Interlacing of Hurwitz Series

Kolchin Seminar in Differential Algebra

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Outline

- Background for Hurwitz series
  - Categorical origins
  - Natural topology
  - Divided powers
  - Composition
  - exp and log
- Interlacing of Hurwitz series
- Examples
- Circular functions
- More examples

Notation

- All rings are commutative with identity and all ring homomorphisms preserve the identity.

- $\mathbf{N} = \{0, 1, 2, \ldots\}$ denotes the natural numbers.

- $\mathbf{N}_+ = \{1, 2, 3, \ldots\}$ denotes the positive integers.

- **Q**, **R** and **C** denote the fields of rational numbers, real numbers and complex numbers respectively.

- For any $m, n \in \mathbf{N}$, δ_n^m will denote the Kronecker delta, i.e., $\delta_n^m = 1$ if $m = n$ and $\delta_n^m = 0$ if $m \neq n$.

More notations

- A differential ring is a ring R with a derivation $d : R \to R$, i.e., $d(x + y) = d(x) + d(y)$ and $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.
- Let R be a differential ring and let $y_1, y_2, \ldots, y_n \in R$. We denote the Wronskian of $y_1, y_2, \ldots, y_n$ by $W(y_1, y_2, \ldots, y_n)$, where $W(y_1, y_2, \ldots, y_n)$ is the determinant of the $n \times n$ matrix $(d^{(i-1)}(y_j))$.
- The set of all $n \times n$ matrices and $n \times n$ invertible matrices over a ring A will be denoted by $M(n, A)$ and $GL(n, A)$ respectively.
- Unless otherwise noted, A will denote a ring of any characteristic.

The Ring of Hurwitz Series over A

Definition: For any ring A, the ring of Hurwitz series over A, denoted by HA, has:

- Elements that are sequences $(a_n) = (a_0, a_1, a_2, \ldots)$, where $a_n \in A$ for each $n \in \mathbf{N}$.
- Addition defined termwise, i.e., for any $(a_n), (b_n) \in HA$.

  $$(a_n) + (b_n) = (c_n), \quad \text{where} \quad c_n = a_n + b_n$$

  for all $n \in \mathbf{N}$.
- The (Hurwitz) product of (a_n) and (b_n) given by

  $$(a_n) \cdot (b_n) = (c_n), \quad \text{where} \quad c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

  for all $n \in \mathbf{N}$.

More about Hurwitz Series

- A sequence (a_n) with $a_n \in A$ is a function $f : \mathbf{N} \to A$.
- Denote the set of all functions $f : \mathbf{N} \to A$ by $A^{\mathbf{N}}$, so that as additive abelian groups $HA \cong A^{\mathbf{N}}$.
- For $f, g \in HA$, we can write $(f+g)(n) = f(n) + g(n)$ and $(f \cdot g)(n) = \sum_{k=0}^{n} \binom{n}{k} f(k) g(n-k)$ for all $n \in \mathbf{N}$.
- The multiplicative identity of HA is $1_{HA} = 1$, the function $\mathbf{N} \to A$ given by $1(n) = \delta_{n,0}$, i.e., $1 = (1, 0, 0, 0, \ldots)$.
- If $\mathbf{Q} \subseteq A$, then $HA \cong A[[t]]$ via the mapping $f \mapsto \sum_{n=0}^{\infty} \frac{f(n)}{n!} t^n$.

Invertibility of Hurwitz Series

- Let $A^{\times}$ denote the multiplicative group of invertible elements in A.
- There is a natural ring homomorphism $\varepsilon : HA \to A$ given by $\varepsilon(f) = f(0)$ for any $f \in HA$.
- A Hurwitz series $f \in HA$ is invertible in HA if and only if $\varepsilon(f)$ is invertible in A.
- $(HA)^{\times} = \varepsilon^{-1}(A^{\times})$.
- If $f \in HA$ with $f(0)$ invertible in A, then $f^{-1} \in HA$ is given by $f^{-1}(0) = f(0)^{-1}$ and $f^{-1}(n) = -f(0)^{-1} \sum_{k=1}^{n} \binom{n}{k} f(k) f^{-1}(n-k)$ for $n \in \mathbf{N}_{+}$.

More about Hurwitz Series

- HA is a differential ring with derivation $\partial : HA \to HA$ given by

$$\partial((a_0, a_1, a_2, \dots)) = (a_1, a_2, a_3, \dots),$$

or by $(\partial(f))(n) = f(n+1)$ for $f \in HA$ and $n \in \mathbf{N}$.

- HA is an A-algebra by the homomorphism $\iota : A \to HA$ where $\iota(a) = (a, 0, 0, 0, \dots)$ or $(\iota(a))(n) = a\delta_n^0$ for $n \in \mathbf{N}$.

Lemma

For any $f, g \in HA$, $f = g$ if and only if $\varepsilon(f) = \varepsilon(g)$ and $\partial(f) = \partial(g)$.

