Category Theory Meets the First Fundamental Theorem of Calculus

Kolchin Seminar in Differential Algebra

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The First Fundamental Theorem of Calculus

One of the most important results in the calculus is the First Fundamental Theorem of Calculus, which states that if $f : [a, b] \to \mathbb{R}$ is a continuous function and if F is defined on $[a, b]$ by $F(x) = \int_a^x f(t)dt$, then

1. F is continuous on $[a, b]$,
2. F is differentiable on (a, b) and
3. $\frac{d}{dx}\left(\int_a^x f(t)dt \right) = f(x)$ on (a, b).

We will investigate (3) from a categorical viewpoint.

Outline

- Notations & Review of Some Category Theory
- Differential Algebras
- Rota-Baxter Algebras
- Mixed Distributive Laws
- Differential Rota-Baxter Algebras

Notations

- Fix **k** a commutative ring with identity and fix $\lambda \in \mathbf{k}$.
- All algebras are commutative **k**-algebras with identity.
- All homomorphisms preserve the identity.
- All linear maps and tensor products are over **k**.
- $\mathbf{N} = \{0, 1, 2, \ldots\}$ will denote the natural numbers.
- $\mathbf{N}_+ = \{1, 2, 3, \ldots\}$.

Natural Transformations

Definition: For categories $\mathbf{A}$ and $\mathbf{B}$ and functors $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{A} \to \mathbf{B}$, a **natural transformation** $\eta : F \to G$ is a family of morphisms in $\mathbf{B}$, $\{\eta_X : FX \to GX\}$, one for each $X \in \mathbf{A}$, such that for any morphism $f : X \to Y$ in $\mathbf{A}$, we have the following in $\mathbf{B}$:

$$\eta_Y \circ Ff = Gf \circ \eta_X,$$

i.e., the diagram

$$\begin{array}{ccc} FX & \xrightarrow{\eta_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow{\eta_Y} & GY \end{array}$$

commutes.

Natural Transformations

Example

- Let **FVS** denote the category of finite-dimensional vector spaces over some field K. Let $V \in \mathbf{FVS}$ and V^* be the dual space of V. Then $V^* \cong V$, but the isomorphism is not "natural" in the sense that it requires the choice of a basis for V.
- However, there is a natural transformation $\eta_V : V \to V^{**}$ defined by $(\eta_V(v))(\varphi) = \varphi(v)$ for any $v \in V$ and $\varphi \in V^*$.

Adjoint Functors

Definition: Given categories $\mathbf{A}$ and $\mathbf{B}$, an **adjunction** between $\mathbf{A}$ and $\mathbf{B}$ consists of functors $F : \mathbf{A} \to \mathbf{B}$ and $U : \mathbf{B} \to \mathbf{A}$ (opposite directions) such that for each $X \in \mathbf{A}$ and $Y \in \mathbf{B}$,

$$\mathbf{B}(FX, Y) \cong \mathbf{A}(X, UY),$$

where the isomorphism is natural in $X \in \mathbf{A}$ and $Y \in \mathbf{B}$.
In this case, we say that F is **left adjoint** to U, or that U is **right adjoint** to F.

Adjoint Functors

Example

- Let $\mathbf{A} = \mathbf{SET}$, the category of sets and functions, and let $\mathbf{B} = \mathbf{GRP}$, the category of groups and group homomorphisms.
- Let $F : \mathbf{SET} \to \mathbf{GRP}$ be the free group functor, and let $U : \mathbf{GRP} \to \mathbf{SET}$ be the underlying set functor, i.e. the "forgetful" functor.
- Then F is left adjoint to U, since we have the natural isomorphism

$$\mathbf{GRP}(FX, G) \cong \mathbf{SET}(X, UG)$$

for any set X and any group G.

Unit-Counit Description of an Adjunction

- Given adjoint functors $F : \mathbf{A} \to \mathbf{B}$ and $U : \mathbf{B} \to \mathbf{A}$ with F left adjoint to U, we can equivalently describe the adjunction by using natural transformations.
- In this case, there are natural transformations $\eta : \mathbf{id_A} \to UF$ and $\varepsilon : FU \to \mathbf{id_B}$ such that

$$\varepsilon F \circ F\eta = \mathbf{id}_F$$

and

$$U\varepsilon \circ \eta U = \mathbf{id}_U.$$

- η is called the **unit** of the adjunction, and ε is called the **counit** of the adjunction.
- We write the adjunction as $\langle F, U, \eta, \varepsilon \rangle : \mathbf{A} \rightharpoonup \mathbf{B}$.

Unit-Counit Description of an Adjunction

Example

- As before, let $F : \mathbf{SET} \to \mathbf{GRP}$ be the free group functor, and $U : \mathbf{GRP} \to \mathbf{SET}$ the forgetful functor.
- The unit is the natural transformation $\eta : \mathbf{id}_{\mathbf{SET}} \to UF$ given for each set X by $\eta_X : X \to UFX$, i.e., η_X is the injection of the generators into the (underlying set of the) free group on X.
- The counit is $\varepsilon : FU \to \mathbf{id}_{\mathbf{GRP}}$ given for each group G by $\varepsilon_G : FUG \to G$, where ε_G maps an element of the free group on UG to the corresponding element in G by using the operation in G to "condense" the string to an element in G.

Monads

Definition: A **monad** $\mathbf{T}$ on a category $\mathbf{A}$ is a triple $\mathbf{T} = (T, \eta, \mu)$ where

- $T : \mathbf{A} \to \mathbf{A}$ is a functor (i.e., an endofunctor on $\mathbf{A}$),
- η is a natural transformation $\eta : \mathbf{id}_{\mathbf{A}} \to T$, and
- μ is a natural transformation $\mu : TT \to T$,

such that

- $\mu \circ T\eta = \mathbf{id}_T = \mu \circ \eta T$, and
- $\mu \circ T\mu = \mu \circ \mu T$

that is, the diagrams

Monads

$$
\begin{array}{ccccc}
T & \xrightarrow{T\eta} & TT & \xleftarrow{\eta T} & T \\
& {\scriptstyle \mathbf{id}_T} \searrow & \downarrow {\scriptstyle \mu} & \swarrow {\scriptstyle \mathbf{id}_T} & \\
& & T & &
\end{array}
$$

and

$$
\begin{array}{ccc}
TTT & \xrightarrow{T\mu} & TT \\
{\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\
TT & \xrightarrow{\mu} & T
\end{array}
$$

commute.

