Generic Base Algebras and Universal Comodule Algebras for some finite-dimensional Hopf algebras

U.N. Iyer, C. Kassel

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- E. Aljadeff and C. Kassel associated to any finite-dimensional Hopf algebra H an algebra $\mathcal{B}_H$ of rational fractions, which is a finitely generated smooth domain of Krull dimension equal to the dimension of H. The algebra $\mathcal{B}_H$ is called the *generic base algebra* associated to H.

Definitions and motivation

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- In this talk, I will describe $\mathcal{B}_H$ for other finite-dimensional Hopf algebras such as the Taft algebras, the Hopf algebras $E(n)$ and certain monomial Hopf algebras, all natural generalizations of the Sweedler algebra. This is joint work with C. Kassel.

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- The generic base algebra is known for very few Hopf algebras: it has been described for finite group algebras in a work by Aljadeff, Haile, Natapov and for the four-dimensional Sweedler algebra by C. Kassel.
- In this talk, I will describe $\mathcal{B}_H$ for other finite-dimensional Hopf algebras such as the Taft algebras, the Hopf algebras $E(n)$ and certain monomial Hopf algebras, all natural generalizations of the Sweedler algebra. This is joint work with C. Kassel.
- A theory of polynomial identities for comodule algebras was also worked out by Aljadeff and Kassel. It leads naturally to a *universal comodule algebra* $\mathcal{U}_H$, the analogue of the "relatively free algebra" in the classical theory of polynomial identities. The subalgebra of coinvariants $\mathcal{V}_H$ of $\mathcal{U}_H$ maps injectively into $\mathcal{B}_H$. In the few known cases, the injection turns $\mathcal{B}_H$ into a localization of $\mathcal{V}_H$. We show that this also holds for the Hopf algebras considered here. Finally for the same Hopf algebras we also describe a suitable central localization of $\mathcal{U}_H$ as a $\mathcal{B}_H$-module.

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- Denote the coproduct of a Hopf algebra by Δ, its counit by ε, and its antipode by S. We also use a Heyneman-Sweedler-type notation

$$\Delta(x) = x_1 \otimes x_2$$

for the image under Δ of an element x of a Hopf algebra H, and we write

$$\Delta^{(2)}(x) = x_1 \otimes x_2 \otimes x_3$$

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- A (right) *H-comodule algebra* over a Hopf k-algebra H is an associative unital k-algebra A equipped with a right H-comodule structure whose (coassociative, counital) *coaction*

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is an algebra map.
- The subalgebra A^H of *coinvariants* of an H-comodule algebra A is the subalgebra

$$A^H = \{a \in A \,|\, \delta(a) = a \otimes 1\}\,.$$

- **Example of an H-comodule algebra: Twisted comodule algebra**
  A *two-cocycle* α on a Hopf algebra H is a bilinear form $\alpha : H \times H \to k$ satisfying the cocycle condition

$$\alpha(x_1, y_1)\,\alpha(x_2 y_2, z) = \alpha(y_1, z_1)\,\alpha(x, y_2 z_2)$$

  for all $x, y, z \in H$. We always assume that α is invertible (with respect to the convolution product) and normalized, i.e., $\alpha(x, 1) = \alpha(1, x) = \varepsilon(x)$ for all $x \in H$.

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- Let u_H be a copy of the underlying vector space of H. Denote the identity map u from H to u_H by $x \mapsto u_x$ ($x \in H$). The *twisted comodule algebra* ${}^{\alpha}H$ is defined as the vector space u_H equipped with the product given by

$$u_x\, u_y = \alpha(x_1, y_1)\, u_{x_2 y_2}$$

  for all $x,\ y \in H$. This product is associative thanks to the cocycle condition. As α is normalized, the unit of ${}^{\alpha}H$ is u_1.

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for all x, $y \in H$. This product is associative thanks to the cocycle condition. As α is normalized, the unit of ${}^{\alpha}H$ is u_1.

- The algebra ${}^{\alpha}H$ is an H-comodule algebra with coaction $\delta \colon {}^{\alpha}H \to {}^{\alpha}H \otimes H$ given for all $x \in H$ by

$$\delta(u_x) = u_{x_1} \otimes x_2 \,.$$

One can check that the subalgebra of coinvariants of ${}^{\alpha}H$ coincides with $k\, u_1$.