Integration Operator

- HA also has a natural integration operator, denoted by $\int : HA \to HA$ and defined for any $f \in HA$ by $(\int(f))(0) = 0$ and $(\int(f))(n) = f(n-1)$ for $n \in \mathbf{N}_+$.
- So $\int((a_0, a_1, a_2, \ldots)) = (0, a_0, a_1, \ldots)$.
- For any $f \in HA$, $\partial(\int(f)) = f$.
- On the other hand, $\int(\partial(f)) = (0, a_1, a_2, \ldots,) = (a_0, a_1, a_2, \ldots,) - (a_0, 0, 0, \ldots,) = f - \iota(\varepsilon(f))$
- For any $f \in HA$, $\varepsilon(\int(f)) = 0$.
- $\int$ satisfies the formula for integration by parts, which may be written as $(\int f)(\int g) = \int(f \int g + g \int f)$.

Solving Differential Equations

Theorem

A complete set of n linearly independent solutions to any n^{th}-order monic homogeneous linear differential equation with coefficients in HA can be found in HA.

Example

As a simple example, consider the differential equation $y' = by$ with initial condition $y(0) = c$, where $b, c \in A$. The unique solution $y \in HA$ is given by $y = (c, cb, cb^2, \ldots)$.

Hurwitz Series as an Adjoint Functor

- Let **Comm** denote the category of commutative rings with identity and **Diff** the category of (ordinary) differential rings.
- There are functors U : **Diff** $\to$ **Comm** and G : **Comm** $\to$ **Diff** with G **right** adjoint to U.
- U is the "forgetful" functor, i.e., $U(R, d) = R$.
- G is the Hurwitz series functor, i.e., $G(A) = (HA, \partial)$.
- $\mathrm{Hom}_{\mathbf{Comm}}(U(R, d), A) \cong \mathrm{Hom}_{\mathbf{Diff}}((R, d), G(A))$.

Universal Embedding Property of Hurwitz Series

- Any differential ring (A, d) can be naturally differentially embedded in the ring of Hurwitz series (HA, ∂).
- The embedding $\eta : (A, d) \to (HA, \partial)$ is given by $(\eta(a))(n) = d^{(n)}(a)$ for any $a \in A$ and $n \in \mathbf{N}$.
- As a sequence, $\eta(a) = (a, d(a), d^{(2)}(a), \ldots)$.
- $\eta : (A, d) \to (HA, \partial)$ is called the Hurwitz homomorphism of (A, d).

Example

As an example, take $(A, d) = (\mathbf{C}(t), \frac{d}{dt})$, and observe that for any $f \in \mathbf{C}(t)$, $\eta : \mathbf{C}(t) \to H(\mathbf{C}(t))$ is given by $(\eta(f))(n) = (\frac{d^n}{dt^n}(f))$.

Definition of Order

- The order of any $h \in HA, h \neq 0$, denoted by $\operatorname{ord}(h)$, is the minimum $i \in \mathbf{N}$ such that $\varepsilon(\partial^i(h)) \neq 0$, and $\operatorname{ord}(0) = \infty$.
- If $h \neq 0$, then $\operatorname{ord}(h)$ is the number of initial zeros occurring in the sequence $(h(0), h(1), h(2), \ldots)$ before the first non-zero entry in the sequence.
- Define $H_0A := \varepsilon^{-1}(0) = \{h \in HA \mid \operatorname{ord}(h) > 0\}$.
- H_0A is an ideal in HA, and as an additive group $H_0A \cong A^{\mathbf{N}}$.

Properties of Order

Lemma

For any $f, g \in HA$,

1. $\mathrm{ord(f+g)} \geq \min\{\mathrm{ord(f)}, \mathrm{ord(f)}\}$.
2. $\mathrm{ord}(fg) \geq \mathrm{ord}(f) + \mathrm{ord}(g)$.
3. *If $f \in H_0A$ and $f \neq 0$, then* $\mathrm{ord}(\partial f) = \mathrm{ord}(f) - 1$.
4. *If $f \neq 0$, then* $\mathrm{ord}(\int f) = \mathrm{ord}(f) + 1$.

The Natural Topology on HA

- Define $\omega : HA \times HA \to \mathbf{N} \cup \{\infty\}$ by $\omega(f, g) := \mathrm{ord}(f - g)$.
- For $n \in \mathbf{N}$, $\omega(f, g) \geq n + 1$ if and only if f and g agree up to at least their nth terms.
- Define $d : HA \times HA \to \mathbf{R}$ by $d(f, g) := (\frac{1}{2})^{\omega(f,g)}$, where $(\frac{1}{2})^{\infty} = 0$.
- This gives a metric on HA that is complete, and for which the addition, multiplication, derivation and integration are continuous.

