

Algebraic theory of integrable PDE

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§1.

Evolution equation is a PDE of the form

$$(1) \quad \frac{du}{dt} = P(u, u', \dots, u^{(n)}),$$

where $u = \begin{pmatrix} u_1 \\ \vdots \\ u_\ell \end{pmatrix}$, $u_i = u_i(t, x)$ is a function in one independent variable x , and t (time) is a parameter;

$$P = \begin{pmatrix} P_1 \\ \vdots \\ P_\ell \end{pmatrix} \in V^\ell, V \text{ algebra of "differential functions"}.$$

Equation (1) is called *compatible* with another evolution equation

$$\frac{du}{dt_1} = Q(u, u', \dots, u^{(m)})$$

if "the corresponding flows commute":

$$\frac{d}{dt} \frac{d}{dt_1} u = \frac{d}{dt_1} \frac{d}{dt} u.$$

Compute the LHS using the chain rule:

$$\frac{d}{dt}Q(u, u', \dots, u^{(m)}) = \sum_{i; n \in \mathbb{Z}_+} \frac{\partial Q}{\partial u_i^{(n)}} \partial^n P_i = X_P Q,$$

where

$$\partial = \frac{d}{dx}$$

is the *total derivative*, and

$$X_P = \sum_{i; n \in \mathbb{Z}_+} (\partial^n P_i) \frac{\partial}{\partial u_i^{(n)}}$$

is the *evolutionary vector field* with characteristic $P \in V^\ell$. (Note that $[\partial, X_P] = 0$.)

Hence,

$$\left[\frac{d}{dt}, \frac{d}{dt_1} \right] u = [X_P, X_Q] = X_{[P, Q]},$$

where

$$(2) \quad [P, Q] = X_P Q - X_Q P$$

is a Lie algebra bracket on V^ℓ .

Thus, equations $\frac{dt}{du} = P, \frac{du}{dt_1} = Q$ are compatible iff the corresponding evolutionary vector fields commute.

Evolution equation is called *integrable* if it can be included in an infinite hierarchy of linearly independent compatible evolution equations:

$$\frac{du}{dt_n} = P_n, \quad [P_m, P_n] = 0, \quad m, n \in \mathbb{Z}_+,$$

called an *integrable hierarchy*.

Thus, classification of integrable evolution equations = classification of infinite-dimensional (maximal) abelian subalgebras \mathcal{L} in the Lie algebra of evolutionary vector fields V^ℓ with the bracket (2).

Trivial examples of integrable hierarchies:

1. linear: $u_{t_n} = u^{(n)},$

since $X_{u^{(m)}}u^{(n)} = u^{(m+n)}$

2. dispersionless: $u_{t_f} = f(u)u',$

since $X_{f(u)u'}(g(u)u') = \left(f \frac{dg}{du} + g \frac{df}{du} \right) u'^2 + f g u''.$

Nontrivial examples of integrable hierarchies:

$$\begin{aligned}
 u_t &= u'' + uu' && (\text{Burgers}) \\
 u_t &= u''' + uu' && (KdV) \\
 u_t &= u''' + u^2u' && (mKdV) \\
 u_t &= u''' + u'^2 && (pKdV) \\
 u_t &= u''' + u'^3 && (LKdV) \\
 u_t &= \underbrace{u''' - \frac{3u''^2}{2u'}}_{\text{Schwarz KdV}} + \frac{h(u)}{u'} && (\text{Krichever–Novikov})
 \end{aligned}$$

$h(u)$ polynomial of degree at most 4.

Shabat, Sokolov, Mikhailov, ..., Meshkov Theorem. Up to automorphism of the algebra of differential functions, there are only nine more integrable equations of the form $u_t = u''' + f(u, u', u'')$. There are many more integrable equations of the third order with a non-constant separant.

Folklore Conjecture. Any order ≥ 7 integrable evolution equation in one function u is contained in the hierarchy of a non-trivial integrable equation of order ≤ 5 . In other words, any maximal infinite-dimensional subalgebra of V with bracket (2) contains a non-central element of order ≤ 5 .

There are partial classificational results on 2-component equations, the most famous among them is the non-linear Schrödinger:

$$\begin{cases} u_t = v'' + 2v(u^2 + v^2) \\ v_t = -u'' - 2u(u^2 + v^2) \end{cases} .$$

I shall now discuss the other part of the problem: how to prove integrability. But first we have to answer the usually neglected question:

What is a differential function $f \in V$?