Monads

Example

- Take $\mathbf{A} = \mathbf{SET}$.
- Let $T : \mathbf{SET} \to \mathbf{SET}$ to be the functor that assigns to each set X the underlying set of the free group on X.
- Let $\eta_X : X \to TX$ be the "insertion of the generators" as before.
- Let $\mu_X : TTX \to TX$ be the underlying set map of the "partial collapse" of a string of strings to a string using the operation in the free group.

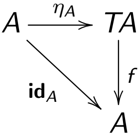
Algebras from a Monad

Given a monad $\mathbf{T} = (T, \eta, \mu)$ on a category $\mathbf{A}$, we can form the category of $\mathbf{T}$-algebras, denoted by $\mathbf{A}^{\mathbf{T}}$, as follows.

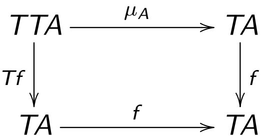
- The objects of $\mathbf{A}^{\mathbf{T}}$ are pairs (A, f), where A is an object of $\mathbf{A}$ and $f : TA \to A$ is a morphism in $\mathbf{A}$ such that
  - $f \circ \eta_A = \mathbf{id}_A$, and
  - $f \circ \mu_A = f \circ Tf$.
- A morphism $g : (A, f) \to (A', f')$ in $\mathbf{A}^{\mathbf{T}}$ is a morphism $g : A \to A'$ in $\mathbf{A}$ such that $g \circ f = f' \circ Tg$.

Algebras from a Monad

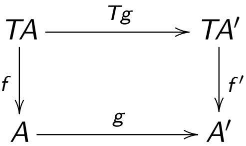
The diagrams for objects:



and



and for morphisms:



Monad from an Adjunction

Given adjoint functors $F : \mathbf{A} \to \mathbf{B}$ and $U : \mathbf{B} \to \mathbf{A}$ with F left adjoint to U, the adjunction $\langle F, U, \eta, \varepsilon \rangle : \mathbf{A} \rightharpoonup \mathbf{B}$ gives rise to a monad $\mathbf{T} = (T, \eta, \mu)$ on the category $\mathbf{A}$ by taking:

- $T = UF : \mathbf{A} \to \mathbf{A}$,
- $\eta = \eta : \mathbf{id}_A \to UF = T$, and
- $\mu = U\varepsilon F : UFUF = TT \to UF = T$.

Adjunction from a Monad

Given a monad $\mathbf{T} = (T, \eta, \mu)$ on the category $\mathbf{A}$, there are adjoint functors $F^{\mathbf{T}} : \mathbf{A} \to \mathbf{A}^{\mathbf{T}}$ and $U^{\mathbf{T}} : \mathbf{A}^{\mathbf{T}} \to \mathbf{A}$ defined as follows:

- For any $X \in \mathbf{A}$, define $F^{\mathbf{T}}X = (TX, \mu_X)$, and define $F^{\mathbf{T}}$ on morphisms in $\mathbf{A}$ similarly.
- For any $(A, f) \in \mathbf{A}^{\mathbf{T}}$, define $U^{\mathbf{T}}(A, f) = A$, and similarly for morphisms in $\mathbf{A}^{\mathbf{T}}$.

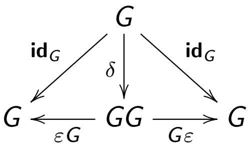
This adjunction gives rise to the same monad $\mathbf{T} = (T, \eta, \mu)$ on the category $\mathbf{A}$.

Comparison of Algebras from an Adjunction

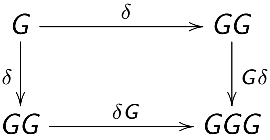
- Given adjoint functors $F : \mathbf{A} \to \mathbf{B}$ and $U : \mathbf{B} \to \mathbf{A}$ with F left adjoint to U, let $\mathbf{T} = (T, \eta, \mu)$ be the monad on the category $\mathbf{A}$ generated by the adjunction, and let $\langle F^{\mathbf{T}}, U^{\mathbf{T}}, \eta^{\mathbf{T}}, \varepsilon^{\mathbf{T}} \rangle : \mathbf{A} \rightharpoonup \mathbf{A}^{\mathbf{T}}$ be the adjunction given from $\mathbf{T}$.
- Then there is a "comparison" functor $K : \mathbf{B} \to \mathbf{A}^{\mathbf{T}}$ given by $KX = (UX, U\varepsilon_X)$, which satisfies $KF = F^{\mathbf{T}}$ and $U^{\mathbf{T}}K = U$.
- In some nice cases, K is an isomorphism, in which case we say that $\mathbf{B}$ is **monadic** over $\mathbf{A}$.

Dualizing Monads and Algebras

A **comonad** $\mathbf{G}$ on a category $\mathbf{B}$ is a (co)triple $\mathbf{G} = (G, \varepsilon, \delta)$ where $G : \mathbf{B} \to \mathbf{B}$ is a functor (i.e., an endofunctor on $\mathbf{B}$) and ε and δ are natural transformations, $\varepsilon : G \to \mathbf{id}_{\mathbf{B}}$ and $\delta : G \to GG$ such that the following diagrams commute:



and



Monads and Comonads from an Adjunction

- We have seen that an adjunction $\langle F, U, \eta, \varepsilon \rangle : \mathbf{A} \rightharpoonup \mathbf{B}$ gives rise to a monad $\mathbf{T} = (T, \eta, \mu)$ on the category $\mathbf{A}$ by taking $T = UF$ and $\mu = U\varepsilon F$.

- The adjunction $\langle F, U, \eta, \varepsilon \rangle : \mathbf{A} \rightharpoonup \mathbf{B}$ also gives rise to a comonad $\mathbf{G} = (G, \varepsilon, \delta)$ on the category $\mathbf{B}$ by taking $G = FU$ and $\delta = F\eta U$.