Divided Powers

- For $n \in \mathbf{N}$ and $f \in HA$, define the n^{th} divided power of f, denoted by $f^{[n]} \in HA$, inductively as follows.
- Set $f^{[0]} = 1_{HA}$ and $f^{[n]} = \int (f^{[n-1]}\partial(f))$ for $n \in \mathbf{N}_+$.
- Let $x = \int 1$. For any $n \in \mathbf{N}$, let $\langle n \rangle$ denote the n^{th} divided power of x, so that, for example,
  - $\langle 0 \rangle = (1, 0, 0, 0, \ldots) = 1$,
  - $\langle 1 \rangle = (0, 1, 0, 0, \ldots) = x$,
  - $\langle 2 \rangle = (0, 0, 1, 0, \ldots)$,

  and so on.
- For any $f \in HA$, $f = \sum_{n \in \mathbf{N}} f(n)\langle n \rangle = \lim_{m \to \infty}(\sum_{n=0}^{m} f(n)\langle n \rangle)$, so that $\{\langle n \rangle | n \in \mathbf{N}\}$ forms a basis for HA.

Properties of Divided Powers

Proposition

Let $f, g \in HA$, and $m, n \in \mathbf{N}$. Then

1. $\partial(f^{[0]}) = 0$ *and* $\partial(f^{[n]}) = f^{[n-1]}\partial(f)$ *for* $n \in \mathbf{N}_+$, *an analogue of the familiar "power rule" from calculus.*
2. $\varepsilon(f^{[0]}) = 1$ *and* $\varepsilon(f^{[n]}) = 0$ *for* $n \in \mathbf{N}_+$.
3. $(f+g)^{[n]} = \sum_{i+j=n} f^{[i]}g^{[j]}$.
4. $(fg)^{[n]} = f^n g^{[n]}$.
5. $f^{[m]} \cdot f^{[n]} = \binom{m+n}{m} f^{[m+n]}$.
6. $(f^{[m]})^{[n]} = \frac{(mn)!}{(m!)^n n!} f^{[mn]}$ *if* $m \in \mathbf{N}_+$.
7. $n! f^{[n]} = f^n$.

A Combinatorial Result

- For $n \in \mathbf{N}$ and $m \in \mathbf{N}_+$, define $\langle n, m \rangle = \frac{(mn)!}{(m!)^n n!}$, as in item 6 in the previous slide.
- For example $\langle 2, 3 \rangle = 10$ and $\langle 3, 2 \rangle = 15$.

Lemma

For any $n \in \mathbf{N}$ and $m \in \mathbf{N}_+$,

$$\langle n, m \rangle = \prod_{k=1}^{n} \binom{km-1}{m-1}.$$

Notes on $\langle n, m\rangle$

- The calculation of $\langle n, m\rangle$ is made much easier using this Lemma.
- It is easier to see when $\langle n, m\rangle$ is divisible by a power of a prime p using Kummer's Theorem.
- **Question** Does $\langle n, m\rangle$ have any other useful properties? Answer - yes, as in the example below and more examples later.

Example

When $m = 2$, $\langle n, 2\rangle = \prod_{k=1}^{n}(2k-1) = (2n-1)!!$, the double factorial of $2n-1$.

Composition of Hurwitz Series

- The divided powers can be used to define a composition of Hurwitz series as follows.
- Suppose that $f \in HA$ and $g \in H_0A$. Then the composition of f and g is denoted by $f \circ g$ and is given by

$$f \circ g = \sum_{n \in \mathbf{N}} f(n) g^{[n]}$$

.

- The condition that $g \in H_0A$ is necessary to insure that the sum $\sum_{n \in \mathbf{N}} f(n) g^{[n]}$ converges in the natural topology on HA.
- This operation of composition satisfies the usual properties as expected of a notion of composition, as in the following.

Properties of Composition

Suppose that $f, g \in HA$, $h \in H_0A$, $m \in \mathbf{N}_+$ and $n \in \mathbf{N}$. Then

1. $(f+g)\circ h = f\circ h + g\circ h$.
2. $(fg)\circ h = (f\circ h)(g\circ h)$.
3. $\langle 1\rangle \circ h = h$ and $f\circ\langle 1\rangle = f$.
4. $\varepsilon(g\circ h) = \varepsilon(g)$
5. Chain Rule: $\partial(g\circ h) = (\partial(g)\circ h)\cdot\partial(h)$.
6. Substitution Rule: $\int((f\circ g)\partial(g)) = (\int(f))\circ g$
7. If $g \in H_0A$, then $(g\circ h)^{[n]} = g^{[n]}\circ h$. In particular, $h^{[n]} = \langle n\rangle\circ h$.
8. $\langle n\rangle\circ\langle m\rangle = \langle n, m\rangle\langle nm\rangle$.
9. If $g \in H_0A$, then $(f\circ g)\circ h = f\circ(g\circ h)$.
10. If $f \in H_0A$, then there is some $g \in H_0A$ such that $f\circ g = \langle 1\rangle$ if and only if $f(1)$ is a unit in A. In this case, g is unique, $g(1)$ is a unit in A and $g\circ f = \langle 1\rangle$.

