On Partitioned Differential Quasifields

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We begin by recalling that a differential ring R is called a **(differential) quasi-field** if every nonunit in R is nilpotent and every nonzero element has some derivative (perhaps of order zero) that is not nilpotent. Every differential field is a differential quasifield, and in characteristic zero, every differential quasi-field is a differential field. Also, the subring of constants of a differential quasifield is a field.

We also recall that there are equivalent ways of describing a differential quasi-field, which illustrate the parallel between fields and differential quasifields.

Recall that a ring R is **reduced** if R has no nonzero nilpotents, and R is **pointed** if every nonunit in R is nilpotent. Observe that a ring R is a field if and only if R is reduced and pointed. An element x in a differential ring R is called **differentially nilpotent** if all order derivatives of x (including order zero) are nilpotent in R. Note that any differentially nilpotent element is nilpotent. A differential ring R is called **quasireduced** if R has no nonzero differentially nilpotent elements. Hence a differential ring R is a differential quasi-field if and only if R is quasireduced and pointed.

Example 1 *Consider the differential ring* $A = \mathbf{Z}\{y\}$ *of differential polynomials with integer coefficients in the differential indeterminate* y. *In* A, *consider the differential ideal* Q *generated by* $\{y' - 1, y^2, 2\}$, *and let* $R = A/Q$. *If we let* x *denote the element* $y + Q \in R$, *then we see that* $R = \{0, 1, x, 1 + x\}$ *has characteristic 2, that* $x^2 = 0$ *and* $x' = 1$. *Hence* R *is quasireduced and* *pointed, i.e., R is a differential quasi-field.*

Proposition 2 *Suppose that E and F are differential quasifields, and that $f : E \longrightarrow F$ is a differential ring homomorphism. Then if C_E denotes the subfield of constants in E, we have:*

1. *f is injective.*

2. *$x \in E$ is nilpotent if and only if $f(x) \in F$ is nilpotent.*

3. $x \in E$ *is constant if and only if* $f(x) \in F$ *is constant.*

4. $x \in E$ *is invertible if and only if* $f(x) \in F$ *is invertible.*

5. $\{x_1, \ldots, x_n\} \subseteq E$ *is linearly independent over* C_E *if and only if* $\{f(x_1), \ldots, f(x_n)\} \subseteq F$ *is linearly independent over* $f(C_E)$.

Theorem 3 *Let* E *be a differential quasi-field of characteristic* $p > 0$ *with derivation* δ *and field of constants* C_E. *If*

$\phi : E \longrightarrow E$ *is any differential automorphism that leaves* C_E *fixed, then* $\phi = \mathrm{id}_E$*, the identity on E.*

Proof: Let $x \in E$, then $x^p \in C_E$ because $\delta(x^p) = px^{p-1}\delta(x) = 0$. If $\overline{x} = \phi(x) - x$, then $\overline{x}$ is nilpotent, since

$$\begin{aligned} \overline{x}^p &= (\phi(x) - x)^p \\ &= \phi(x)^p - x^p \\ &= \phi(x^p) - x^p \\ &= x^p - x^p \\ &= 0. \end{aligned}$$

Now for any $m \in \mathbf{N}$, we have

$$\begin{aligned}
(\delta^m(\overline{x}))^p &= (\delta^m(\phi(x) - x))^p \\
&= (\delta^m(\phi(x)) - \delta^m(x))^p \\
&= (\phi(\delta^m(x)) - \delta^m(x))^p \\
&= (\phi(\delta^m(x)))^p - (\delta^m(x))^p \\
&= \phi((\delta^m(x))^p) - (\delta^m(x))^p \\
&= (\delta^m(x))^p - (\delta^m(x))^p \\
&= 0.
\end{aligned}$$

Thus for all $m \in \mathbf{N}$, $\delta^m(\overline{x})$ is nilpotent. But E is a quasifield, so it must be that $\overline{x} = 0$, that is, $\phi(x) = x$. $\square$

We recall that for any commutative ring R with identity, the **ring of Hurwitz series** over R, denoted by HR, is defined as follows. The elements of HR are sequences $(a_n) = (a_0, a_1, a_2, \ldots)$, where $a_n \in R$ for each $n \in \mathbf{N}$. Let $(a_n), (b_n) \in HR$. Addition in HR is defined termwise, i.e.,

$$(a_n)+(b_n) = (c_n), \quad \text{where} \quad c_n = a_n+b_n$$

for all $n \in \mathbf{N}$. The (Hurwitz) product of (a_n) and (b_n) is given by $(a_n) \cdot (b_n) = (c_n)$, where

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

Moreover, HR is a differential ring with derivation $\partial_R : HR \longrightarrow HR$ given by

$$\partial_R((a_0, a_1, a_2, \ldots)) = (a_1, a_2, a_3, \ldots).$$

We will often write ∂ in place of ∂_R. We will denote, for any $j \in \mathbf{N}$, the additive mapping $\pi_j : HR \longrightarrow R$ defined by $\pi_j((a_n)) = a_j$.