Dualizing Monads and Algebras

- A comonad $\mathbf{G} = (G, \varepsilon, \delta)$ on $\mathbf{B}$ gives a category of $\mathbf{G}$-coalgebras, denoted by $\mathbf{B}_\mathbf{G}$, as follows.
- The objects of $\mathbf{B}_\mathbf{G}$ are pairs (B, g), where B is an object of $\mathbf{B}$ and $g : B \to GB$ is a morphism in $\mathbf{B}$ such that $\varepsilon_B \circ g = \mathbf{id}_B$ and $Gg \circ g = \delta_B \circ g$.
- A morphism $f : (B, g) \to (B', g')$ in $\mathbf{B}_\mathbf{G}$ is a morphism $f : B \to B'$ in $\mathbf{B}$ such that $g' \circ f = Gf \circ g$.
- Et cetera, et cetera, et cetera.

Derivations with Weight

Let $\mathbf{k}$ be a ring, $\lambda \in \mathbf{k}$, and let R be an algebra.

- A **derivation of weight λ on R over k** or more briefly, a **λ-derivation on R over k** is a module endomorphism d of R satisfying both

$$d(xy) = d(x)y + xd(y) + \lambda d(x)d(y), \quad \text{for all } x,\ y \in R$$

  and

$$d(\mathbf{1}_R) = 0.$$

- A **λ-differential algebra** is a pair (R, d) where R is an algebra and d is a λ-derivation on R over $\mathbf{k}$.
- Let (R, d) and (S, e) be two λ-differential algebras. A **homomorphism of λ-differential algebras** $f : (R, d) \to (S, e)$ is a homomorphism $f : R \to S$ of algebras such that $f(d(x)) = e(f(x))$ for all $x \in R$.

Note that a 0-derivation is a derivation in the usual sense.

An Example of a λ-Derivation

Example

Let $\mathbb{R}$ denote the field of real numbers, and let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Let A denote the $\mathbb{R}$-algebra of $\mathbb{R}$-valued continuous functions on $\mathbb{R}$, and consider the usual "difference quotient" operator d_λ on A defined by

$$(d_\lambda(f))(x) = (f(x+\lambda) - f(x))/\lambda.$$

Then a simple calculation shows that d_λ is a λ-derivation on A.

Leibniz' Rule

Proposition (Leibniz' Rule): Let (R, d) be a λ-differential algebra, let $x, y \in R$, and let $n \in \mathbf{N}$. Then

$$d^{(n)}(xy) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k d^{(n-j)}(x) d^{(k+j)}(y).$$

When $\lambda = 0$, this reduces to the familiar

$$d^{(n)}(xy) = \sum_{k=0}^{n} \binom{n}{k} d^{(k)}(x) d^{(n-k)}(y).$$

A Simplification

For the remainder of the talk, to simplify notation, we will assume that

$$\lambda = 0.$$

The Hurwitz Product

For any algebra A, let $A^{\mathbf{N}}$ denote the $\mathbf{k}$-module of all functions $f : \mathbf{N} \to A$. On $A^{\mathbf{N}}$, we define the **Hurwitz product** fg of any $f, g \in A^{\mathbf{N}}$ by

$$(fg)(n) = \sum_{k=0}^{n} \binom{n}{k} f(k)g(n-k).$$

Compare with Leibniz' Rule:

$$d^{(n)}(xy) = \sum_{k=0}^{n} \binom{n}{k} d^{(k)}(x)d^{(n-k)}(y).$$

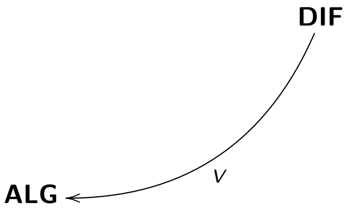
The Algebra of Hurwitz Series

- $A^{\mathbf{N}}$ (with the Hurwitz product) is called the **algebra of Hurwitz series** over A.
- The map $\partial_A : A^{\mathbf{N}} \to A^{\mathbf{N}}$, defined by $\partial_A(f)(n) = f(n+1)$ for any $f \in A^{\mathbf{N}}$, is a derivation on $A^{\mathbf{N}}$.
- Hence $(A^{\mathbf{N}}, \partial_A)$ is a differential algebra for any algebra A.
- For any algebra homomorphism $h : A \to B$, the map $h^{\mathbf{N}} : A^{\mathbf{N}} \to B^{\mathbf{N}}$ defined by $(h^{\mathbf{N}}(f))(n) = h(f(n))$ for any $f \in A^{\mathbf{N}}$ and $n \in \mathbf{N}$ is a differential algebra homomorphism from $(A^{\mathbf{N}}, \partial_A)$ to $(B^{\mathbf{N}}, \partial_B)$.

Functors for Differential Algebras

- Let **DIF** denote the category of differential algebras, and let **ALG** denote the category of algebras.
- We have a functor $G : \mathbf{ALG} \to \mathbf{DIF}$ given on objects $A \in \mathbf{ALG}$ by $G(A) = (A^{\mathbf{N}}, \partial_A)$ and on morphisms $h : A \to B$ in **ALG** by $G(h) = h^{\mathbf{N}}$ as defined above.
- Let $V : \mathbf{DIF} \to \mathbf{ALG}$ denote the forgetful functor defined on objects $(R, d) \in \mathbf{DIF}$ by $V(R, d) = R$ and on morphisms $g : (R, d) \to (S, e)$ in **DIF** by $V(g) = g$.

Beginning of the Big Picture



(1)

The Natural Transformations

There are two natural transformations $\eta : \mathbf{id_{DIF}} \to GV$ and $\varepsilon : VG \to \mathbf{id_{ALG}}$.

- For any $(R, d) \in \mathbf{DIF}$, define

$$\eta_{(R,d)} : (R, d) \to (GV)(R, d) = (R^{\mathbf{N}}, \partial_R)$$

by $(\eta_{(R,d)}(x))(n) := d^{(n)}(x), x \in R, n \in \mathbf{N}$.