Definition of exp

- Recall $\varepsilon : HA \to A$ is given by $\varepsilon(f) = f(0)$ for any $f \in HA$.
- Also $H_0A = \varepsilon^{-1}(0)$.
- $f \in HA$ is invertible in HA if and only if $\varepsilon(f)$ is invertible in A.
- Define $H_1A := \varepsilon^{-1}(1)$, so that $H_1A \subset (HA)^\times$.
- Define $\exp : H_0A \to H_1A$ by

$$\exp(h) := \sum_{n \in \mathbf{N}} h^{[n]}$$

for any $h \in H_0A$.

Properties of exp

Proposition

Let $g, h \in H_0A$. Then

1. $\exp(g+h) = \exp(g) \cdot \exp(h)$
2. $\exp(0) = 1_{HA}$
3. $\exp(-h) = \exp(h)^{-1}$
4. $\partial(\exp(h)) = \exp(h)\partial h$
5. $\varepsilon(\exp(h)) = 1$

Observe that $\exp : (H_0A, +) \to (H_1A, \cdot)$ is a group homomorphism.

Notes about exp

Remark

- *Note that* $\exp(x) = (1, 1, 1, 1, \ldots)$*, which we denote by* $e \in HA$*, where* $x = (0, 1, 0, 0, \ldots,)$.
- *For all* $k \in \mathbf{Z}$*,* $e^k(n) = k^n$*.*
- *For any* $a \in A$*,* $\exp(ax) = (1, a, a^2, a^3, \ldots) = (a^n)$*.*
- *Also* $\exp(h) = \exp(x) \circ h = e \circ h$*.*
- *For example,* $\exp(\langle 2\rangle) = e \circ \langle 2\rangle = \sum_{n\in\mathbf{N}} \langle 2\rangle^{[n]} = \sum_{n\in\mathbf{N}} \langle n\rangle \circ \langle 2\rangle = \sum_{n\in\mathbf{N}} \langle n, 2\rangle\langle 2n\rangle = \sum_{n\in\mathbf{N}} (2n-1)!!\langle 2n\rangle$*.*
- *Similarly,* $\exp(\langle 3\rangle) = \sum_{n\in\mathbf{N}} \langle n, 3\rangle\langle 3n\rangle$*, etc.*

An Example of exp(f)

- Let $f \in HA$ be defined by $f(n) = n$ for $n \in \mathbf{N}$.
- Then $f \in H_0A$ and so we have $\exp(f) = \sum_{n\in\mathbf{N}} f^{[n]} \in H_1A$.
- Computing $\exp(f)$ by hand, we see that the first few terms of $\exp(f)$ are $(1, 1, 3, 10, 41, 196, 1057, 6322, \ldots)$.
- This is sequence A000248 in OEIS.
- The nth term a_n in this sequence counts the number of forests with n nodes and height at most 1.
- a_n is the number of idempotents in $\mathrm{End}(\{1, 2, \ldots, n\})$.
- Also, $(\exp(f))(n) = \sum_{k=0}^{n} \binom{n}{k}(n-k)^k$.

Definition and Properties of log

Define $\log : (HA)^{\times} \to H_0 A$ by

$$\log(g) := \int (g^{-1} \cdot \partial(g))$$

for any $g \in (HA)^{\times}$.

Proposition

Suppose that $g, h \in (HA)^{\times}$. Then

1. $\log(gh) = \log(g) + \log(h)$.
2. $\log(1_{HA}) = 0$.
3. $\log(g^{-1}) = -\log(g)$.
4. $\log(g^n) = n \cdot \log(g)$ *for any* $n \in \mathbf{Z}$.
5. $\partial(\log(g)) = g^{-1} \cdot \partial(g)$.
6. $\varepsilon(\log(g)) = 0$.

exp and log are Inverse

Proposition

For any $f \in H_0A$ *and any* $g \in H_1A$,

- $\log(\exp(f)) = f$
- $\exp(\log(g)) = g$

Remark

Hence $\log : (H_1A, \cdot) \to (H_0A, +)$ *and* $\exp : (H_0A, +) \to (H_1A, \cdot)$ *are inverse group isomorphisms.*

Naturality of exp and log

Proposition

For any ring homomorphism $h : A \to B$, the following diagrams commute:

$$\begin{array}{ccc} H_0 A & \xrightarrow{Hh} & H_0 B \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ H_1 A & \xrightarrow[Hh]{} & H_1 B \end{array} \quad \text{and} \quad \begin{array}{ccc} H_1 A & \xrightarrow{Hh} & H_1 B \\ \downarrow{\scriptstyle \log} & & \downarrow{\scriptstyle \log} \\ H_0 A & \xrightarrow[Hh]{} & H_0 B \end{array}$$

