Finite Tensor Categories and a certain class of Frobenius algebras

Miodrag C Iovanov
University of Southern California, LA

CUNY Graduate Center
April 30th, 2010

Frobenius and QF Algebras

Definition

(i) An algebra A is Frobenius if $A \cong A^\vee$ where $V^\vee$ denotes the dual of the vector space V.

(ii) A f.d. (Artinian...) algebra is called quasi-Frobenius if the following equivalent conditions hold:

- A is left (or right) injective.
- Every left (or right) injective module is projective.
- Every left (or right) projective module is injective.
- The function { left (right) f.d. projectives } $P \longmapsto P^\vee$ { right (left) f.d. injectives } is well defined (and then, consequently, bijective).
- The function { left (right) f.d. injectives } $Q \longmapsto Q^\vee$ { right (left) f.d. projectives } is well defined (and then, consequently, bijective).
- A is weakly isomorphic to $A^\vee$, that is, there are some coproduct powers of these modules which are isomorphic: $A^{(I)} \cong (A^\vee)^{(J)}$.

Frobenius and QF Algebras

Note

Every QF algebra is Morita equivalent to a Frobenius algebra.

Frobenius and QF Algebras

Note

Every QF algebra is Morita equivalent to a Frobenius algebra.

So then, what's the difference? Let $S_1, ..., S_n$ be "the" simple left A-modules. $A/Jac(A) = \bigoplus S_i^{n_i}$,
$A = \bigoplus_i P_i^{n_i}$, where P_i are projective covers of S_i (i.e. P_i local and $P_i \to S_i \to 0$). Let $P_i' \to S_i^\vee$ be the projctive right modules.
$A^\vee = \bigoplus_j (P_j'^\vee)^{n_j}$.

Frobenius and QF Algebras

Note

Every QF algebra is Morita equivalent to a Frobenius algebra.

So then, what's the difference? Let $S_1, ..., S_n$ be "the" simple left A-modules. $A/Jac(A) = \bigoplus S_i^{n_i}$,
$A = \bigoplus_i P_i^{n_i}$, where P_i are projective covers of S_i (i.e. P_i local and $P_i \to S_i \to 0$). Let $P_i' \to S_i^\vee$ be the projctive right modules.
$A^\vee = \bigoplus_j (P_j'^\vee)^{n_j}$.

Suppose A is QF. Let $j = \sigma(i)$ if $P_j'^\vee \cong P_i$.
An isomorphism $A \cong A^\vee$ means
$A = \bigoplus_i P_i^{n_i} \cong A^\vee = \bigoplus_j (P_j'^\vee)^{n_j} \cong \bigoplus_i P_i^{n_{\sigma(i)}}$. So we need $n_i = n_{\sigma(i)}$

Frobenius and QF Algebras

n_i= multiplicity of S_i;
$n_{\sigma(i)}$=multiplicity of $S_{\sigma(i)}$.

Since $P_i^\vee = P'^\vee_{\sigma(i)} \to S^\vee_{\sigma(i)} \to 0$ (a proj cover), we have $0 \to S_{\sigma(i)} \to P_i$ (an injective envelope). Thus we need that
the multiplicity of the socle of P_i (the bottom) equals **the multiplicity of the cosocle** of P_i (the top).

Frobenius and QF Algebras

n_i= multiplicity of S_i;
$n_{\sigma(i)}$=multiplicity of $S_{\sigma(i)}$.

Since $P_i^\vee = P'^\vee_{\sigma(i)} \to S^\vee_{\sigma(i)} \to 0$ (a proj cover), we have $0 \to S_{\sigma(i)} \to P_i$ (an injective envelope). Thus we need that
the multiplicity of the socle of P_i (the bottom) equals **the multiplicity of the cosocle** of P_i (the top).

Note

When the basefield K is algebraically closed (or $A/Jac(A)$ is a product of blocs of K-matrices), **multiplicity** of the simple module S_i is the **dimension** of S_i. So
Frobenius $\Leftrightarrow$ QF + "dim of the top of P_i equals dim of the bottom of P_i".