- For any $A \in \mathbf{ALG}$, define

$$\varepsilon_A : (VG)(A) = A^{\mathbf{N}} \to A$$

by $\varepsilon_A(f) := f(0), f \in A^{\mathbf{N}}$.

The Forgetful Functor Has a Right Adjoint

Proposition:

- The functor $G : \mathbf{ALG} \to \mathbf{DIF}$ defined above is the **right** adjoint of the forgetful functor $V : \mathbf{DIF} \to \mathbf{ALG}$.
- It follows that $(A^{\mathbf{N}}, \partial_A)$ is a **cofree** differential algebra on the algebra A.

The Comonad from the Adjunction

The adjunction $\langle V, G, \eta, \varepsilon \rangle : \mathbf{DIF} \rightharpoonup \mathbf{ALG}$ gives rise to a comonad $\mathbf{C} = (C, \varepsilon, \delta)$ on the category **ALG**, where

- C is the functor $C := VG : \mathbf{ALG} \to \mathbf{ALG}$ given by $C(A) = A^{\mathbf{N}}$ for any $A \in \mathbf{ALG}$, and
- $\delta : C \to CC$ is the natural transformation defined by $\delta := V\eta G$.

It follows that for any $A \in \mathbf{ALG}$,

$$\delta_A : A^{\mathbf{N}} \to (A^{\mathbf{N}})^{\mathbf{N}}, \quad (\delta_A(f)(m))(n) = f(m+n), \quad f \in A^{\mathbf{N}}, m, n \in \mathbf{N}.$$

Note that as a **k**-module, $(A^{\mathbf{N}})^{\mathbf{N}} \cong A^{\mathbf{N}\times\mathbf{N}}$, the set of sequences of sequences, or equivalently, doubly-indexed sequences with values in A.

The Comonad Comes from the Monoid $(\mathbf{N}, +, 0)$

Observe that comonad $\mathbf{C} = (C, \varepsilon, \delta)$ on **ALG** come from the monoid of natural numbers $(\mathbf{N}, +, 0)$ in the following sense:

- The functor C is given by $C(A) = A^{\mathbf{N}}$.
- $\varepsilon_A : A^{\mathbf{N}} \to A \cong A^{\{*\}}$ is induced by $0 : \{*\} \to \mathbf{N}$, i.e., $\varepsilon_A \cong A^0$.
- $\delta_A : A^{\mathbf{N}} \to (A^{\mathbf{N}})^{\mathbf{N}} \cong A^{\mathbf{N}\times\mathbf{N}}$ is induced by $+ : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$, i.e., $\delta_A \cong A^+$.

Coalgebras for the Differential Comonad C on ALG

- The comonad **C** induces a category of **C**-coalgebras, denoted by $\mathbf{ALG}_{\mathbf{C}}$.
- The objects in $\mathbf{ALG}_{\mathbf{C}}$ are pairs $\langle A, f\rangle$ where $A \in \mathbf{ALG}$ and $f : A \to A^{\mathbf{N}}$ is an algebra homomorphism satisfying the two properties
$$\varepsilon_A \circ f = \mathbf{id}_A, \quad \delta_A \circ f = f^{\mathbf{N}} \circ f$$
.
- A morphism $\varphi : \langle A, f\rangle \to \langle B, g\rangle$ in $\mathbf{ALG}_{\mathbf{C}}$ is an algebra homomorphism $\varphi : A \to B$ such that $g \circ \varphi = \varphi^{\mathbf{N}} \circ f$.

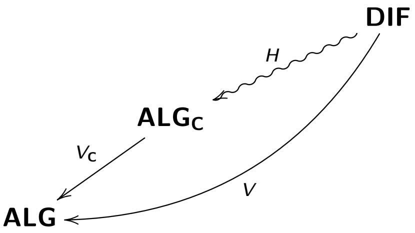
DIF is comonadic over ALG

Proposition: The cocomparison functor $H : \mathbf{DIF} \to \mathbf{ALG_C}$ is an isomorphism, i.e., **DIF** is comonadic over **ALG**.

Corollary: For any algebra A, there is a one-to-one correspondence among

- derivations d on A over $\mathbf{k}$;
- **C**-costructures f on A, i.e., algebra homomorphisms $f : A \to A^{\mathbf{N}}$ satisfying $\varepsilon_A \circ f = \mathbf{id}_A$ and $\delta_A \circ f = f^{\mathbf{N}} \circ f$;
- sequences of $\mathbf{k}$-module homomorphisms $(f_n) : A \to A$ for $n \in \mathbf{N}$ that satisfy $f_0 = \mathbf{id}_A$, $f_m \circ f_n = f_{m+n}$ and $f_n(ab) = \sum_{k=0}^{n} \binom{n}{k} f_k(a) f_{n-k}(b)$ for all $a, b \in A$.

The Picture Grows



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Definitions

Let R be an algebra.

- A **Rota-Baxter operator** on R is a $\mathbf{k}$-linear endomorphism P of R satisfying

$$P(x)P(y) = P(xP(y)) + P(yP(x)), \quad \text{for all } x, \ y \in R.$$

- A **Rota-Baxter algebra** is a pair (R, P) where R is an algebra and P is a Rota-Baxter operator on R.
- Let (R, P) and (S, Q) be two Rota-Baxter algebras. A **homomorphism of Rota-Baxter algebras** $f : (R, P) \to (S, Q)$ is a homomorphism $f : R \to S$ of algebras with the property that $f(P(x)) = Q(f(x))$ for all $x \in R$.

An Example

Example

Let $R = \mathrm{Cont}(\mathbb{R})$ denote the $\mathbb{R}$-algebra of continuous functions on $\mathbb{R}$. Let P_0 be the operator on R given by

$$P_0(f)(x) = \int_0^x f(t)dt.$$

Then P_0 is a Rota-Baxter operator on R.