- **Another example of an H-comodule algebra: The tensor algebra** Take a copy X_H of H; the identity map from H to X_H sends an element $x \in H$ to the symbol X_x. The map $x \mapsto X_x$ is linear and is determined by its values on a linear basis of H. Now consider the tensor algebra on X_H:

$$T(X_H) = \bigoplus_{i \geq 0} T^i(X_H)\,,$$

where $T^i(X_H) = (X_H)^{\otimes i}$.
There is a tautological H-comodule algebra structure on $T(X_H)$ with coaction $\delta : T(X_H) \to T(X_H) \otimes H$ given on each generator X_x by

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- **Definition: H-identity** Given an H-comodule algebra A, we say that an element $P \in T(X_H)$ is an H-identity for A if $\mu(P) = 0$ for all H-comodule algebra maps $\mu : T(X_H) \to A$.

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- Denote the set of all H-identities for A by $\mathrm{Id}_H(A)$. The set $I_H(A)$ is a two-sided ideal, right H-coideal of $T(X_H)$, and it is preserved by all comodule algebra endomorphisms of $T(X_H)$.

- The quotient algebra $\mathcal{U}_H(A) = T(X_H)/I_H(A)$ is an H-comodule algebra such that the canonical surjection $T(X_H) \to \mathcal{U}_H(A)$ is a comodule algebra map. By definition, all H-identities for A vanish in $\mathcal{U}_H(A)$, which is the biggest quotient of $T(X_H)$ for which this happens. We call $\mathcal{U}_H(A)$ the *universal H-comodule algebra* attached to the H-comodule algebra A (in the classical literature on polynomial identities, $\mathcal{U}_H(A)$ is called the *relatively free algebra*).

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- **H-identities when A is a twisted comodule algebra:** Aljadeff and Kassel have shown that the the H-identities for ${}^{\alpha}H$ can be detected by a comodule algebra map

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  where the details are as follows:
- Consider a copy t_H of H, identifying each $x \in H$ linearly with the symbol $t_x \in t_H$. Define $S(t_H)$ to be the symmetric algebra on the vector space t_H.
  The algebra $S(t_H) \otimes {}^{\alpha}H$ is generated by the symbols $t_x u_y$ $(x, y \in H)$ as a k-algebra (we drop the tensor product sign $\otimes$ between the t-symbols and the u-symbols).

- It is a comodule algebra whose coaction is $S(t_H)$-linear and extends the coaction of ${}^{\alpha}H$:

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- **Aljadeff-Kassel:** An element $P \in T(X_H)$ is an H-identity for ${}^\alpha H$ if and only if $\mu_\alpha(P) = 0$. In other words, $I_H({}^\alpha H) = \operatorname{Ker} \mu_\alpha$.

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- Let $\mathcal{U}_H^\alpha = \mathcal{U}_H({}^{\alpha}H)$ and $I_H^\alpha = I_H({}^{\alpha}H)$. It follows from the previous item that μ_α induces an injection of comodule algebras

$$\mathcal{U}_H^\alpha = T(X_H)/I_H^\alpha \hookrightarrow S(t_H) \otimes {}^{\alpha}H \,,$$

  and by abuse of notation denote this map by μ_α.

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- **Localizing the symmetric algebra:** Aljadeff-Kassel showed that there is a unique linear map $x \mapsto t_x^{-1}$ from H to the field of fractions $\operatorname{Frac} S(t_H)$ of the symmetric algebra $S(t_H)$ such that for all $x \in H$,

$$\sum_{(x)} t_{x_1}\, t_{x_2}^{-1} = \sum_{(x)} t_{x_1}^{-1}\, t_{x_2} = \varepsilon(x)\, 1\,.$$

When x is a *group-like* element of H, i.e., such that $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$, then $t_x^{-1} = 1/t_x$.

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- Denote by $S(t_H)_\Theta$ the subalgebra of $\operatorname{Frac} S(t_H)$ generated by all elements t_x and t_x^{-1} $(x \in H)$.

- When H is a *pointed* Hopf algebra, $S(t_H)_\Theta$ has a simple description as the following localization of $S(t_H)$:

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- Note, the algebra $S(t_H)_\Theta$ carries a commutative Hopf algebra structure with coproduct Δ, counit ε and antipode S given for all $x \in H$ by

$$\Delta(t_x) = t_{x_1} \otimes t_{x_2} , \quad \Delta(t_x^{-1}) = t_{x_2}^{-1} \otimes t_{x_1}^{-1} ,$$

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- This Hopf algebra is Takeuchi's free commutative Hopf algebra on the coalgebra underlying H; it satisfies the following universal property: for any coalgebra map $f : H \to H'$ into a *commutative Hopf algebra* H', there is a unique Hopf algebra map $\tilde{f} : S(t_H)_\Theta \to H'$ extending f, i.e., such that $\tilde{f}(t_x) = f(x)$ for all $x \in H$.