An *algebra of differential functions* is a differential algebra V with the derivation ∂ (total derivative), endowed with commuting derivations

$$\frac{\partial}{\partial u_i^{(n)}}, \quad i = 1, \dots, \ell; \quad n \in \mathbb{Z}_+,$$

subject to two axioms:

1. $\frac{\partial}{\partial u_i^{(n)}} f = 0$ for all but finite number of i, n .
2. $\left[\frac{\partial}{\partial u_i^{(n)}}, \partial \right] = \frac{\partial}{\partial u_i^{(n-1)}}$ (the basic identity).

Axiom 1. is needed, otherwise $X_P Q$ is divergent.

Axiom 2. is satisfied for the main example, the algebra of differential polynomials:

$$V = \mathbb{F}[u_i^{(n)} | i = 1, \dots, \ell; n \in \mathbb{Z}_+] \\ \partial u_i^{(n)} = u_i^{(n+1)}.$$

Arbitrary V is its extension, for example, for KN we need to invert u' .

Note: ∂^{-1} cannot be defined if we want both axioms to hold!

S.-S. Chern. In life both men and women are important. Likewise in geometry both vector fields and differential forms are important.

In our theory vector fields are evolutionary vector fields

$$X_P \quad (P \in V^\ell).$$

They commute with $\partial = X_{u'}$. This tells us how to define *variational differential forms*.

§2.

Ordinary differential forms (dual to all vector fields) are

$$\omega = \sum f_{i_1, \dots, i_k}^{n, \dots, n_k} du_{i_1}^{(u_1)} \wedge \dots \wedge du_{i_k}^{(n_k)}$$

with the usual de Rham differential d :

$$\tilde{\Omega}^0 = V \xrightarrow{d} \tilde{\Omega}^1 \xrightarrow{d} \tilde{\Omega}^2 \rightarrow \dots$$

and the derivation ∂ , extended by $\partial(du_i^{(n)}) = du_i^{(n+1)}$.

Axiom 2. of V (the basic identity) is equivalent to the property that ∂ commutes with d . Therefore we can define the *variational complex* by letting

$$\begin{aligned} \Omega^k &= \tilde{\Omega}^k / \partial \tilde{\Omega}^k : \\ V / \partial V &\xrightarrow{d} \tilde{\Omega}^1 / \partial \tilde{\Omega}^1 \xrightarrow{d} \tilde{\Omega}^2 / \partial \tilde{\Omega}^2 \xrightarrow{d} \dots \end{aligned}$$

Here $V / \partial V$ is the space (not algebra any more) of *local functionals*, the universal space where we can perform integration by parts. Now we can describe the variational complex more explicitly:

$V/\partial V \rightarrow V^{\oplus \ell} \rightarrow$ skew-adjoint matrix diff. operators on $V^\ell \rightarrow \dots$

The first map is $\int f \mapsto \frac{\delta f}{\delta u}$.

The second map is $F \mapsto D_F - D_F^*$, where

$$\frac{\delta}{\delta u} = \left(\frac{\delta}{\delta u_j} \right)_j, \quad \frac{\delta f}{\delta u_i} = \sum_{n \in \mathbb{Z}_+} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}$$

is the variational derivative;

$$(D_F)_{ij} = \sum_{n \in \mathbb{Z}_+} \frac{\partial F_i}{\partial u_j^{(n)}} \partial^n$$

is the Frechet derivative.

Note that

(a) $\frac{\delta}{\delta u} \circ \partial = 0$ (\Leftrightarrow basic identity) (Euler)

(b) $D_{\frac{\delta f}{\delta u}}$ is self-adjoint (Helmholtz), is the condition on $F \in V^{\oplus \ell}$

to be a variational derivative (exact 1-form is closed)

One can describe the whole variational complex explicitly in terms of polydifferential operators.

Theorem (Barakat-De Sole-Kac [2009]). Let

$$V_{m,i} = \left\{ f \in V \mid \frac{\partial}{\partial u_j^{(n)}} f = 0, \quad (n, j) > (m, i) \right\}$$

and suppose that $\frac{\partial}{\partial u_i^{(m)}} V_{m,i} = V_{m,i}$. Then the variational complex is exact. One can always embed V in a larger algebra of differential functions \tilde{V} s.t. the variational complex becomes exact.

Note that we have a non-degenerate *pairing* between the space of evolutionary vector fields $= V^\ell$ and the space of variational 1-forms $\Omega^1 = V^{\oplus \ell}$, induced from the usual pairing of vector fields with differential 1-forms:

$$(3) \quad (X_P | \omega_Q) = (P | Q) := \int P \cdot Q \in V / \partial V.$$

§3.

An effective way of constructing an integrable equation is to use Poisson structures. What is a local (or non-local) Poisson structure on V ?

