebra for Derivations with Nontrivial Commutation Rules, Journal of Pure and Applied Algebra. Volume 200, Issues 1-2, 1 August 2005, Pages 163-190.