Rota-Baxter Categories and Functors

- Let **RBA** denote the category of commutative Rota-Baxter **k**-algebras, and let **ALG** denote the category of commutative algebras.
- Let $U : \mathbf{RBA} \to \mathbf{ALG}$ denote the forgetful functor given on objects $(R, P) \in \mathbf{RBA}$ by $U(R, P) = R$ and on morphisms $f : (R, P) \to (S, Q)$ in **RBA** by $U(f) = f : R \to S$.
- We proved earlier that U has a left adjoint, and below we give an explicit description of the left adjoint, the free commutative Rota-Baxter algebra functor.
- There are earlier constructions (e.g., by Cartier and by Rota) of free commutative Rota-Baxter algebras on sets, that is, as a left adjoint of the forgetful functor from **RBA** to **SET**.

The Free Rota-Baxter Algebra

We begin with some general observations about the free commutative Rota-Baxter algebra on a commutative algebra A with identity $\mathbf{1}_A$.

- The product for this free Rota-Baxter algebra on A is constructed in terms of a generalization of the shuffle product, called the **mixable shuffle product**, which we will describe below and which in its recursive form is a natural generalization of the quasi-shuffle product.
- This free commutative Rota-Baxter algebra on A is denoted by $Ш(A)$.
- As a module, we have

$$Ш(A) = \bigoplus_{i \geq 1} A^{\otimes i} = A \oplus (A \otimes A) \oplus (A \otimes A \otimes A) \oplus \cdots$$

where the tensors are defined over $\mathbf{k}$.

The Mixable Shuffle Product

The multiplication on $\text{Ш}(A)$ is the product $\diamond$ defined as follows.

Let $\mathfrak{a} = a_0 \otimes \cdots \otimes a_m \in A^{\otimes(m+1)}$ and $\mathfrak{b} = b_0 \otimes \cdots \otimes b_n \in A^{\otimes(n+1)}$.

If $mn = 0$, define

$$\mathfrak{a} \diamond \mathfrak{b} = \begin{cases} (a_0 b_0) \otimes b_1 \otimes \cdots \otimes b_n, & m = 0, n > 0, \\ (a_0 b_0) \otimes a_1 \otimes \cdots \otimes a_m, & m > 0, n = 0, \\ a_0 b_0, & m = n = 0. \end{cases}$$

The Mixable Shuffle Product

If $m > 0$ and $n > 0$, then $\mathfrak{a} \diamond \mathfrak{b}$ is defined inductively on $m + n$ by

$$\begin{aligned} \mathfrak{a} \diamond \mathfrak{b} &= (a_0 b_0) \otimes \Big((a_1 \otimes \cdots \otimes a_m) \diamond (\mathbf{1}_A \otimes b_1 \otimes \cdots b_n) \\ &\quad + (\mathbf{1}_A \otimes a_1 \otimes \cdots \otimes a_m) \diamond (b_1 \otimes \cdots b_n) \Big). \end{aligned}$$

Extending by additivity, $\diamond$ gives a $\mathbf{k}$-bilinear map

$$\diamond : Ш(A) \times Ш(A) \to Ш(A).$$

An Example of the Product

Example

$$(a_0 \otimes a_1)(b_0 \otimes b_1 \otimes b_2) = (a_0 b_0) \otimes \Big(a_1(\mathbf{1}_A \otimes b_1 \otimes b_2) + (\mathbf{1}_A \otimes a_1)(b_1 \otimes b_2) \Big).$$

Now the first term in the right tensor factor is just $a_1 \otimes b_1 \otimes b_2$. For the second term, we have

$$\begin{aligned}(\mathbf{1}_A \otimes a_1)(b_1 \otimes b_2) &= b_1 \otimes (a_1(\mathbf{1}_A \otimes b_2) + (\mathbf{1}_A \otimes a_1)b_2) \\ &= b_1 \otimes (a_1 \otimes b_2 + b_2 \otimes a_1).\end{aligned}$$

Thus we obtain

$$(a_0 \otimes a_1)(b_0 \otimes b_1 \otimes b_2) = (a_0 b_0) \otimes (a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1).$$

The Rota-Baxter Operator on $Ш(A)$

Define a linear endomorphism P_A on $Ш(A)$ by assigning

$$P_A(x_0 \otimes x_1 \otimes \ldots \otimes x_n) = \mathbf{1}_A \otimes x_0 \otimes x_1 \otimes \ldots \otimes x_n,$$

for all $x_0 \otimes x_1 \otimes \ldots \otimes x_n \in A^{\otimes(n+1)}$ and extending by additivity.

The Rota-Baxter Functors

- Let $F : \mathbf{ALG} \to \mathbf{RBA}$ denote the functor given on objects $A \in \mathbf{ALG}$ by $F(A) = (Ш(A), P_A)$ and on morphisms $f : A \to B$ in $\mathbf{ALG}$ by

$$F(f)\left(\sum_{i=1}^{k} a_{i0} \otimes a_{i1} \otimes \cdots \otimes a_{in_i}\right) = \sum_{i=1}^{k} f(a_{i0}) \otimes f(a_{i1}) \otimes \cdots \otimes f(a_{in_i})$$

  which we also denote by $Ш(f)$.
- As above, $U : \mathbf{RBA} \to \mathbf{ALG}$ denotes the forgetful functor.

The Natural Transformations

Next define two natural transformations $\eta : \mathbf{id}_{\mathbf{ALG}} \to UF$ and $\varepsilon : FU \to \mathbf{id}_{\mathbf{RBA}}$.

- For any $A \in \mathbf{ALG}$, define $\eta_A : A \to (UF)(A) = Ш(A)$ to be just the natural embedding $j_A : A \to Ш(A) = \bigoplus_{i\geq 1} A^{\otimes i}$.
- For any $(R, P) \in \mathbf{RBA}$, define

$$\varepsilon_{(R,P)} : (FU)(R, P) = (Ш(R), P_R) \to (R, P)$$

by

$$\varepsilon_{(R,P)} \left(\sum_{i=1}^{k} a_{i0} \otimes a_{i1} \otimes \cdots \otimes a_{in_i} \right) = \sum_{i=1}^{k} a_{i0} P(a_{i1} P(\cdots P(a_{in_i}) \cdots)),$$

for any $\sum_{i=1}^{k} a_{i0} \otimes a_{i1} \otimes \cdots \otimes a_{in_i} \in Ш(R)$.