Physicists define it by the following formula:

$$(4) \quad \{u_i(x), u_j(y)\} = H_{ij}(u(y), u'(y), \dots, u^{(n)}(y); \partial/\partial y) \delta(x - y),$$

where $\int f(y) \delta(x - y) = f(x)$ and $H = (H_{ij})$ is an $\ell \times \ell$ matrix differential (or pseudo-differential) operator, whose coefficients are functions in $u, u', \dots, u^{(n)}$.

Extending this formula (4) by Leibniz's rule and bilinearity to $f, g \in V$, we obtain

$$(5) \quad \{f(x), g(y)\} = \sum_{i,j} \sum_{m,n \in \mathbb{Z}_+} \frac{\partial f(x)}{\partial u_i^{(m)}} \frac{\partial g(y)}{\partial u_j^{(n)}} \partial_x^m \partial_y^n \{u_i(x), u_j(y)\}.$$

Integrating (5) by parts in x , we obtain (for $g = u_j$):

$$(6) \quad \left\{ \int f, u \right\}_H = H \frac{\delta}{\delta u} \int f.$$

Integrating (5) by parts in x and in y , we obtain:

$$(7) \quad \left\{ \int f, \int g \right\}_H = \int \frac{\delta \int g}{\delta u} \cdot H(\partial) \frac{\delta \int f}{\delta u}.$$

Definition (a) An $\ell \times \ell$ matrix differential operator H is called a (local) *Poisson structure* on V if (7) is a Lie algebra bracket on $V/\partial V$. This happens iff $H^* = -H$ and certain cubic relations (explained further) on the entries of H hold.

(b) Given a Poisson structure H on an algebra of differential functions V and a local functional $\int h$ (Hamiltonian), the corresponding *Hamiltonian evolution equation* is

$$(8) \quad \frac{du}{dt} = \left\{ \int h, u \right\}_H$$

(the corresponding evolutionary vector field is $X_{H \frac{\delta \int h}{\delta u}}$).

(c) Two local functionals are in *involution* if their commutator (7) is zero.

Remark. The map $V/\partial V \rightarrow$ Lie algebra of evolutionary vector fields V^ℓ given by

$$\int f \mapsto X_{H \frac{\delta \int f}{\delta u}}$$

is a Lie algebra homomorphism. In particular, local functionals in involution correspond to commuting evolutionary vector fields.

Corollary. If $\int h$ is contained in an infinite-dimensional abelian subalgebra of the Lie algebra $(V/\partial V, \{ , \}_H)$ and $\dim \text{Ker } H < \infty$ (i.e. H non-degenerate), then equation (8) is integrable.

An alternative approach is to apply the Fourier transform $\int dx e^{\lambda(x-y)}$ to both sides of (5). Denoting $\{f_\lambda g\} = \int dx e^{\lambda(x-y)} \{f(x), g(y)\}$, we get the Master Formula:

$$(9) \quad \{f_\lambda g\} = \sum_{i,j=1}^{\ell} \sum_{m,n \in \mathbb{Z}_+} \frac{\partial g}{\partial u_j^{(n)}} (\lambda + \partial)^n H_{ji} (\lambda + \partial) (-\lambda - \partial)^m \frac{\partial f}{\partial u_i^{(m)}}.$$

This λ -bracket satisfies:

- (i) (Leibniz rules) $\{f_\lambda gh\} = g\{f_\lambda h\} + h\{f_\lambda g\}; \quad \{fg_\lambda h\} = \{f_{\lambda+\partial} h\} \rightarrow g + \{g_{\lambda+\partial} h\} \rightarrow f;$
- (ii) (sesquilinearity) $\{\partial f_\lambda g\} = -\lambda\{f_\lambda g\}, \quad \{f_\lambda \partial g\} = (\lambda + \partial)\{f_\lambda g\}.$

Theorem. (a) The bracket (7) is a Lie algebra bracket iff:

- (iii) (skewcommutativity) $\{g_\lambda f\} = -\{f_{-\lambda-\partial} g\},$
- (iv) (Jacobi identity) $\{f_\lambda \{g_\mu h\}\} - \{g_\mu \{f_\lambda h\}\} = \{\{f_\lambda g\}_{\lambda+\mu} h\}.$

(b) It suffices to check skewcommutativity of any pair (u_i, u_j) and Jacobi identity for any triple (u_i, u_j, u_k) .

Definition (a) A $\mathbb{F}[\partial]$ -module R is called a *Lie conformal algebra* if $\{R_\lambda R\} \subset R[\lambda]$ and (ii), (iii), (iv) hold.

(b) A unital differential algebra (V, ∂) is called a (local) *Poisson vertex algebra* (PVA) if $\{V_\lambda V\} \subset V[\lambda]$ and (i)–(iv) hold.