Weak Hopf algebras and quasi-Hopf algebras

quasi-Hopf algebras = Hopf algebras which are only coassociative only up to an invertible twist
coquasi-Hopf algebras, or **dual quasi-Hopf algebras** = Hopf algebras which are only associative up to a twist
weak Hopf algebras = "Hopf algebras over a base", that is an algebra and a coalgebra, onlu with "weaker" axioms for the antipode and counit.

Notations

General starting category: $\mathcal{C} = Rep(A)$, for an algebra A.
$\mathcal{C}$ - finite tensor category:

finite - finitely many simples, semiperfect (projective covers exist for simples), all objects have finite length and $\mathrm{Hom}(S, S)$ finite dimensional for simple objects S (or all $\mathrm{Hom}(M, N)$ are f.d.). tensor - monoidal, rigid, and 1 is **simple** (in general, 1 is semisimple - multitensor category)
category - K-category

Tannakian reconstruction

Notions: Monoidal category, dual objects (rigid category), tensor functor = faithful and $F(A \otimes B) \cong F(A) \otimes F(B)$, $F(I) = I$ with "compatible" isomorphisms. quasi-tensor functor = we do not require compatibility for these isomorphisms.

Tensor category = Monoidal + ridgid. + 1 is simple.

Tannakian reconstruction

Notions: Monoidal category, dual objects (rigid category), tensor functor $=$ faithful and $F(A \otimes B) \cong F(A) \otimes F(B)$, $F(I) = I$ with "compatible" isomorphisms. quasi-tensor functor $=$ we do not require compatibility for these isomorphisms.

Tensor category $=$ Monoidal $+$ ridgid. $+$ 1 is simple.

Let γ be a tensor category. Tannakian duality: a certain one-one correspondence (equivalence of suitably defined categories) between Tensor Functors $\mathcal{C} \to \gamma$ and Hopf algebras in γ. In general,

$$\mathrm{Hopf}(\gamma) \ni H \longmapsto (\mathrm{Rep}(H) = H - \mathrm{mod} \to \gamma) \in (* \to \gamma)$$

has a *left* adjoint.

In many situations, there is a Tannakian reconstruction $C \longrightarrow R - Bimod$ $< --\text{to}-- >$ $(\mathrm{comod} - H)_c)$ (must be f.g. proj over R...) with fibre functors, where H is a Hopf algebra in $R - Bimod$, for a ring R.

Examples

1. compact groups G / $Rep_c(G)$ ¡− − −¿ **semisimple** symmetric tensor $\mathbb{C}$-categories, with $\dim \mathrm{Hom}(S,S) = 1$ for simple objects, together with fibre functor to $Vect_{\mathbb{C}}$ / comod-$R_c(G)$.
2. algebraic groups G / $Rep_a(G)$ /f.g. commutative Hopf algebras ¡− − −¿ **symmetric** rigid tensor categories (and equal left and right duals), together with fibre functor to $Vect_k$ / comod-$R_a(G)$.
3. algebraic group schemes / commutative Hopf algebras H (representing the scheme as a representable functor) $<--->$ neutral Tannakian categories (ridgid tensor categories, with same left and right duals) with fibre functor to $Vect_K$ / comod-H.
4. Hopf algebras H $<--->$ finite tensor categories C with fibre functor $C \to Vect_k$ / comod-H / mod-H^*.

Examples

5. quasi-Hopf algebras / co-quasi Hopf algebras $<--->$ finite tensor categories C with fibre **quasi**-tensor functor $C \to Vect_k$ / comod-H / mod-H^*

6. weak-Hopf algebras L / Hopf algebras H in $A-\mathrm{mod}$ $<--->$ finite tensor categories C with fibre functor $C \to A-\mathrm{mod}$, A-**semisimple separable** / comod-H / mod-L

7. "weak quasi-Hopf algebras" L / Hopf algebras H in $A-\mathrm{mod}$ $<--->$ finite tensor categories C with fibre **quasi**-tensor functor $C \to A-\mathrm{mod}$ / comod-H / mod-L

?. differential algebraic groups / a "suitable" commutative Hopf algebra of representative functions / commutative Hopf algebras in the category $K[\delta_1, ..., \delta_n] - Mod_c$ $<--->$ neutral Tannakian categories C with fibre functors to $K[\delta_1, ..., \delta_n] - Mod$.