Hence exp *and* log *are natural isomorphisms between the functors H_0 and H_1.*

An Observation

- $\Phi : (H_0A, \circ, x) \to (\mathrm{End}_+(H_0A, +, 0), \circ, x)$ defined by $(\Phi(g))(f) = f \circ g$ is a monoid antihomomorphism, but it is not natural.
- Similarly $\Psi : (H_1A, \circ, x) \to (\mathrm{End}_\times(H_1A, \times, 1), \circ, x)$ defined by $(\Psi(g))(f) = f \circ g$ is a monoid antihomomorphism, but it is not natural.
- Also, the following diagrams commute:

$$\begin{array}{ccc} H_0A & \xrightarrow{\Phi(g)} & H_0A \\ \exp \downarrow & & \downarrow \exp \\ H_1A & \xrightarrow[\Psi(g)]{} & H_1A \end{array} \quad \text{and} \quad \begin{array}{ccc} H_1A & \xrightarrow{\Psi(g)} & H_1A \\ \log \downarrow & & \downarrow \log \\ H_0A & \xrightarrow[\Phi(g)]{} & H_0A \end{array}$$

Exponentiation

Let $f \in (HA)^\times$ and $g \in HA$. Then define $f^g \in H_1A$ by

$$f^g := \exp(g \cdot \log(f)).$$

The usual properties of exponentiation hold.

Proposition

Let $f, f_1, f_2 \in (HA)^\times$ and $g, g_1, g_2 \in HA$. Then

1. $f^{g_1+g_2} = f^{g_1} \cdot f^{g_2}$.
2. $(f_1 \cdot f_2)^g = f_1^g \cdot f_2^g$.
3. $(f^{g_1})^{g_2} = f^{g_1 \cdot g_2}$.
4. $\log(f^g) = g \cdot \log(f)$.
5. $\partial(f^g) = f^g \left[\partial(g) \cdot \log(f) + g \cdot f^{-1} \cdot \partial(f)\right]$.
6. $\varepsilon(f^g) = 1$.
7. *If $h \in H_0A$, then $\exp(g \cdot h) = (\exp(h))^g$.*

Notations

- Let $n \in \mathbf{N}_+$ be a (fixed) positive integer.
- Let $\mathbf{N_n} = \{0, 1, \ldots, n-1\} \subset \mathbf{N}$.
- For any $m \in \mathbf{N}$, define $\widehat{m} := \lfloor \frac{m}{n} \rfloor \in \mathbf{N}$
- For any $m \in \mathbf{N}$, define $\overline{m} := m - n\widehat{m} \in \mathbf{N_n}$.
- As in the Division Algorithm, $m = n\widehat{m} + \overline{m}$.
- $\widehat{m + nq} = \widehat{m} + q$ for any $q \in \mathbf{N}$
- $\overline{m + nq} = \overline{m}$ for any $q \in \mathbf{N}$.

Definition of Interlacing

Definition

Let $z_0, z_1, \ldots, z_{n-1} \in HA$. The interlacing of $z_0, z_1, \ldots, z_{n-1}$, denoted by $\mathrm{int}(z_0, z_1, \ldots, z_{n-1})$, or more compactly by $\mathrm{int}_{j \in \mathbf{N}_n}(z_j)$, is the Hurwitz series $x \in HA$ defined by $x(m) = z_{\overline{m}}(\widehat{m})$.

- Observe that if $n = 1$, then $\mathrm{int}(z_0) = z_0$, since if $n = 1$, then $\overline{m} = 0$ and $\widehat{m} = m$. So for the interesting cases we may assume that $n \geq 2$.
- For example, if $z_0 = (1, 2, 3, \ldots)$, $z_1 = (4, 8, 16, \ldots)$, and $z_2 = (0, 0, 0, \ldots)$, then $\mathrm{int}(z_0, z_1, z_2) = (1, 4, 0, 2, 8, 0, 3, 16, 0, \ldots)$.
- Note that $\mathrm{int}(z_0, z_1, \ldots, z_{n-1}) = \sum_{n \in \mathbf{N}} \sum_{i=0}^{n-1} z_i(k)\langle nk + i\rangle$.