- **The generic base algebra:** To a pair (H, α) consisting of a Hopf algebra H and a normalized convolution invertible two-cocycle α, we attach a bilinear map $\sigma_\alpha : H \times H \to S(t_H)_\Theta$ with values in the previously defined algebra $S(t_H)_\Theta$.

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- The map σ_α is given for all $x, y \in H$ by

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- We call σ_α the *generic cocycle* associated to α. The cocycle α being invertible, so is σ_α, with inverse σ_α^{-1} given for all $x, y \in H$ by

$$\sigma_\alpha^{-1}(x, y) = t_{x_1 y_1}\, \alpha^{-1}(x_2, y_2)\, t^{-1}_{x_3}\, t^{-1}_{y_3} .$$

- Define the *generic base algebra* $\mathcal{B}_H^\alpha$ attached to the pair (H, α) to be the subalgebra of $S(t_H)_\Theta$ generated by the values of the generic cocycle σ_α and of its inverse σ_α^{-1}.

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- (1) $\mathcal{B}_H^\alpha$ is a finitely generated smooth Noetherian domain of Krull dimension equal to $\dim_k H$; and
- (2) $S(t_H)_\Theta$ is a finitely generated projective $\mathcal{B}_H^\alpha$-module, from which it follows that $S(t_H)_\Theta$ is integral over $\mathcal{B}_H^\alpha$.

- It can be shown that the map μ_α induces an embedding of the subalgebra of coinvariants $\mathcal{V}_H^\alpha$ of $\mathcal{U}_H^\alpha$ into $\mathcal{B}_H^\alpha$:

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- A presentation by generators and relations of $\mathcal{B}_H^\alpha$ when H is the four-dimensional Sweedler algebra was given by Aljadeff and Kassel. They also showed that any two-cocycle on a group algebra or on the Sweedler algebra is nice.

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- Kassel and Masuoka showed that any two-cocycle on a cocommutative Hopf algebra is nice.

- **Functoriality:**
  Consider pairs (H, α), where H is a Hopf algebra and α is a normalized convolution invertible two-cocycle. We define a map of such pairs $(H, \alpha) \to (H', \alpha')$ to be a Hopf algebra map $\varphi : H \to H'$ such that $\alpha' \circ (\varphi \times \varphi) = \alpha$.

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- Then, the map φ_T induces a comodule algebra map $\varphi_U : \mathcal{U}_H^{\alpha} \to \mathcal{U}_{H'}^{\alpha'}$.
- Moreover, if φ_S is injective, then so is φ_U.

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  The *trivial* two-cocycle is given by

$$\alpha_0 : (x, y) \mapsto \varepsilon(x)\varepsilon(y) \qquad (x, y \in H)\,.$$

In this case it follows from (4) that ${}^{\alpha}H$ coincides as a H-comodule algebra with H itself, the coaction being the coproduct, and that the linear isomorphism $u : H \to u_H$ is a Hopf algebra map. This allows us to write x instead of u_x $(x \in H)$.

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- When $\alpha = \alpha_0$, we write I_H for I_H^α, $\mathcal{U}_H$ for $\mathcal{U}_H^\alpha$, $\mathcal{V}_H$ for $\mathcal{V}_H^\alpha$, and $\mathcal{B}_H$ for $\mathcal{B}_H^\alpha$.

- Note, I_H is the kernel of the comodule algebra map $\mu_0 : T(X_H) \to S(t_H) \otimes H$ given by

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- Also when $\alpha = \alpha_0$, we write $\sigma^{\pm 1}$ instead of $\sigma_\alpha^{\pm 1}$. In this case we have

$$\sigma(x, y) = t_{x_1}\, t_{y_1}\, t_{x_2 y_2}^{-1} \quad \text{and} \quad \sigma^{-1}(x, y) = t_{x_1 y_1}\, t_{x_2}^{-1}\, t_{y_2}^{-1} \,.$$

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- Kassel-Masuoka showed that the study of the generic base algebra attached to a non-trivial cocycle can be reduced to the study of the generic base algebra attached to the trivial cocycle.