Notations

$F : \mathcal{C} = \mathrm{Rep}(H) \to \mathrm{Bimod}(A)$ (quasi)tensor functor.
$(V_i)_{i \in I}$ - the simple objects in $\mathcal{C}$, with P_i covers.
The vector space dimension of an H-module M is $\dim_K(F(M))$.
Then $H = \bigoplus_{i \in I} P_i^{\dim(F(V_i))}$.
$(S_j)_{j=1,p}$ the simple right A-modules; $S_{ij} = S_i^\vee \otimes_K S_j$ are the simple A-bimodules. $d_i := \dim_K(S_i)$.
For X of $\mathcal{C}$ define N_{Xj}^k defined by the left multiplication by X, where N_{Xj}^k is the multiplicity $[X \otimes V_j : V_k]$ of V_k in the Jordan-Hölder series of $X \otimes V_j$ in $\mathcal{C}$.

Some fairly imediate facts

(Etingof,Ostrik)

For P_i, $i \in I$, there is $D(i) \in I$ be such that $P_i^* \simeq P_{D(i)}$ (here $(-)^*$ denotes the **categorical right dual**).

Also, there is an invertible object V_ρ of $\mathcal{C}$ such that $P_{D(i)} = P_{*i} \otimes V_\rho$ and $V_{D(i)} = V_{*i} \otimes V_\rho = {}^*V_i \otimes V_\rho$, where we convey $V_{*i} = {}^*V_i$.

Counting dimensions

Proposition

Denote $[F(X) : S_{ij}]$ *the multiplicity of* S_{ij}. *Then*

$$\begin{aligned}\dim_K(F(soc(P_k))) &= \sum_{i,j}[F(V_{D(k)}) : S_{ij}]d_i d_j \\ \dim_K(F(cosoc(P_k))) &= \sum_{i,j}[F({}^*V_k) : S_{ij}]d_i d_j\end{aligned}$$

A Proof

We have $P_k^* \to cosoc(P_k^*) \to 0$, equivalently, by taking left duals, we get $0 \to {}^*cosoc(P_k^*) \to {}^*(P_k^*) = P_k$ so $soc(P_k) = {}^*cosoc(P_k^*)$. Also, $\dim_K(F({}^*X)) = \dim_K(F(X^*)) = \dim_K(F(X))^\vee = \dim_K(F(X))$ (in Bimod(A) left and right duals are the same). Therefore

$$
\begin{aligned}
\dim_K(F(soc(P_k))) &= \dim_K(F({}^*cosoc(P_k^*))) \quad \text{(by duality)} \\
&= \dim_K(F({}^*cosoc(P_{D(k)}))) \quad (P_{D(k)} \simeq P_k^*) \\
&= \dim_K(F({}^*V_{D(k)})) = \dim_K(F(V_{D(k)})) \\
&= \sum_{i,j} [F(V_{D(k)}) : S_{ij}] d_i d_j
\end{aligned}
$$

The second equality follows similarly.

A first result

If X, Y are objects of $\mathcal{C}$, then the matrix $M_X = [F(X) : S_{ij}]_{i,j=1,n}$ has integer coefficients, and moreover, $M_{X\otimes Y} = M_X M_Y$.

A first result

If X, Y are objects of $\mathcal{C}$, then the matrix $M_X = [F(X) : S_{ij}]_{i,j=1,n}$ has integer coefficients, and moreover, $M_{X \otimes Y} = M_X M_Y$.

Theorem

Let H be a weak quasi-Hopf algebra with the base algebra A. If the dimensions of the simple components of A are all equal, then H is a Frobenius algebra. In particular, this is true if the base algebra A is commutative, so also when H is a quasi-Hopf algebra.