Properties of Interlacing

Proposition

Let $(u_0, \ldots, u_{n-1}), (v_0, \ldots, v_{n-1}) \in (HA)^n$ *and let* $a \in A$. *Then*

1. $\mathrm{int}(u_0, \ldots, u_{n-1}) = 0$ *if and only if* $u_0 = 0, \ldots, u_{n-1} = 0$,
2. $\mathrm{int}(u_0, \ldots, u_{n-1}) + \mathrm{int}(v_0, \ldots, v_{n-1}) = \mathrm{int}(u_0 + v_0, \ldots, u_{n-1} + v_{n-1})$,
3. $a(\mathrm{int}(u_0, \ldots, u_{n-1})) = \mathrm{int}(au_0, \ldots, au_{n-1})$,
4. $\varepsilon(\mathrm{int}(u_0, u_1, \ldots, u_{n-1})) = \varepsilon(u_0)$,
5. $\partial(\mathrm{int}(u_0, u_1, \ldots, u_{n-1})) = \mathrm{int}(u_1, \ldots, u_{n-1}, \partial(u_0))$,
6. $\int(\mathrm{int}(u_0, u_1, \ldots, u_{n-1})) = \mathrm{int}(\int(u_{n-1}), u_0, \ldots, u_{n-2})$

Main Result on Interlacing

Theorem

Suppose that

- $n \in \mathbf{N}_+$,
- $a_0, a_1, \ldots, a_{n-1} \in A$,
- $z = (z_0, z_1, \ldots, z_{n-1}) \in (HA)^n$,
- $u = \mathrm{int}(z_0, z_1, \ldots, z_{n-1}) = \mathrm{int}_{j \in \mathbf{N}_n}(z_j) \in HA$.

Then

- u *is a solution in* HA *of* $L(y) = \partial^n y + \sum_{i=0}^{n-1} a_i \partial^i y = 0$ *if and only if*
- $Z = z^t$ *is a solution of the* $n \times n$ *matrix equation* $\mathcal{L}Z' = \mathcal{U}Z$,

where $Z' = (\partial(z_0), \partial(z_1), \ldots, \partial(z_{n-1}))^t$, *and* $\mathcal{L}$ *and* $\mathcal{U}$ *are the matrices*

Main Result (Continued)

$$\mathcal{L} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ a_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ a_{n-2} & a_{n-1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & 1 & 0 \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & 1 \end{pmatrix} \quad \text{and}$$

$$\mathcal{U} = \begin{pmatrix} -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \\ 0 & -a_0 & -a_1 & \cdots & -a_{n-3} & -a_{n-2} \\ 0 & 0 & -a_0 & \cdots & -a_{n-4} & -a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -a_0 & -a_1 \\ 0 & 0 & 0 & \cdots & 0 & -a_0 \end{pmatrix}$$

respectively.

An Example

Consider $y'' - y' - y = 0$, with $a_0 = a_1 = -1$. Then $\mathcal{L}Z' = \mathcal{U}Z$ is

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} z_0' \\ z_1' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}.$$

Solving for Z', we obtain

$$\begin{pmatrix} z_0' \\ z_1' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}. \tag{1}$$

To find two linearly independent solutions $u, v \in HA$ to $y'' - y' - y = 0$, we first find $u_0, u_1 \in HA$ solutions to (1) with $\varepsilon(u_0) = 1$ and $\varepsilon(u_1) = 0$. In this way we obtain $u_0 = (1, 1, 2, 5, \ldots)$ and $u_1 = (0, 1, 3, 8, \ldots)$, so that

The Example Continues

$$u = \mathrm{int}(\mathrm{u}_0, \mathrm{u}_1) = (1, 0, 1, 1, 2, 3, 5, 8, \ldots).$$

Similarly we find $v_0, v_1 \in HA$ solutions to (1) with $\varepsilon(v_0) = 0$ and $\varepsilon(v_1) = 1$, and we get

$$v = \mathrm{int}(\mathrm{v}_0, \mathrm{v}_1) = (0, 1, 1, 2, 3, 5, 8, \ldots).$$

Note that these two solutions are Fibonacci sequences. While $u = \mathrm{int}(\mathrm{u}_0, \mathrm{u}_1)$ is a solution to $y'' - y' - y = 0$, $\mathrm{int}(\mathrm{u}_1, \mathrm{u}_0) = (0, 1, 1, 1, 3, 2, 8, 5, \ldots)$ is not a solution, so the order of the interlacing matters. Also, since $u' = v$ and $v' = u'' = u' + u = u + v$, it follows that the Wronskian is $W(u, v) = uv' - u'v = u^2 + uv - v^2$. Since $\partial(u^2 + uv - v^2) = u^2 + uv - v^2$ and $\varepsilon(u^2 + uv - v^2) = 1$, it follows that $u^2 + uv - v^2 = e = (1, 1, 1, \ldots)$.

Another Example

Let $L(y) = y''' - 3y'' + 3y' - y$, so that

$$\mathcal{L} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & -3 & 1 \end{pmatrix}, \qquad \mathcal{U} = \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix},$$

and $\mathcal{M} = \mathcal{L}^{-1}\mathcal{U} = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -8 & 6 \\ 6 & -15 & 10 \end{pmatrix}$. As above we get

$$u_1 = (1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots) = e$$

is one solution of $L(y) = 0$. Also

$$u_2 = (0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots) = \langle 1 \rangle \cdot e = x \cdot e$$

is another solution and

$$u_3 = (0, 0, 1, 3, 6, 10, 15, 21, 28, \ldots) = \langle 2 \rangle \cdot e$$

is a third solution, and $W(u_1, u_2, u_3) = u_1^3 = e^3$.

