Representations of an Algebra of Quantum Differential Operators

U.N. Iyer, Bronx Community College

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- The Generalized Weyl Algebras (GWAs) were introduced independently by A. Rosenberg (1995) and V. Bavula (1993).
- **GWA of rank 1:** Let R be a ring, $\sigma : R \to R$ be an automorphism and t be a central element of R. The GWA $A = A(R, \sigma, t)$ is the ring extension of R generated by X, Y modulo the relations

$$YX = t, \quad XY = \sigma(t),$$
$$Xr = \sigma(r)X, \quad Yr = \sigma^{-1}(r)Y \quad \forall r \in R.$$

- A is free as an R-module with basis $\{1, X^n, Y^n \mid n \in \mathbb{N}\}$.

- Let $R = \Bbbk[T]$ the polynomial ring over some field $\Bbbk$. Let $t = T$ and $\sigma : R \to R$ defined by $\sigma(T) = T + 1$.
- Then the GWA $A = A(R, \sigma, t)$ is generated over $\Bbbk[T]$ by varaibles X, Y and relations $YX = T, XY = T + 1$, and $Xf = \sigma(f)X$, $Yf = \sigma^{-1}(f)Y$.
- That is, $XY - YX = 1$.
- That is, A is the first Weyl Algebra over $\Bbbk$, denoted A_1.

Examples of Generalized Weyl algebras

- Let $R = \Bbbk[T]$ the polynomial ring over some field $\Bbbk$, and $q \in \Bbbk^*$. Let $t = T$ and $\sigma : R \to R$ defined by $\sigma(T) = qT + 1$.
- Then the GWA $A = A(R, \sigma, t)$ is generated over $\Bbbk[T]$ by varaibles X, Y and relations $YX = T, XY = qT + 1$, and $Xf = \sigma(f)X,\ Yf = \sigma^{-1}(f)Y$.
- That is, $XY - qYX = 1$.
- That is, A is the first quantum Weyl Algebra over $\Bbbk$, denoted A_1^q.

- Let $R = \mathbb{k}[H, T]$, the polynomial algebra in two variables. Let $t = T$ and $\sigma : R \to R$ defined by $\sigma(H) = H + 2$ and $\sigma(T) = Y + H$.
- In this case, $A = A(R, \sigma, t)$ is generated over R by two variables X, Y with $YX = T, XY = T + H$, $Xf = \sigma(f)X$, and $Yf = \sigma^{-1}(f)Y$.
- That is, $XY - YX = H$, $XH - HX = 2X$, $YH - HY = -2Y$.
- A is the universal enveloping algebra over $sl_2(\mathbb{k})$.
- That is, $A = U(sl_2)$.

Examples of Generalized Weyl algebras

- Let $R = \mathbb{k}[T, Z, Z^{-1}]$, the localization of polynomial algebra in two variables $\mathbb{k}[T, Z]$ with respect to the multiplicative set $\{Z^i\}$. Let $q \in \mathbb{k} \setminus \{0, -1, 1\}$ and $t = T$. Define $\sigma : R \to R$ by $\sigma(Z) = q^2 Z$ and $\sigma(T) = \dfrac{Z - Z^{-1}}{q - q^{-1}}$.
- In this case, $A = A(R, \sigma, t)$ is generated over R by two variables X, Y with $YX = T, XY = T + \dfrac{Z - Z^{-1}}{q - q^{-1}}$, $XZ = q^2 ZX$, and $YZ = q^{-2} Y$.
- That is, $ZXZ^{-1} = q^2 X$, $ZYZ^{-1} = q^{-2} Y$, and $XY - YX = \dfrac{Z - Z^{-1}}{q - q^{-1}}$.
- A is the quantum group over $sl_2(\mathbb{k})$.
- That is, $A = U_q(sl_2)$.

Weight Modules

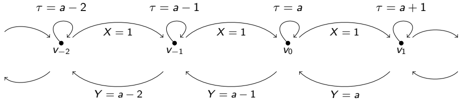
- Let R be a commutative ring. Let V be an R-module.
- Denote by $Max(R)$, the set of all maximal ideals of R.
- For $m \in Max(R)$, let $V_m = \{v \in V \mid mv = 0\}$, the *weight space* of V corresponding to *weight* m.
- The *support* of V is the set $Supp(V) = \{m \in Max(R) \mid V_m \neq 0\}$.
- We say that V is a *weight module* if $V = \sum_{m \in Max(R)} V_m = \bigoplus_{m \in Max(R)} V_m$.
- Now suppose A is a ring with R a fixed commutative subalgebra of A, and let V be an A-module.
- Then we call V a weight module over A if V is a weight module over R.