Adjoint Functors for RBA

Theorem: The functor $F : \mathbf{ALG} \to \mathbf{RBA}$ defined above is the left adjoint of the forgetful functor $U : \mathbf{RBA} \to \mathbf{ALG}$. More precisely, there is an adjunction $\langle F, U, \eta, \varepsilon \rangle : \mathbf{ALG} \rightharpoonup \mathbf{RBA}$.

The Monad for Rota-Baxter Algebras

The adjunction $\langle F, U, \eta, \varepsilon \rangle : \mathbf{ALG} \rightharpoonup \mathbf{RBA}$ gives rise to a monad $\mathbf{T} = \langle T, \eta, \mu \rangle$ on **ALG**.

- T is the functor defined for any $A \in \mathbf{ALG}$ by $T(A) = Ш(A)$.
- μ is the natural transformation $\mu_A : Ш(Ш(A)) \to Ш(A)$ extended additively from

$$\begin{aligned} & \mu_A((a_{00} \otimes \cdots \otimes a_{0n_0}) \otimes \cdots \otimes (a_{k0} \otimes \cdots \otimes a_{kn_k})) \\ = \; & (a_{00} \otimes \cdots \otimes a_{0n_0}) P_A(\cdots P_A(a_{k0} \otimes \cdots \otimes a_{kn_k}) \cdots), \end{aligned}$$

where $(a_{00} \otimes \cdots \otimes a_{0n_0}) \otimes \cdots \otimes (a_{k0} \otimes \cdots \otimes a_{kn_k}) \in Ш(Ш(A))$ with $a_{i0} \otimes \cdots \otimes a_{in_i} \in A^{\otimes(n_i+1)}$ for $n_0, \ldots, n_k \geq 0$ and $0 \leq i \leq k$.

Algebras for the Rota-Baxter Monad T on ALG

- The monad $\mathbf{T}$ induces a category of $\mathbf{T}$-algebras, denoted by $\mathbf{ALG}^{\mathbf{T}}$.
- The objects in $\mathbf{ALG}^{\mathbf{T}}$ are pairs $\langle A, h\rangle$ where $A \in \mathbf{ALG}$ and $h : Ш(A) \to A$ is an algebra homomorphism satisfying the two properties

$$h \circ \eta_A = \mathbf{id}_A, \qquad h \circ T(h) = h \circ \mu_A.$$

- A morphism $\phi : \langle R, f\rangle \to \langle S, g\rangle$ in $\mathbf{ALG}^{\mathbf{T}}$ is an algebra homomorphism $\phi : R \to S$ such that $g \circ T(\phi) = \phi \circ f$.

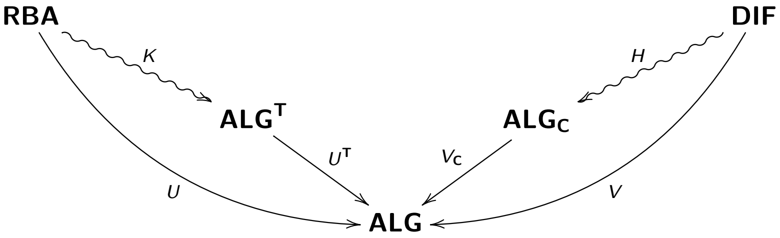
RBA is Monadic over ALG

Theorem: The comparison functor $K : \mathbf{RBA} \to \mathbf{ALG}^{\mathbf{T}}$ is an isomorphism, i.e., **RBA** is monadic over **ALG**.

Corollary: For any algebra A, there is a one-to-one correspondence between

1. Rota-Baxter operators P on A;
2. **T**-structures on A, i.e., algebra homomorphisms $h : Ш(A) \to A$ satisfying both $h \circ \eta_A = \mathbf{id}_A$ and $h \circ T(h) = h \circ \mu_A$;
3. Sequences of linear maps $h_n : Ш(A) \to A, n \in \mathbf{N}_+$, satisfying certain conditions.

The Big Picture Grows a Little More



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Rota-Baxter Operators on A Extend to $A^{\mathbf{N}}$

Proposition: Let (A, P) be a Rota-Baxter algebra.

- Define a $\mathbf{k}$-linear mapping $\widetilde{P} : A^{\mathbf{N}} \to A^{\mathbf{N}}$ by

  $$\widetilde{P}(f)(0) = P(f(0)), \quad \widetilde{P}(f)(n) = f(n-1), \quad f \in A^{\mathbf{N}}, n \in \mathbf{N}_+.$$

- Then $\widetilde{P}$ is a Rota-Baxter operator on $A^{\mathbf{N}}$,

  $$\varepsilon_A \circ \widetilde{P} = P \circ \varepsilon_A$$

  and

  $$\partial_A \circ \widetilde{P} = \mathbf{id}_{A^{\mathbf{N}}}.$$

- Compare with the First Fundamental Theorem of Calculus.

Definition of Mixed Distributive Law

Definition: Given a category **A**, a monad $\mathbf{T} = (T, \eta, \mu)$ on **A** and a comonad $\mathbf{C} = (C, \varepsilon, \delta)$ on **A**, then a **mixed distributive law of T over C** is a natural transformation $\beta : TC \to CT$ such that

- $\beta \circ \eta C = C\eta$;
- $\varepsilon T \circ \beta = T\varepsilon$;
- $\delta T \circ \beta = C\beta \circ \beta C \circ T\delta$ and
- $\beta \circ \mu C = C\mu \circ \beta T \circ T\beta$.

The Lifting Theorem

Theorem: Given a category $\mathbf{A}$, a monad $\mathbf{T} = (T, \eta, \mu)$ on $\mathbf{A}$, a comonad $\mathbf{C} = (C, \varepsilon, \delta)$ on $\mathbf{A}$, and a mixed distributive law of $\mathbf{T}$ over $\mathbf{C}$. Then:

- there is a comonad $\widetilde{\mathbf{C}}$ on the category $\mathbf{A}^{\mathbf{T}}$ of $\mathbf{T}$-algebras which lifts $\mathbf{C}$,
- there is a monad $\widetilde{\mathbf{T}}$ on the category $\mathbf{A}_{\mathbf{C}}$ of $\mathbf{C}$-coalgebras which lifts $\mathbf{T}$, and
- there is an isomorphism of categories $(\mathbf{A}_{\mathbf{C}})^{\widetilde{\mathbf{T}}} \cong (\mathbf{A}^{\mathbf{T}})_{\widetilde{\mathbf{C}}}$ over $\mathbf{A}$.