Restating the Main Theorem

The main theorem can be reformulated as follows:

- Let $\mathfrak{l} \in M(n, A)$ denote the matrix (δ^i_{j+1})
- Let $\mathfrak{u} \in M(n, A)$ denote the matrix (δ^i_{j-1}).
- Then the matrices $\mathcal{L}$ and $\mathcal{U}$ in the theorem can be written as $\mathcal{L} = I + \sum_{k=1}^{n-1} a_{n-k}\mathfrak{l}^k$ and $\mathcal{U} = -\sum_{k=0}^{n-1} a_k \mathfrak{u}^k$.
- Note also that $\mathfrak{l}$ and $\mathfrak{u}$ are nilpotent of order at most n.
- Hence $\mathcal{L} \in GL(n, A)$.
- Also $\mathcal{U} \in GL(n, A)$ if and only if a_0 is invertible in A.

Circulant Matrices

Note that the matrix

$$\mathcal{L} - I - \mathcal{U} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_0 & a_1 \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_0 \end{pmatrix}$$

is a circulant matrix. Properties of and applications of circulant matrices are well-known. For example, eigenvectors and eigenvalues and determinants of circulant matrices can be written in terms of n^{th} roots of unity. Also, the eigenvectors of a circulant matrix are the columns of the unitary discrete Fourier transform matrix of the same size. Linear equations involving circulant matrices can be solved using techniques involving fast Fourier transforms much faster than ordinary Gaussian elimination.

Even and Odd Hurwitz Series

- Given $f \in HA$, we can write $f = \mathrm{int}(f_0, f_1)$, where $f_0(n) = f(2n)$ and $f_1(n) = f(2n+1)$ for all $n \in \mathbf{N}$.
- Note that $\mathrm{int}(f, 0) = \sum_{n\in\mathbf{N}} f(n)\langle 2n\rangle$ and $\mathrm{int}(0, f) = \sum_{n\in\mathbf{N}} f(n)\langle 2n+1\rangle$.
- We say that $f \in HA$ is even if $f \circ (-x) = f$, and f is odd if $f \circ (-x) = -f$, where $x = (0, 1, 0, 0, \dots,)$.
- Then f is even if and only if $f = \mathrm{int}(f_0, 0)$ and f is odd if and only if $f = \mathrm{int}(0, f_1)$, where $f_0, f_1 \in HA$ are as above.
- We can split any $f \in HA$ into its even and odd parts by $f = \mathrm{int}(f_0, 0) + \mathrm{int}(0, f_1)$.

sin and cos

- Recall that $e = \exp(x) = (1, 1, 1, \ldots)$.
- Since $e \in H_1 A$, e^{-1} denotes the multiplicative inverse of e in HA.
- As functions $\mathbf{N} \to A$, $e(n) = 1^n = 1$, while $e^{-1}(n) = (-1)^n$ for all $n \in \mathbf{N}$.
- Also, $\partial(e) = e$ and $\partial(e^{-1}) = -e^{-1}$.
- Define:
  $\mathsf{sin} := \mathrm{int}(0, e^{-1})$ and $\mathsf{cos} := \mathrm{int}(e^{-1}, 0)$.
- It follows that $\partial(\sin) = \cos$ and $\partial(\cos) = -\sin$.
- Also $W(\cos, \sin) = \cos^2 + \sin^2 = 1$.

More about sin and cos

- Both sin and cos are solutions in HA of $L(y) = \partial^2(y) + y = 0$.
- In the notation of the main theorem, $a_0 = 1$ and $a_1 = 0$, and the matrix equation $\mathcal{L}Z' = \mathcal{U}Z$ becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_0' \\ z_1' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}.$$

- Both $\begin{pmatrix} e^{-1} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ e^{-1} \end{pmatrix}$ are linearly independent solutions to the matrix equation.
- Of course one can see by direct calculation that both sin and cos are linearly independent solutions in HA of $L(y) = \partial^2(y) + y = 0$.