(c) If the λ -bracket is given on V by the Master formula, and it is a PVA, the (skewadjoint) differential operator $H = (H_{ij})$ is called a (local) *Poisson structure*.

(d) Two Poisson structures H_0 and H_1 on V are called *compatible* if their sum is a Poisson structure. This pair is then called a bi-Poisson structure.

Example 1) (a) $H = \partial$ (GFZ structure). The corresponding LCA:

$$\{u_\lambda u\} = \lambda.$$

(b) $H = u' + 2u\partial + c\partial^3$ (Virasoro–Magri structure). The corresponding LCA:

$$\{u_\lambda u\} = u' + 2u\lambda + c\lambda^3.$$

They are compatible Poisson structures on $\mathbb{F}[u, u', u'', \dots]$, and we get a 3-parameter family of pairwise compatible Poisson structures:

$$H = a(u' + 2u\partial) + c\partial^3 + \varepsilon\partial.$$

Example 2) ADKP [1988] family of pairwise compatible (local) Poisson structures on differential polynomials in two variables u, v :

$$\begin{pmatrix} a(u' + 2u\partial) + \alpha\partial + c\partial^3 & au\partial + \beta\partial + \gamma\partial^2 \\ a\partial \circ v + \beta\partial - \gamma\partial^2 & \varepsilon\partial \end{pmatrix}$$

where $a, c, \alpha, \beta, \gamma, \varepsilon$ are arbitrary constants.

It appeared in the study of cohomology of the moduli spaces of curves M_g of genus g . Namely:

$$H^2(M_g, \mathbb{C}) \simeq H^2(LCA, \mathbb{C}) \text{ for } g \geq 5,$$

where $LCA = \mathbb{C}[\partial]u + \mathbb{C}[\partial]v$ with λ -brackets

$$[u_\lambda u] = (\partial + 2\lambda u), \quad [u_\lambda v] = (\partial + \lambda)v, \quad [v_\lambda v] = 0.$$

Reformulation of the bracket (7) on V and the Hamiltonian equation (8) in terms of the λ -bracket on V :

$$\left\{ \int a, \int b \right\} = \int \{a_\lambda b\} \Big|_{\lambda=0}$$

defines on $V/\partial V$ a Lie algebra structure, and

$$\left\{ \int a, b \right\} = \{a_\lambda b\} \Big|_{\lambda=0}$$

is its representation by derivation of V commuting with ∂ , i.e. have homomorphism of the Lie algebra $V/\partial V$ in the Lie algebra of evolutionary vector fields.

Hamiltonian equation (8) is:

$$\frac{du}{dt} = \left\{ \int h, u \right\}.$$

§4. A digression: Lie conformal algebras and differential Lie algebras.

Let R be a Lie conformal algebra. Given a differential algebra (A, δ) , we construct a new Lie conformal algebra $\tilde{R}_{A, \delta}$ (“tensor product” of R with A):

$$\tilde{R}_{A, \delta} = R \otimes_{\mathbb{F}} A, \quad \tilde{\partial} = \partial \otimes 1 + 1 \otimes \delta$$

with the following λ -bracket:

$$[a \otimes f \lambda b \otimes g] = ([a_{\lambda+\delta} b]_{\rightarrow} \otimes f) g.$$

A little more explicitly:

$$[a \otimes f \lambda b \otimes g] = \sum_{j \in \mathbb{Z}_+} \frac{\lambda^j}{j!} (a_{(n+j)} b) \otimes (\delta^j f) g,$$

where

$$[a_{\lambda} b] = \sum_{j \in \mathbb{Z}_+} \frac{\lambda^j}{j!} (a_{(j)} b).$$

The Lie algebra

$$\text{Lie}_{A,\delta}R = \widetilde{R}_{A,\delta}/\widetilde{\partial}\widetilde{R}_{A,\delta},$$

obtained as above, carries the derivation, induced by $-1 \otimes \delta$, thereby making it a differential Lie algebra, Thus any LCA R defines a functor from the category of differential algebras (A, δ) to the category of differential Lie algebras over (A, δ) , given by

$$R \rightarrow \text{Lie}_{A,\delta}R$$

(very much like a group scheme defines a functor from the category of unital commutative associative algebras to the category of groups).