- This works as follows: given a convolution invertible two-cocycle α on H, define the Hopf algebra $L = {}^{\alpha}H^{\alpha^{-1}}$ as the coalgebra H with the product

$$x * y = \alpha(x_1, y_1)\, x_2 y_2\, \alpha^{-1}(x_3, y_3) \qquad (x, y \in H)\,.$$

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- We concentrate on Hopf algebras equipped with the trivial cocycle.

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- When $n = 2$, the algebra H is the four-dimensional Sweedler algebra.

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$$S(t_H)_\Theta = S(t_H)\left[\frac{1}{t_{x^i}}\right]_{0 \leq i < n}.$$

- $S(t_H)_\Theta$ is a Hopf algebra. The coproduct on an element $t_{x^i y^j}$ is given by

$$\Delta(t_{x^i y^j}) = \sum_{r=0}^{j} \begin{bmatrix} j \\ r \end{bmatrix} t_{x^i y^r} \otimes t_{x^{i+r} y^{j-r}}\,.$$

- Consider the set Γ_0 consisting of the following n elements of $S(t_H)$: $t_x t_{x^{n-1}}$ and $t_{x^i}/(t_x)^i$ for $0 \leq i < n$, $i \neq 1$; these elements are invertible in $S(t_H)_\Theta$ and we denote by Γ_0^{-1} the set of inverses of the elements of Γ_0.

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- Let Γ_1 be the set consisting of the elements $t_{x^i y^j} t_{x^k}$, where $j \not\equiv 0$ and $i + j + k \equiv 0 \pmod{n}$; the cardinality of Γ_1 is $n(n-1)$.

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- **Iyer-Kassel:** Let $n \geq 3$.
  (a) The n^2 elements of $\Gamma_0 \cup \Gamma_1$ are algebraically independent.
  (b) The generic base algebra $\mathcal{B}_H$ is given by

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- When $n = 2$, the elements t_1, t_x^2, $t_x t_y$, t_{xy} are algebraically independent and

$$\mathcal{B}_H = k[(t_1)^{\pm 1}, (t_x^2)^{\pm 1}, t_x t_y, t_{xy}] .$$

- As a $\mathcal{B}_H$-algebra, we have

$$S(t_H)_\Theta \cong \mathcal{B}_H[t]/\left(t^n - (t_x)^n\right) .$$

That is, $S(t_H)_\Theta$ is a finite étale (hence integral) extension of $\mathcal{B}_H$ and it is a free $\mathcal{B}_H$-module of rank n.

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- **The universal comodule algebra:**
The tensor algebra $T(X_H)$ is the free algebra on the indeterminates $X_{x^i y^j}$ $(0 \leq i, j < n)$. We will use the same notation for the image of $X_{x^i y^j}$ in $\mathcal{U}_H = T(X_H)/I_H$.

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- Note that the comodule algebra map $\mu_0 : T(X_H) \to S(t_H) \otimes H$ defined by induces an embedding of $\mathcal{U}_H$ into $S(t_H) \otimes H$ and

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- Note that the comodule algebra map $\mu_0 : T(X_H) \to S(t_H) \otimes H$ defined by induces an embedding of $\mathcal{U}_H$ into $S(t_H) \otimes H$ and
- an element P belongs to the the subalgebra $\mathcal{V}_H$ of coinvariants of $\mathcal{U}_H$ if and only if $\mu_0(P) \in S(t_H) \otimes 1$.
- Moreover, since the center of H is one-dimensional, $\mathcal{V}_H$ coincides with the center of $\mathcal{U}_H$.

- Now, $\mu_0(X_1) = t_1$, hence $X_1 \in \mathcal{V}_H$, and

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- Let $\mathcal{V}'_H$ (resp. $\mathcal{U}'_H$) be the localization of $\mathcal{V}_H$ (resp. of $\mathcal{U}_H$) obtained by inverting the central elements X_1, $X^n_{x^i}$ for all $i = 1, \ldots, n-1$.

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- We have $\mathcal{V}'_H = \mathcal{B}_H$. That is, the trivial cocycle of a Taft algebra is nice.
- Moreover, There is an isomorphism of algebras

$$\mathcal{U}'_H \cong \mathcal{B}_H \langle \xi, \eta \,|\, \xi^n = X^n_x \,,\; \eta^n = 0 \,,\; \eta\xi - q\xi\eta = 0 \rangle \,,$$

where, the isomorphism is given by

$$f(\xi) = X_x \quad \text{and} \quad f(\eta) = X_y - \frac{t_y}{t_x} X_x \,.$$

- **The Hopf algebras $E(n)$:** Fix an integer $n \geq 1$. The algebra $H = E(n)$ is generated by $n+1$ elements x, $y_1, \ldots, y_n$ subject to the relations

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- By convention, $y_I = 1$ if $I = \emptyset$.