Another proof

Since $V_{D(k)} = {}^*V_k \otimes V_\rho$, $M_{V_{D(k)}} = M_{{}^*V_k} \cdot M_{V_\rho}$.
V_ρ is invertible $\Rightarrow M_{V_\rho} \cdot M_{V_\rho^{-1}} = M_{V_\rho^{-1}} \cdot M_{V_\rho} = M_{V_\rho \otimes V_\rho^{-1}} = M_{\mathbf{1}} = \mathrm{Id}$, so M_{V_ρ} is a permutation matrix (has $\mathbb{Z}$-coefficients and so does its inverse $M_{V_\rho^{-1}}$)
col's and elements of $M_{V_{D(k)}} = [F(V_{D(k)}) : S_{ij}]_{i,j=1,p}$ are a permutation of the col's and elements of $M_{{}^*V_k} = [F({}^*V_k) : S_{ij}]$.
$d = d_i = d_j$ for all i, j (e.g. A-commutative)

$$\begin{aligned} \dim_K(F(soc(P_k))) &= d^2 \sum_{i,j} [F(V_{D(k)}) : S_{ij}] \\ &= d^2 \sum_{i,j} [F({}^*V_k) : S_{ij}] \\ &= \dim_K(F(cosoc(P_k))) \end{aligned}$$

The Main Example

(Taft algebras)

$B=$ Taft algebra of dimension p^2:
has generators g, x with $g^p = 1$, $x^p = 0$, $xg = \lambda gx$ with λ a primitive p'th root of unity, and comultiplication $\Delta(g) = g \otimes g$, $\Delta(x) = g \otimes x + x \otimes 1$, counit $\varepsilon(g) = 1$, $\varepsilon(x) = 0$ and antipode $S(g) = g^{-1}$, $S(x) = -g^{-1}x$.

The Main Example

(Taft algebras)

$B=$ Taft algebra of dimension p^2:
has generators g, x with $g^p = 1$, $x^p = 0$, $xg = \lambda gx$ with λ a primitive p'th root of unity, and comultiplication $\Delta(g) = g \otimes g$, $\Delta(x) = g \otimes x + x \otimes 1$, counit $\varepsilon(g) = 1$, $\varepsilon(x) = 0$ and antipode $S(g) = g^{-1}$, $S(x) = -g^{-1}x$.

Denote V_k the 1-dimensional B-module K with structure $x \cdot \alpha = 0$ and $g \cdot \alpha = \lambda^k \alpha$ - the simple B-modules.

Taft's algebra

$P_k \to V_k \to 0$ projective covers are in fact chain modules:
There is a Jordan-Hölder series of $I_k^k = P_k$,
$0 =\subset I_k^1 \subseteq I_k^2 \subseteq \cdots \subseteq I_k^{p-1} \subseteq I_k^p$ and the terms of these series are
$I_k^i / I_k^{i-1} \simeq V_{k+i-p}$

So we have the quotients of the J-H series:
$[V_{k+1}, \dots, V_p, V_1, \dots, V_{k-1}, V_k]$.

The tensor functor

We now build a tensor functor $F : \mathrm{Rep}(B) \to \mathrm{Bimod}(A)$ in several steps.

$$\begin{array}{ccc}
\mathrm{Rep}(B) & \xrightarrow{F_1} & \mathrm{Rep}(\mathbb{Z}/p) \\
\Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle F_2} \\
\mathrm{Bimod}(A) & \xleftarrow{F_3} & \mathrm{Bimod}(K[\mathbb{Z}/p])
\end{array}$$

The tensor functor

• $F_1 : \mathrm{Rep}(B) \to \mathrm{Rep}(\mathbb{Z}/p)$ be the forgetful functor, given by $< 1, g, ..., g^{p-1} > \simeq K[\mathbb{Z}/p] \hookrightarrow B$.