Examples. If $R = \text{Cur}(\mathfrak{g}) = \mathbb{F}[\partial] \otimes \mathfrak{g} + \mathbb{F}K$, $[a_\lambda b] = [a, b] + \lambda(a|b)K$, then

$$\text{Lie}_{A,\delta}R = (A \otimes \mathfrak{g}) \oplus (A/\delta A)$$

with the Lie algebra bracket

$$[f \otimes a, g \otimes b] = (fg \otimes [a, b]) \oplus \int f'g, \quad A/\delta A \text{ central.}$$

If R is the Virasoro LCA $\mathbb{F}[\partial]L \oplus \mathbb{F}C$, $[L_\lambda L] = (\partial + 2\lambda)L + \frac{1}{12}\lambda^3 C$, then

$$\text{Lie}_{A,\delta}R = A \oplus (A/\delta A)$$

with the Lie algebra bracket

$$[f, g] = (f'g - fg') \oplus \frac{1}{2} \int f'''g, \quad A/\delta A \text{ central.}$$

Here: $\int A \rightarrow A/\delta A$ is the quotient map.

Theorem. (D'Andrea–Kac, 1998) (a) These are all finite, simple LCA with their central extensions (\mathbb{F} alg. closed, char. 0).

(b) All finite semisimple LCA are direct sums of the simple ones and the semidirect sums of Vir and Cur \mathfrak{g} : $[L_\lambda a] = (\partial + \lambda)a$, $a \in \mathfrak{g}$.

Easy theorem. All non-trivial rank 1 LCA is Virasoro. cf Ritt (1950).

More difficult theorem. Complete description of all rank 2 LCA:

(a) $\mathbb{F}[\partial]u \oplus \mathbb{F}[\partial]v$ nilpotent;

(b) $\mathbb{F}[\partial]u \oplus \mathbb{F}[\partial]v$ solvable;

(c) $\text{Vir} \oplus \mathbb{F}[\partial]v$, where $\text{Vir} = \mathbb{F}[\partial]L$, $[L_\lambda L] = (\partial + 2\lambda)L$;

(d) $\text{Vir} \oplus \text{Vir}$;

(e) $\mathbb{F}[\partial]L + \mathbb{F}[\partial]v$, $[L_\lambda L] = (\partial + 2\lambda)L$,
 $[L_\lambda v] = (\partial + \Delta\lambda)v$, $[v_\lambda v] = 0$, $\Delta \in \mathbb{F}$;

(f) $\mathbb{F}[\partial]L + \mathbb{F}[\partial]v$, $[L_\lambda v] = (\partial + \Delta\lambda)v$, $[v_\lambda v] = 0$, $[L_\lambda L] = (\partial + 2\lambda)L + P_\Delta(\partial, \lambda)v$.

In cases (a)-(c), (e), (f), $\mathbb{F}[\partial]v$ is an abelian ideal, which is central in cases (a) and (c). In case (f) there exists such non-zero $P_\Delta(\partial, \lambda)$ if and only if

$$\Delta = 1, \quad \underbrace{0, -1}, \quad -4, \quad -6$$

1-parameter family

It is because (Bakalov–Kac–Voronov [1999])

$$\dim H^2(\text{Vir}, V_\Delta) = \begin{cases} 0 & \text{if } \Delta \neq 1 - \frac{3r^2 \pm r}{2}, \quad r \in \mathbb{Z}_+ \\ 2 & \text{if } q = r + 1, \quad \Delta = 1 - \frac{3r^2 \pm r}{2} \\ 1 & \text{if } q = r \text{ or } r + 2, \quad \Delta = 1 - \frac{3r^2 \pm r}{2} \end{cases}$$

Polynomials P_Δ , where $x = \partial + \lambda$, $y = -\lambda$, are as follows
($\deg P_\Delta(\partial, \lambda) \leq 3 - \Delta$) \therefore

$$\begin{aligned} P_1 &= x - y \\ P_0 &= x^2 - y^2, \quad xy(x - y) \\ P_{-1} &= x^3 - y^3, \quad xy(x^2 - y^2) \\ P_{-4} &= x^3 y^3 (x - y) \\ P_{-6} &= x^3 y^3 (x - y)(2x^2 - 7xy + 2y^2) \end{aligned}$$

cf. Ritt (1951) and Cassidy (1979). Cases, containing Vir are called of “substitutional type”, the rest are of “finite type”.

Given a differential algebra (A, δ) , a *differential Lie algebra* is a free (for simplicity) A -module $L = A^n = \sum_i A e^i$ with the Lie bracket:

$$\left[\sum_i u_i e^i, \sum_j v_j e^j \right] = \sum_{i,j,k} Q_k^{ij}(\delta_1, \delta_2) u_i v_j e^k$$

where $Q_k^{ij}(\delta_1, \delta_2)$ are polynomials.

Letting $P_k^{ij}(\partial, \lambda) = Q_k^{ij}(\partial + \lambda, -\lambda)$ and defining λ -bracket by

$$[e_\lambda^i e^j] = \sum_k P_k^{ij}(\partial, \lambda) e^k,$$

we get a Lie conformal algebra.