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is a basis of H.
- The elements $y_1, \ldots, y_n$ generate a subalgebra that is isomorphic to an exterior (or Grassmann) algebra.
- The algebra H is a Hopf algebra with coproduct Δ and counit ε defined by
$$\Delta(x) = x \otimes x\,, \qquad \Delta(y_i) = 1 \otimes y_i + y_i \otimes x\,,$$
$\varepsilon(x) = 1$ and $\varepsilon(y_i) = 0$ for all $i = 1, \ldots, n$.
- For this Hopf algebra, $S(t_H)$ is the polynomial algebra on the indeterminates t_{y_I} and t_{xy_I}, where I runs over all subsets of $\{1, \ldots, n\}$.

- The localization $S(t_H)_\Theta$ of $S(t_H)$ is obtained from $S(t_H)$ by inverting t_1 and t_x:

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The Hopf Algebras $E(n)$

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- Set $\Gamma_0 = \{t_1, (t_x)^2\}$ and $\Gamma_0^{-1} = \{t_1^{-1}, (t_x)^{-2}\}$.
- Set Γ_1 consists of the elements t_{y_I}, $t_x t_{xy_I}$, where $|I|$ is even ≥ 2, and of the elements $t_x t_{y_I}$, t_{xy_I}, where $|I|$ is odd. Observe that these elements are all of degree 0.

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- The 2^{n+1} elements of $\Gamma_0 \cup \Gamma_1$ are algebraically independent and the generic base algebra $\mathcal{B}_H$ is given by

$$\mathcal{B}_H = k[\Gamma_0, \Gamma_0^{-1}, \Gamma_1].$$

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- Let $\mathcal{V}'_H$ (resp. $\mathcal{U}'_H$) be the localization of $\mathcal{V}_H$ (resp. of $\mathcal{U}_H$) obtained by inverting the central elements X_1 and X_x^2.

- We have $\mathcal{B}_H = \mathcal{V}'_H$. That is, the trivial cocycle on $E(n)$ is nice.

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- Let $\mathcal{A}_H$ be the $\mathcal{B}_H$-algebra generated by $\xi, \eta_1, \ldots, \eta_n$ subject to the relations

$$\xi^2 = X_x^2\,, \quad \eta_1^2 = \cdots = \eta_n^2 = 0\,, \quad \eta_i\xi + \xi\eta_i = 0\,, \quad \eta_i\eta_j + \eta_j\eta_i = 0$$

for all $i, j = 1, \ldots, n$. There is an isomorphism of algebras $\mathcal{U}'_H \cong \mathcal{A}_H$.

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- **Monomial Hopf algebras of type *I*:** Consider a triple (G, x, χ), where G is a finite group, x a central element of G of order $n \geq 2$, and χ a character $G \to k^{\times}$ such that $\chi^n = 1$ and $\chi(x) = q$.

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- To such a triple one associates a Hopf algebra H, which is defined as an algebra with generators the elements g of G and an additional generator y; the defining relations are those of the group algebra kG as well as

$$y^n = 0\,, \qquad yg = \chi(g)gy$$

for all $g \in G$.

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- If $G = \mathbb{Z}/n$ and x is a generator of G, then H is the n^2-dimensional Taft algebra.
- Observe that$\mathcal{U}_{kG}$ (resp. $\mathcal{B}_{kG}$) splits off $\mathcal{U}_H$ (resp. $\mathcal{B}_H$). Similary, passing to the coinvariants, $\mathcal{V}_{kG}$ splits off $\mathcal{V}_H$.

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- The set Γ is algebraically independent over $\mathcal{B}_{kG}$ and

$$\mathcal{B}_H = \mathcal{B}_{kG}[\Gamma].$$

- **The universal comodule algebra** There is a localization $\mathcal{V}'_{kG}$ of $\mathcal{V}_{kG}$ such that $\mathcal{B}_{kG} = \mathcal{V}'_{kG}$ We define a localization $\mathcal{V}'_H$ of $\mathcal{V}_H$ by

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$$\mathcal{U}'_H = \mathcal{U}'_{kG} * k[\eta] \,/\, (\eta^n = 0,\ \eta X_g - \chi(g) X_g \eta = 0 \mid g \in G) \,.$$

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