The Mixed Distributive Law

- We want to apply this theorem to the case where $\mathbf{A} = \mathbf{ALG}$, $\mathbf{T}$ is the Rota-Baxter monad and $\mathbf{C}$ is the differential comonad. So we need a mixed distributive law of $\mathbf{T}$ over $\mathbf{C}$.
- This means that for each $A \in \mathbf{ALG}$, we need a natural homomorphism $\beta_A : Ш(A^{\mathbf{N}}) \to (Ш(A))^{\mathbf{N}}$.
- By an earlier result, the Rota-Baxter operator P_A on $Ш(A)$ extends to a Rota-Baxter operator $\widetilde{P_A}$ on $(Ш(A))^{\mathbf{N}}$.

The Key Lemma

Lemma: For any algebra A, there is a unique Rota-Baxter algebra homomorphism

$$\beta_A : (Ш(A^{\mathbf{N}}), P_{A^{\mathbf{N}}}) \to ((Ш(A))^{\mathbf{N}}, \widetilde{P_A})$$

such that the equation

$$(\eta_A)^{\mathbf{N}} = \beta_A \circ \eta_{A^{\mathbf{N}}}$$

holds.

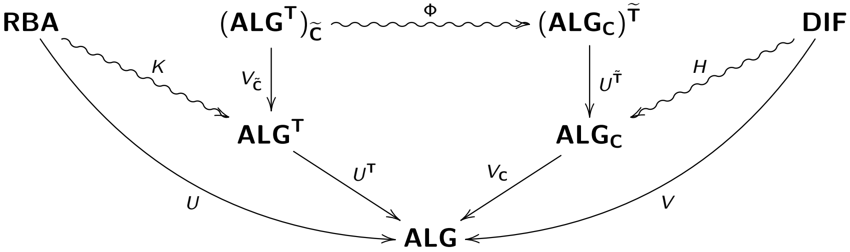
The Main Theorem

- The natural transformation $\beta : TC \to CT$ given by $\beta_A : Ш(A^{\mathbf{N}}) \to (Ш(A))^{\mathbf{N}}$ is a mixed distributive law of $\mathbf{T}$ over $\mathbf{C}$.
- $\beta : TC \to CT$ gives rise to a comonad $\widetilde{\mathbf{C}}$ on the category $\mathbf{ALG}^{\mathbf{T}}$ of $\mathbf{T}$-algebras which **lifts C** in the sense that the underlying functor $U^{\mathbf{T}} : \mathbf{ALG}^{\mathbf{T}} \to \mathbf{ALG}$ commutes with $\widetilde{\mathbf{C}}$ and $\mathbf{C}$, that is,

$$U^{\mathbf{T}}\widetilde{C} = CU^{\mathbf{T}}, \quad U^{\mathbf{T}}\widetilde{\varepsilon} = \varepsilon U^{\mathbf{T}} \quad \text{and} \quad U^{\mathbf{T}}\widetilde{\delta} = \delta U^{\mathbf{T}}.$$

- Similarly, β gives rise to a monad $\widetilde{\mathbf{T}}$ on the category $\mathbf{ALG}_{\mathbf{C}}$ of $\mathbf{C}$-coalgebras which lifts $\mathbf{T}$.
- There is an isomorphism $\Phi : (\mathbf{ALG}_{\mathbf{C}})^{\widetilde{\mathbf{T}}} \to (\mathbf{ALG}^{\mathbf{T}})_{\widetilde{\mathbf{C}}}$ of categories.

The Big Picture Grows Larger



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Differential Rota-Baxter Algebras

Definition: We say that (R, d, P) is a **differential Rota-Baxter algebra** if

- (R, d) is a differential algebra,
- (R, P) is a Rota-Baxter algebra, and
- $d \circ P = \mathbf{id}_R$.

If (R, d, P) and (R', d', P') are differential Rota-Baxter algebras, then a morphism of differential Rota-Baxter algebras $f : (R, d, P) \to (R', d', P')$ is an algebra homomorphism $f : R \to R'$ such that $d'(f(x)) = f(d(x))$ and $P'(f(x)) = f(P(x))$ for all $x \in R$. The category of differential Rota-Baxter algebras will be denoted by **DRB**.

DRB is Added to the Big Picture

There are forgetful functors:

- $U' : \mathbf{DRB} \to \mathbf{DIF}$ and
- $V' : \mathbf{DRB} \to \mathbf{RBA}$

such that $UV' = VU'$. We will see that:

- U' has a left adjoint,
- V' has a right adjoint, and
- $\mathbf{DRB} \cong (\mathbf{ALG_C})^{\widetilde{\mathbf{T}}} \cong (\mathbf{ALG^T})_{\widetilde{\mathbf{C}}}$, where $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{C}}$ come from the Main Theorem.

The Right Adjoint to V'

- Suppose that $(A, P) \in \mathbf{RBA}$.
- Let $\partial_A : A^{\mathbf{N}} \to A^{\mathbf{N}}$ and $\widetilde{P} : A^{\mathbf{N}} \to A^{\mathbf{N}}$ be as above.
- Since $\partial_A \circ \widetilde{P} = \mathbf{id}_{A^{\mathbf{N}}}$, the triple $(A^{\mathbf{N}}, \partial_A, \widetilde{P})$ is a differential Rota-Baxter algebra.
- Thus we have a functor $G' : \mathbf{RBA} \to \mathbf{DRB}$ given on objects $(A, P) \in \mathbf{RBA}$ by $G'(A, P) = (A^{\mathbf{N}}, \partial_A, \widetilde{P})$ and on morphisms $\varphi : (A, P) \to (A', P')$ in $\mathbf{RBA}$ by $(G'(\varphi)(f))(n) = \varphi(f(n))$ for $f \in A^{\mathbf{N}}$ and $n \in \mathbf{N}$.
- G' is the right adjoint to V'.