Other Circular Functions

- In the real world, e^{-x} is asymptotic to 0, while $\sin x$ and $\cos x$ are periodic functions.
- Perhaps the "interlaced" nature of sin and cos may explain their periodic behavior.
- It certainly explains their even or odd behavior, since $\sin = \mathrm{int}(0, e^{-1})$ and $\cos = \mathrm{int}(e^{-1}, 0)$.
- Since $\cos \in H_1 A$, we can further define $\sec := \cos^{-1}$ and $\tan := \sin \cdot \sec$.
- A straightforward calculation shows that $\partial(\sec) = \sec \cdot \tan$ and that $\partial(\tan) = 1 + \tan^2 = \sec^2$.
- However, since $\sin \in H_0 A$, we can define neither csc nor cot as in the real world.

sec and tan

- Computing $\sec = \cos^{-1}$ "by hand," the first few terms of sec are $\sec = (1, 0, 1, 0, 5, 0, 61, 0, \ldots)$.
- Similarly, the first few terms of $\tan = \sin \cdot \sec$ are $\tan = (0, 1, 0, 2, 0, 16, 0, 272, \ldots)$.
- In particular, if A_n denotes the sequence of Euler zigzag numbers, i.e, the number of alternating permutations of $\mathbf{N_n}$, then $\sec = \mathrm{int}(A_{2n}, 0)$ and $\tan = \mathrm{int}(0, A_{2n+1})$.
- Hence $\sec + \tan = A_n$.
- Also, $\partial(\log(\sec + \tan)) = \sec$ as usual.

Inverse Functions

- Observe that $\sin \in H_0A$ has $\sin(1) = 1$.
- Hence there is a unique $h \in H_0A$ with $h(1)$ a unit in A such that $\sin \circ h = x = \langle 1 \rangle$.
- Denote this unique h by arcsin.
- Since $\sin \circ \arcsin = x$, $\partial(\sin \circ \arcsin) = \partial(x)$.
- By the chain rule, $(\cos \circ \arcsin)\partial(\arcsin) = 1$.
- It follows that $\partial(\arcsin) = (\cos \circ \arcsin)^{-1}$.
- Similarly there is $\arctan \in HA$ with $\partial(\arctan) = (1 + x^2)^{-1}$.

Hyperbolic Functions

- In a similar way we can define the hyperbolic functions sinh and cosh by

$$\mathsf{sinh} = \mathrm{int}(0, e)$$

and

$$\mathsf{cosh} = \mathrm{int}(e, 0)$$

- The usual properties follow, including $W(\cosh, \sinh) = \cosh^2 - \sinh^2 = 1$.

First Order Linear Equations

- For any $h \in HA$, a solution of $Y' - hY = 0$ is $y = \exp(\int h)$.
- This can be shown by direct calculation, using the chain rule.
- For example, a solution of $Y' - \langle 2 \rangle Y = 0$ is $y = \exp(\langle 3 \rangle)$, which is an interlacing of three Hurwitz series, two of which are zero.
- More precisely, $\exp(\langle 3 \rangle) = \mathrm{int}(f, 0, 0)$, where $f(n) = \langle n, 3 \rangle$.

Airy's Equation $y'' - xy = 0$

Assume that $h \in HA$ is a solution. Then using $\partial^2(h)(n) = h(n+2)$ and $(x \cdot h)(n) = nh(n-1)$, we get

$$h(n+2) = nh(n-1)$$

for every $n \in \mathbf{N}_+$. Solving this system of equations recursively, we see that for all $n \in \mathbf{N}$,

$$h(3n) = h(0) \prod_{k=1}^{n} (3k-2),$$

$$h(3n+1) = h(1) \prod_{k=1}^{n} (3k-1),$$

and

$$h(3n+2) = 0$$

.

Airy's Equation Continued

- Define $h_0(n) = \prod_{k=1}^{n}(3k-2)$.
- Define $h_1(n) = \prod_{k=1}^{n}(3k-1)$.
- Then any solution of Airy's equation is given by $h = \mathrm{int}(ah_0, bh_1, 0)$, where $a, b \in A$ are arbitrary constants.
- Note that h is the interlacing of 3 elements in HA.
- $(\partial(h_0))(n) = (3n+1)h_0(n)$ and $(\partial(h_1))(n) = (3n+2)h_1(n)$.
- By computing the Wronskian $W(h_0, h_1)$, it follows that h_0 and h_1 are linearly independent over A.

Third Order Analog of sin and cos

- Let $z = e^{-1}$ and define $u = \text{int}(z, 0, 0)$, $v = \text{int}(0, z, 0)$ and $w = \text{int}(0, 0, z)$.
- Then $\partial(u) = -w$, $\partial(v) = u$ and $\partial(w) = v$.
- Also, $\{u, v, w\}$ is a complete set of linearly independent solutions of $\partial^3 Y + Y = 0$.
- The Wronskian $W(u, v, w) = u^3 - v^3 + w^3 + 3uvw$.
- Also $\partial(u^3 - v^3 + w^3 + 3uvw) = 0$, and $\varepsilon(u^3 - v^3 + w^3 + 3uvw) = 1$, so that

$$u^3 - v^3 + w^3 + 3uvw = 1.$$

Third Order Analog of sin and cos

- Hence we have an algebraic relation among a fundamental set of solutions of $\partial^3 Y + Y = 0$.
- Are these functions u, v, w well-known functions, e.g., are they elementary functions?
- Note that this example can be extended to any order n in the obvious way.

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