§5. Lenard–Magri scheme in the local case.

Magri lemma. If

$$H_0 \frac{\delta h_{n+1}}{\delta u} = H_1 \frac{\delta h_n}{\delta u}, \quad n \in \mathbb{Z}_+,$$

where $H_0(\partial), H_1(\partial)$ are skew-adjoint, then all the local functionals $\int h_n$ are in involution for both Poisson structures:

$$\left\{ \int h_m, \int h_n \right\}_{H_0, H_1} = 0, \quad m, n \in \mathbb{Z}_+.$$

Hence all $\int h_m$ are integrals of motion of each of the equations

$$\frac{du}{dt_n} = H_0 \frac{\delta h_{n+1}}{\delta u} \left(= H_1 \frac{\delta h_n}{\delta u} \right).$$

Therefore, this is an integrable hierarchy, provided that $H_0 \frac{\delta h_n}{\delta u}, n \in \mathbb{Z}_+$, span an infinite-dimensional subspace in V^ℓ .

Theorem. [DeSole–Kac–Turhan, 2014]. Let H_0, H_1 be (local) compatible Poisson structures and assume that H_0 is *strongly skewadjoint* over \mathcal{V} , i.e.

- i. skewadjoint,
- ii. $(\text{Ker}H_0)^\perp = \text{Im}H_0$ (in general contains),
- iii. $\text{Ker}H_0 \subset \frac{\delta}{\delta u}\mathcal{V}$.

Suppose that $(\text{Ker} H_0) \cap (\text{Ker} H_1)$ has $\text{codim} = 1$ in $\text{Ker}H_0$. Then, beginning with $\int h_0 \in \mathcal{V}/\partial\mathcal{V}$, s.t. $\frac{\delta}{\delta u}h_0 \in \text{Ker}H_0 \setminus \text{Ker}H_1$, there is always a solution to the LM scheme: $\int h_n \in \tilde{\mathcal{V}}/\partial\tilde{\mathcal{V}}$ for some extension $\tilde{\mathcal{V}}$ of \mathcal{V} s.t. $\frac{\delta \int h_n}{\delta u} \in \mathcal{V}^\ell$.

Consequently, we have an infinite hierarchy of compatible Hamiltonian equations (provided that $H_0 \frac{\delta h_n}{\delta u}, n \in \mathbb{Z}_+$ span an infinite-dimensional space):

$$\frac{du}{dt_n} = H_0 \frac{\delta \int h_n}{\delta u}, \quad n \in \mathbb{Z}_+.$$

Example 1.

$H_0 = \partial$ is strongly skewadjoint,

$H_1 = u' + 2u\partial + c\partial^3$ (not strongly skewadjoint if $c \neq 0$).

DSKT conditions hold. Lenard–Magri scheme:

$$\left(\frac{du}{dt_n} = \right) H_0 \frac{\delta h_{n+1}}{\delta u} = H_1 \frac{\delta h_n}{\delta u}, \quad n \in \mathbb{Z}_+.$$

Begin with $h_0 = u$, then $\frac{\delta h_0}{\delta u} = 1 \in \text{Ker} \partial$, get equation

$$\frac{du}{dt_0} \left(= \partial \frac{\delta h_0}{\delta u} \right) = 0.$$

Next, h_1 is computed from $H_0 \frac{\delta h_1}{\delta u} = H_1 \frac{\delta h_0}{\delta u} = u'$, so $h_1 = \frac{1}{2}u^2$, the corresponding equation:

$$\frac{du}{dt_1} \left(= \partial \frac{\delta h_1}{\delta u} \right) = u'.$$

Next, h_2 is computed from $H_0 \frac{\delta h_2}{\delta u} = H_1 \frac{\delta h_1}{\delta u}$ get $h_2 = \frac{1}{1}u^3 + \frac{1}{2}cu''^2$, and the corresponding equation

$$\frac{du}{dt_2} \left(= \partial \frac{\delta h_2}{\delta u} \right) = 3uu' + cu''' \quad KdV.$$

Thus, KdV is integrable (order of n^{th} equation is $2n - 1$).

Example 2.

$$H_0 = \begin{pmatrix} \alpha\partial & \beta\partial + \gamma\partial^2 \\ \beta\partial - \gamma\partial^2 & \varepsilon\partial \end{pmatrix}, \quad H_1 = \begin{pmatrix} u' + 2u\partial & v\partial + \gamma_1\partial^2 \\ \partial \circ v - \gamma_1\partial^2 & \varepsilon_1\partial \end{pmatrix}$$

a pair from the ADKP family.