- Let R be a commutative ring, $t \in R$, $\sigma : R \to R$ be an automorphism and $A = A(R, \sigma, t)$ be a GWA of rank 1.
- The cyclic group $< \sigma >$ acts on $Max(R)$. Let Ω be the orbit set of $Max(R)$ under this action.
- Note: $X \cdot V_m \subset V_{\sigma(m)}$ and $Y \cdot V_m \subset V_{\sigma^{-1}(m)}$ for $m \in Max(R)$.
- Therefore, any weight module V decomposes into a direct sum:

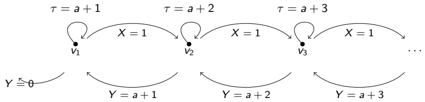
$$V = \oplus_{\omega \in \Omega} V_\omega$$

  with $Supp(V_\omega) \subset \omega$.
- For $m \in Max(R)$, we call m a break if $t \in m$.
- Let B be the set of all breaks.
- For $\omega \in \Omega$, denote $B_\omega = B \cap \omega$, the set of breaks in ω.

- The Indecomposable Weight Modules over a GWA of rank 1 were classified by Drozd, Guzner, and Ovsienko in 1996.
- The indecomposable weight modules fall in five families.
- **Family I:** Here, $|\omega| = \infty$ and $B_\omega = \emptyset$.
- An example: Let $a \in \mathbb{C} \setminus \mathbb{Z}$. Then the action of A_1 on the vector space $x^a\mathbb{C}[x, x^{-1}]$ is indecomposable with linear orbit and no breaks.



- **Family II:** Here, $|\omega| = \infty$, $B_\omega \neq \emptyset$.
- An example of this family is as follows:
- Let the characteristic of $\mathbb{k}$ be $p > 0$. Then $\mathbb{k}[x]$ is an A_1 indecomposable module which is not irreducible.



- **Family III:** Here, $|\omega| < \infty$ and $B_\omega = \emptyset$.
- Here we obtain finite dimensional representations. The orbit is *circular*.
- For example, when the characteristic of $\Bbbk$ is $p > 0$, then A_1 has finite dimensional irreducible modules.
- **Family IV, Family V:** These families occur when $|\omega| < \infty$ and $B_\omega \neq \emptyset$.

We follow here definitions and constructions given by V. Lunts and A. L. Rosenberg in a series of papers (1997-1999).

- Let $\Bbbk$ be a field, Γ be an abelian group, R be a Γ-graded $\Bbbk$-algebra. Fix a bicharacter $\beta : \Gamma \times \Gamma \longrightarrow \Bbbk^*$.
- For each $a \in \Gamma$, define $\sigma_a \in \mathrm{grHom}_{\Bbbk}(R, R)$ defined by $\sigma_a(r) = \beta(a, d_r)r$ for homogeneous $r \in R$, and extend σ_a linearly on all of R. We call σ_a the **grading operator.** Note, σ_a is an automorphism with $\sigma_a^{-1} = \sigma_{-a}$.
- Let $\mathcal{Z}_q$ denote the *quantum-center* of $\mathrm{grHom}_{\Bbbk}(R, R)$ defined as the $\Bbbk$-span of homogeneous homomorphisms φ for which there exists an $a \in \Gamma$ such that
$$\varphi r = \sigma_a(r)\varphi \quad \text{for } r \in R.$$
- Let $[\varphi, r]_a = \varphi r - \sigma_a(r)\varphi$. Using these notations,
$$\mathcal{Z}_q = \Bbbk - span\{homogeneous\ \varphi \mid \exists a \in \Gamma \ such\ that\ [\varphi, r]_a = 0 \forall r \in R\}.$$

- Let $D_q^0 = R\mathcal{Z}_q R$.
- For $i \geq 1$, D_q^i denotes the R-bimodule generated by the set

$$\mathbb{k} - span\{homogeneous\ \varphi \mid \exists a \in \Gamma\ such\ that\ [\varphi, r]_a \in D_q^{i-1} \forall r \in R\}.$$

- $D_q^0 \subset D_q^1 \subset \cdots$ and $D_q = \cup_{i\geq 0} D_q^i$.
- The algebra $D_q^0(R)$ is generated by the set $\{\lambda_r, \rho_s, \sigma_a \mid r, s \in R, a \in \Gamma\}$ where

$$\lambda_r(t) = rt,\ \ \rho_r(t) = tr\ \forall t \in R.$$

- We see the following relations:

$$\lambda_r \rho_s = \rho_s \lambda_r,\ \ \sigma_a \lambda_r = \lambda_{\sigma_a(r)} \sigma_a,\ \ and\ \sigma_a \rho_r = \rho_{\sigma_a(r)} \sigma_a.$$