• $F_2 : K[\mathbb{Z}/p]-\mathrm{mod} = \mathrm{Rep}(\mathbb{Z}/p) \to \mathrm{Bimod}(K[\mathbb{Z}/p])$,
$F_2(V_k) = \bigoplus_{i+j=k} V_i^* \otimes_K V_j = \bigoplus_i V_{-i} \otimes_K (V_{-i} \otimes V_k)$. Left adjoint of

$$G : \mathrm{Bimod}(\mathbb{Z}/\mathrm{p}) = \mathrm{Rep}(\mathbb{Z}/\mathrm{p} \times \mathbb{Z}/\mathrm{p}) \longrightarrow K[\mathbb{Z}/p]-\mathrm{mod} = \mathrm{Rep}(\mathbb{Z}/p)$$

, induced by the diagonal map $K[Z/p] \to K[Z/p] \otimes K[Z/p]$ (from $\mathbb{Z}/p \ni i \mapsto (-i, i) \in \mathbb{Z}/p \times \mathbb{Z}/p$).

• $A = \bigoplus_{i=1}^{p} M_{d_i}(K)$ and $F_3 : \mathrm{Bimod}(\mathbb{Z}/p) \to \mathrm{Bimod}(A)$,
$F_3(V_i^* \otimes V_j) = S_i^\vee \otimes S_j = S_{ij}$.

When $\mathrm{Taft}(d_1, d_2, \ldots, d_n)$ is Frobenius

Proposition

With the notations above, the weak Hopf algebra H is a Frobenius algebra if and only if $d_1, \ldots, d_p$ are all equal. Also, the algebra H has dimension $(\sum_i d_i)^4$. Thus, if the d_i's are not all equal, H is a weak Hopf algebra which is not a Frobenius algebra.

Proof

$\dim_K(F(soc(P_k))) = \dim_K(F(V_{k+1})) = \dim_K(\bigoplus_{i+j=k+1} S_i^\vee \otimes S_j) = \sum_{i+j=k+1} d_i d_j$ and also $\dim_K(F(cosoc(P_k))) = \dim_K(F(V_k)) = \sum_{i+j=k} d_i d_j$.
H is Frobenius $\Leftrightarrow$ these two are equal for all k. Let $1 \neq \omega$ be a p'th root of 1 and $t(x) = \sum_{k=0}^{p-1} d_k x^k$.

$t(\omega)^2 = \sum_{i,j} d_i d_j \omega^{i+j} = \sum_{k=0}^{p-1} \sum_{i+j=k} d_i d_j \omega^k = (\sum_i d_i d_{-i}) \cdot (\sum_k \omega^k) = 0$ (indices are mod p). So t is divisible by $\sum_{k=0}^{p-1} x^p$, i.e. they are proportional. Hence all d_i are equal.

The "correct" version of weak Hopf algebras are Frobenius

Proposition

If $\mathcal{C}$ is a finite tensor category, then $d_+(soc(P_k)) = d_+(cosoc(P_k))$, where d_+ represents the Frobenius-Perron dimension in $\mathcal{C}$.

Proof. As in Proposition 12, $soc(P_k) = {}^*L_{D(k)}$, so we compute
$d_+(soc(P_k)) = d_+({}^*L_{D(k)}) = d_+(L_{D(k)}) = d_+({}^*L_k \otimes L_\rho) =$
$d_+({}^*L_k)d_+(L_\rho) = d_+(L_k) = d_+(cosoc(P_k))$. □

Perhapse Frobenius in other way?

Frobenius Extensions: $k \hookrightarrow H$

$A \hookrightarrow H$

$A \otimes A^{op} \hookrightarrow H$

None of the above... reason: transitivity of Frobenius Extensions.

$k \hookrightarrow A \hookrightarrow H$ and $k \hookrightarrow A \otimes A^{op} \hookrightarrow H$

Article

MCI, Lars Kadison, * *When Weak Hopf Algebras are Frobenius*, **Proceedings of the American Mathematical Society**, Volume 138, Number 3, March 2010, Pages 837845.

*with special thanks to P.Etingof, MIT

T

T H

T H A

T H A N

T H A N K

T H A N K Y

THANK YO

THANK YOU

THANK YOU!