H_0 is always strongly skew-adjoint and $\text{Ker}H_0 \cap \text{Ker} H_1 = \mathbb{C} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Can apply DSKT, taking $h_0 = u$, the corresponding equation is

$$\frac{d}{dt_0} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u' \\ v' \end{pmatrix},$$

which is the x -translation symmetry.

Next, after a change of variables, the equation, corresponding to h_1 , becomes the following integrable equation:

$$\frac{d}{dt_1} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u''' + 3uu' + vv' + \gamma v'' \\ (uv)' + \kappa v' - \gamma u'' \end{pmatrix}$$

Some special cases, namely $\gamma\kappa = 0$ are well known water wave equations (Ito, Kupershmidt, and Fokas-Liu), but $\gamma\kappa \neq 0$ is new.

§6. How to extend these notions to the non-local case (i.e. $H(\partial)$ is a pseudodifferential operator)? In this case we see from (9) that

$$\{V_\lambda V\} \subset V((\lambda^{-1})).$$

It is easy to interpret the identities (i)–(iii): expand in positive powers of ∂ each time when we encounter $\frac{1}{(\lambda+\partial)^n}$. However, in order for the Jacobi identity to make sense we must impose *admissibility* property:

$$\{f_\lambda\{g_\mu h\}\} \subset V[[\lambda^{-1}, \mu^{-1}, (\lambda + \mu)^{-1}][[\lambda, \mu].$$

Proposition [DSK 2013]. The λ -bracket (9), given by the Master Formula, is admissible, provided that $H(\partial)$ is a *rational* pseudodifferential operator, i.e. it is contained in the subalgebra of the algebra of pseudodifferential operators $V((\partial^{-1}))$, generated by differential operators and their inverses.

Then our basic definitions extend to the non-local case: non-local Lie conformal algebra, non-local PVA, non-local Poisson structure.

Examples: $H = \partial^{-1}$

$H = u' \partial^{-1} \circ u'$ (Sokolov)

$H = \partial^{-1} \circ u' \partial^{-1} \circ u' \partial^{-1}$ (Dorfman)

$$H = \partial I_2 + \begin{pmatrix} v \partial^{-1} \circ v & -v \partial^{-1} \circ u \\ -u \partial^{-1} \circ v & u \partial^{-1} \circ u \end{pmatrix}$$

(Magri: non-local Poisson structure for NLS)

A theory of rational pseudodifferential operators.

Let (V, ∂) be a unital differential algebra, assume V is a domain, \mathcal{K} field of fractions. Let $\mathcal{K}((\partial))$ be the skewfield of pseudodifferential operators, $\mathcal{K}(\partial)$ the sub-skewfield of rational ones (i.e. the sub-skewfield, generated by $\mathcal{K}[\partial]$). Then

Theorem (Carpentier–De Sole–Kac [2013].) (a) Any $H \in \text{Mat}_n(\mathcal{K}(\partial))$ can be represented as AB^{-1} , where $A, B \in \text{Mat}_n\mathcal{K}[\partial]$, B non-degenerate.

(b) There exists a *minimal* such representation $A_0B_0^{-1}$ so that any other is $(A_0C)(B_0C)^{-1}$, C non-degenerate.

(c) AB^{-1} is minimal iff $\text{Ker } A \cap \text{Ker } B = 0$ in any differential field extension of \mathcal{K} .

The best *proof*. Use the theory of non-commutative *principal ideal rings*.

§7. What is a Hamiltonian equation

$$(10) \quad \frac{du}{dt} = H(\partial) \frac{\delta}{\delta u} \int h$$

when H is a non-local Poisson structure?

Fix a fractional decomposition $H = AB^{-1}$. We define the *association relation*:

$$V/\partial V \ni \int h \overset{H}{\leftrightarrow} P \in V^\ell$$

if $P = A(\partial)F$, $\frac{\delta}{\delta u} \int h = B(\partial)F$ for some $F \in \mathcal{K}^\ell$. Then equation (10) is interpreted as

$$\frac{du}{dt} = P \quad \left(\approx A(\partial)B(\partial)^{-1} \frac{\delta}{\delta u} \int h \right).$$

Again, the equality $P = (AB^{-1})\xi$ is interpreted as:

$$P = AF, \xi = BF \text{ for some } F \in V^\ell,$$

and we write $\xi \overset{H}{\leftrightarrow} P$ where $H = AB^{-1}$.