- Example (*Joint with T.C.McCune*): Let q be transcendental over $\mathbb{Q}$ and $\mathbb{Q}(q) \subset \mathbb{k}$. Let $R = \mathbb{k}[x]$, $\Gamma = \mathbb{Z}$, $deg(x) = 1$, and $\beta : \Gamma \times \Gamma \to \mathbb{k}^*$ be given by $\beta(n, m) = q^{nm}$.
- Let $\sigma : R \to R$ be the authormorphism given by $\sigma(x^n) = q^n x^n$, and extend σ linearly. Denote by σ^{-1} the inverse of σ. That is, $\sigma^{-1}(x^n) = q^{-n}x^n$ and extended linearly.
- Let $\partial_1, \partial, \partial_{-1} : R \to R$ be the linear maps defined by

$$\partial_1(x^n) = \left(\frac{q^n - 1}{q - 1}\right) x^{n-1}, \quad \partial(x^n) = nx^{n-1}, \quad \partial_{-1}(x^n) = \left(\frac{q^{-n} - 1}{q^{-1} - 1}\right) x^{n-1}.$$

- The algebra $D_q(R)$ is generated by the set $\{\lambda_x = x, \partial_1, \partial, \partial_{-1}\}$.
- In a joint work with D.A. Jordan, we have proved that this algebra is left and right Noetherian algebra. Moreover, $D_q(R)$ is a simple domain of GK dimension 3.
- The defining relations among the generators $\{x, \partial_1, \partial, \partial_{-1}\}$ are: $(a, b \in \{-1, 0, 1\})$

$$\partial_a x - q^a x \partial_a = 1, \quad \partial_a x \partial_b = \partial_b x \partial_a, \quad \partial_{-1}\partial_1 = q\partial_1\partial_{-1}.$$

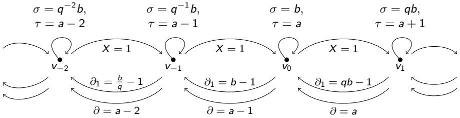
- Note, $\sigma = \partial_1 x - x\partial_1$ and $\sigma^{-1} = \partial_{-1}x - x\partial_{-1}$. One can check that $\partial_{-1} = \sigma^{-1}\partial_1$.
- Let $\tau = \partial x$.
- Then, $D_q = D_q(\mathbb{k}[x])$ is generated over $\mathbb{k}[\tau, \sigma, \sigma^{-1}]$ with generating set $\{x, \partial, \partial_1\}$.
- Note that, D_q has two GWAs of rank 1 generated over $\mathbb{k}[\tau, \sigma, \sigma^{-1}]$:
- A_1 is generated by $\{x, \partial\}$ over $\mathbb{k}[\tau, \sigma, \sigma^{-1}]$;
- A_q is generated by $\{x, \partial_1\}$ over $\mathbb{k}[\tau, \sigma, \sigma^{-1}]$.

Weight Modules of D_q

- The following is joint work with V. Futorny.
- We study the weight modules over D_q as an algebra over $\mathbb{k}[\tau, \sigma, \sigma^{-1}]$.
- Let $\mathbb{k} = \overline{\mathbb{k}}$.
- **Family I:** Irreducible D_q modules which are extended from irreducible A_1 modules.
- An example: Suppose the characteristic of $\mathbb{k}$ is 0. Then, the natural action of A_1 on $\mathbb{k}[x]$ is irreducible. If q is a root of 1, then the natural action of A_q on $\mathbb{k}[x]$ is not irreducible.

Weight Modules of D_q

- **Family II:** Irreducible D_q modules which are extended from irreducible A_q modules.
- An example: Suppose the characteristic of $\mathbb{k}$ is $p > 0$. Then, the natural action of A_1 on $\mathbb{k}[x]$ is not irreducible. If q is not a root of 1, then the natural action of A_q on $\mathbb{k}[x]$ is irreducible.
- **Family III:** Irreducible D_q modules which do not descend to irreducible A_1 or A_q modules.
- An example:



When $a \in \mathbb{Z}$ and $b = q^i$ for some $i \in \mathbb{Z}$, this example descends to indecomposable (but not irreducible) modules over A_1 and A_q. As long as $(\partial, \partial_1) \neq (0,0)$ simultaneously, this is an irreducible D_q module.

- **Theorem [V. Futorny, U.I]** Let $\mathbb{k} = \overline{\mathbb{k}}$. Every irreducible D_q weight module is indecomposable as an A_1 and as an A_q module.
- **Theorem [V. Futorny, U.I]** Let $\mathbb{k} = \overline{\mathbb{k}}$. The irreducible D_q weight modules are described in Families I, II, and III.

Future Work

- With V. Futorny, we are investigating the irreducible weight modules of $D_q(\mathbb{k}[x_1, x_2, \cdots, x_n])$.
- Several works have been done on weight modules of A_n and A_q^n.
- Some of the authors being: V. Bekkert, G. Benkart, V. Futorny, J. Hartwig.
- Thank you.