Another Comonad

The adjunction $\langle V', G', \eta', \varepsilon' \rangle$: **DRB** $\rightharpoonup$ **RBA** gives a comonad $\mathbf{C}' = \langle C', \varepsilon', \delta' \rangle$ on the category **RBA**, where

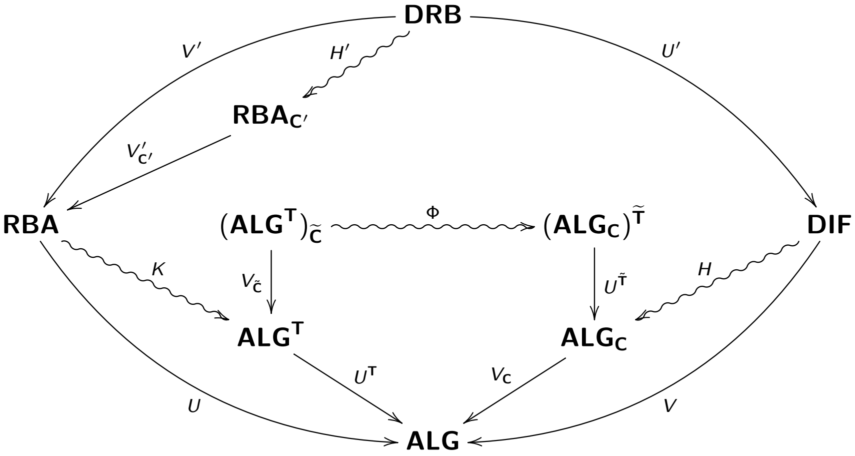
- $C' := V'G'$: **RBA** $\to$ **RBA** is given by $C'(A, P) = (A^{\mathbf{N}}, \widetilde{P})$,
- δ' is a natural transformation from C' to $C'C'$ defined by $\delta' := V'\eta' G'$.
- In other words, for any $(A, P) \in$ **RBA**,

$$\delta'_{(A,P)} : (A^{\mathbf{N}}, \widetilde{P}) \to ((A^{\mathbf{N}})^{\mathbf{N}}, \widetilde{\widetilde{P}}), \quad \delta'_{(A,P)}(f) = \delta_A(f), \quad f \in A^{\mathbf{N}}.$$

And Another Category of Coalgebras

- The comonad $\mathbf{C}'$ on **RBA** gives a category of $\mathbf{C}'$-coalgebras, denoted by $\mathbf{RBA}_{\mathbf{C}'}$.
- The comonad $\mathbf{C}'$ also induces an adjunction.
- There is a uniquely defined cocomparison functor $H' : \mathbf{DRB} \to \mathbf{RBA}_{\mathbf{C}'}$, and
- H' is an isomorphism, so that $\mathbf{DRB} \cong \mathbf{RBA}_{\mathbf{C}'}$.

More Functors for the Big Picture



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Some Folklore from Category Theory

Lemma: Suppose that $\mathbf{A}$ and $\mathbf{B}$ are categories, $K : \mathbf{A} \to \mathbf{B}$ is an isomorphism of categories, $\mathbf{C} = \langle C, \varepsilon, \delta \rangle$ is a comonad on $\mathbf{A}$ and $\mathbf{C}' = \langle C', \varepsilon', \delta' \rangle$ is a comonad on $\mathbf{B}$. If K commutes with $\mathbf{C}$ and $\mathbf{C}'$, i.e., $KC = C'K$, $K\varepsilon = \varepsilon' K$ and $K\delta = \delta' K$, then there exists a unique isomorphism $\widetilde{K} : \mathbf{A}_{\mathbf{C}} \to \mathbf{B}_{\mathbf{C}'}$ that lifts K, i.e., $U_{\mathbf{C}'}\widetilde{K} = KU_{\mathbf{C}}$.

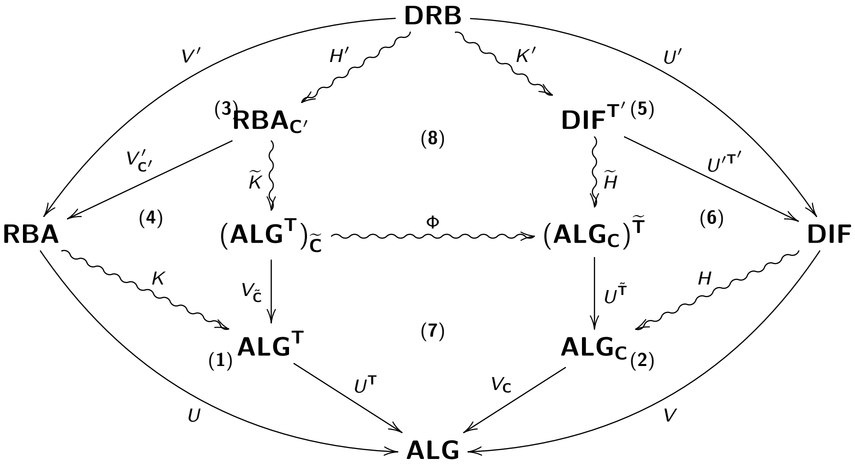
Corollary: There is an isomorphism of categories $\widetilde{K} : \mathbf{RBA}_{\mathbf{C}'} \to (\mathbf{ALG}^{\mathbf{T}})_{\widetilde{C}}$ such that $V_{\widetilde{\mathbf{C}}}\widetilde{K} = KV'_{\mathbf{C}'}$.

The Left Adjoint to U'

Let (A, d) be a differential algebra.

- There is a derivation $\widetilde{d}$ on $Ш(A) \to Ш(A)$ extending d.
- $(Ш(A), \widetilde{d}, P_A)$ is a free differential Rota-Baxter algebra on the differential algebra (A, d).
- There is a functor $F' : \mathbf{DIF} \to \mathbf{DRB}$ that is left adjoint to the forgetful $U' : \mathbf{DRB} \to \mathbf{DIF}$.
- There is a monad $\mathbf{T}'$ on $\mathbf{DIF}$ such that $\mathbf{DIF}^{\mathbf{T}'} \cong \mathbf{DRB}$.

To Complete the Picture



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