§8 *Lenard–Magri* scheme for the (non-local) bi-Poisson structure (H, K) i.e. both H, K are Poisson and also $H + K$ is Poisson (all above examples are such). A bi-Hamiltonian equation:

$$(11) \quad \frac{du}{dt} = \underbrace{H(\partial) \frac{\delta}{\delta u} \int h_0 = K(\partial) \frac{\delta}{\delta u} \int h_1}_{\text{means}} := P_1$$

$$\int h_0 \overset{H}{\leftrightarrow} P_1 \overset{K}{\leftrightarrow} \int h_1.$$

Then under certain conditions the Hamiltonian equation (11) is integrable:

Theorem (DSK [2013]). Let $H = AB^{-1}$, $K = CD^{-1}$ be skewadjoint. Let $\{\xi_n\}_{n=-1}^N$, $\{P_n\}_{n=0}^N$ be sequences such that

$$(*) \quad \xi_{n-1} \overset{H}{\leftrightarrow} P_n \overset{H}{\leftrightarrow} \xi_n, \quad n = 0, \dots, N.$$

Then

(a) $(P_n | \xi_m) = 0$, $m \geq -1, n \geq 0$ (i.e. the $\int h_m$ are in involution if $\xi_m = \int h_m$ are exact)

(b) Provided that $H = AB^{-1}, K = CD^{-1}$ is a bi-Poisson structure, K non-degenerate, and ξ_{-1}, ξ_0 closed, we have: ξ_n are closed, hence exact in some differential algebra extension of V , and

$$[P_m, P_n] \subset \text{Ker } B^* \cap \text{Ker } D^*, \quad m, n \geq 0.$$

(c) If the orthogonality conditions hold:

$$\begin{aligned} (\text{span } \{\xi_m\}_{m=-1}^N)^\perp &\subset \text{Im } C \\ (\text{span } \{P_m\}_{m=0}^N)^\perp &\subset \text{Im } B, \end{aligned}$$

we can extend (*) to infinity.

(d) If also $\text{ord } P_n \rightarrow \infty$, then each of the equations

$$\frac{du}{dt_n} = P_n$$

is integrable and has infinitely many linearly independent integrals of motion in involution.

§9 *Classical Hamiltonian reduction* for PVA V .

Let $\mu : R \rightarrow V$ be a Lie conformal algebra homomorphism; it extends to the PVA homomorphism $\mu : S(R) \rightarrow V$. Let $I_0 \subset S(R)$ be a PVA ideal. Let $I = V\mu(I_0)$ be the differential algebra ideal of V , generated by $\mu(I_0)$. The classical Hamiltonian reduction is the differential algebra

$$\mathcal{W}(V, R, I_0) = (V/I)^{\mu(R)}$$

with the λ -bracket

$$\{f + I_\lambda g + I\} = \{f_\lambda g\} + I[\lambda].$$

Examples. Classical W -algebra, associated to $(\mathfrak{g}, \text{nilpotent } f)$,

$\mathcal{W}(\mathfrak{g}, f)$ is obtained by taking

$$V = S(\mathbb{F}[\partial]\mathfrak{g}) \text{ with } [a_\lambda b] = [a, b] + (a|b)\lambda,$$

$$R = \mathbb{F}[\partial]\mathfrak{g}_{>0}, [a_\lambda b] = [a, b],$$

I_0 ideal of $S(R)$, generated by $m - (f|m)$, where $m \in \mathfrak{g}_{\geq 1}$.

Drinfeld–Sokolov, using $f = \text{principal nilpotent}$, constructed the integrable DS hierarchy. One can construct the generalized DS hierarchies for any nilpotent f , such that $f + s$ is a semisimple element of \mathfrak{g} , where s has maximal $(\text{ad } h)$ -eigenvalue, using the language of PVA (De Sole–Kac–Valeri).

§10 *Dirac reduction* for PVA. (De Sole-Kac-Valeri)

Let V be a non-local PVA, let $\theta_1, \dots, \theta_m \in V$ (constraints), let I be the differential ideal of V generated by them. Consider the rational pseudodifferential operator $C(\partial)$ with symbol $(C_{\alpha\beta}(\lambda)) = (\{\theta_{\beta\lambda}\theta_\alpha\})$.

Theorem. Assume that $C(\partial)$ is an invertible matrix pseudodifferential operator. Then

(a) $\{f_\lambda g\}^D := \{f_\lambda g\} - \sum_{\alpha,\beta=1}^m \{\theta_\alpha \lambda + \partial g\} \rightarrow (C^{-1})_{\alpha\beta}(\partial + \lambda) \{f_\lambda \theta_\beta\}$ is again a (non-local) PVA structure on V .

(b) θ_i are central: $\{\theta_i \lambda f\}^D = 0$.

(c) V/I with the induced λ -bracket is again a (non-local) PVA.

Corollary. If $H = \begin{smallmatrix} m & B \\ -B^* & D \end{smallmatrix}$ is a (non-local) Poisson structure in $m + n$ variables, then $A + BC^{-1}B^*$ is a non-local Poisson structure in m variables.