Recall also that for any ring R of positive characteristic, R is a field if and only if the differential ring HR of Hurwitz series over R is a differential quasi-field.

Proposition 4 *Let E be a differential quasifield of positive characteristic, N_E the nilradical of E and C_E the subfield of constants in E. Then there is a natural injective differential ring homomorphism $\eta_E : E \longrightarrow Hk$ of E into the quasifield of Hurwitz series Hk, where $k = E/N_E$. Moreover, we have:*

1. *$x \in E$ is invertible in E if and only if $\eta_E(x) \in Hk$ is invertible in Hk.*

2. *$x \in E$ is nilpotent in E if and only if $\eta_E(x) \in Hk$ is nilpotent in Hk.*

3. *$x \in E$ is constant in E if and only if $\eta_E(x) \in Hk$ is constant in Hk.*

4. *Let $X = \{x_1, x_2, \ldots, x_n\} \subseteq E$. Then X is linearly independent over C_E if and only if $\eta_E(X) \subseteq Hk$ is linearly independent over $\eta_E(C_E)$.*

Here η_E is defined by

$$\begin{aligned}\eta_E(x) \;=\; (x + N_E, \delta_E(x) + N_E,\\ \delta_E^2(x) + N_E, \ldots).\end{aligned}$$

Let E be a differential quasifield, C_E the subfield of constants of E, and N_E the nilradical of E. We say that E is **partitioned** if, as an additive group, $E = C_E \oplus N_E$, i.e., if for any $x \in E$, there exist unique $c_x \in C_E$ and $n_x \in N_E$ such that $x = c_x + n_x$.

Proposition 5 *Let E be a differential quasifield with positive characteristic p and field of constants C_E. If C_E is perfect, then E is partitioned.*

Proof: First recall that if $x \in E$, then $x^p \in C_E$. Now let N_E be the nilradical of E and let

$$k = E/N_E.$$

Then k is an extension of C_E, in the obvious way, that is,

$$0 \longrightarrow C_E \longrightarrow E \longrightarrow k.$$

If

$$x \in k \quad \text{then} \quad x^p \in C_E,$$

and thus, since C_E is perfect,

$$k = C_E.$$

Thus N_E has codimension one, in E, as a C_E-vector space and so

$$E = C_E \oplus N_E,$$

that is, E is partitioned. $\square$

Proposition 6 *Let* (E, δ_E) *be a differential quasifield of positive characteristic,* C_E *the subfield of constants of* E*,* N_E *the nilradical of* E*,*

$$k = E/N_E$$

the reduced field of E *and*

$$\eta : E \longrightarrow Hk$$

the canonical embedding . Then E *is partitioned if and only if*

$$\eta(C_E) = C_{Hk} \cong k.$$

Proof: Suppose first that E is partitioned, and let

$$c = (c_0, 0, \ldots, 0, \ldots) \in C_{Hk},$$

so that $c_0 \in k = E/N_E$. Hence there exists $x \in E$ such that

$$\tau(x) = c_0,$$

where $\tau : E \longrightarrow k$ is the canonical surjection. Since E is partitioned, there exists $c_x \in C_E$ such that $x - c_x \in N_E$. Then

$$\begin{aligned} \eta(c_x) &= (\tau(c_x), \tau(\delta_E(c_x)), \tau(\delta_E^2(c_x)), \ldots) \\ &= (c_0, 0, \ldots, 0, \ldots), \end{aligned}$$

showing that $\eta(C_E) = C_{Hk} \cong k$.

Conversely, suppose that $\eta(C_E) = C_{Hk}$, so that for any $x \in E$, there exists $c_x \in C_E$ such that

$$\eta(c_x) = (x + N_E, 0 + N_E, 0 + N_E, \ldots) \in C_{Hk}.$$

It is clear that $c_x \in C_E$ satisfying $\eta(c_x) = (x + N_E, 0 + N_E, 0 + N_E, \ldots)$ is unique, since η is injective. Also, $x - c_x$ is nilpotent in E, since

$$
\begin{aligned}
\eta(x - c_x) &= (x + N_E, \delta_E(x) + N_E, \\
&\qquad \delta_E^2(x) + N_E, \ldots) \\
&- (x + N_E, 0 + N_E, \\
&\qquad 0 + N_E, \ldots) \\
&= (0 + N_E, \delta_E(x) + N_E, \\
&\qquad \delta_E^2(x) + N_E, \ldots)
\end{aligned}
$$

is nilpotent in Hk. Hence E is partitioned. □

A basic result in differential algebra (of characteristic zero) is that if (F, δ_F) is a differential field of characteristic zero with field of constants C, then $y_1, \ldots, y_n \in F$ are linearly dependent over C if and only if the Wronskian $w(y_1, \ldots, y_n) = 0$. The **Wronskian** $w(y_1, \ldots, y_n)$ of $y_1, \ldots, y_n \in F$ is defined by

$$w(y_1, \ldots, y_n) = \det(\delta_F^{i-1}(y_j)).$$

This result does not carry over directly to differential quasifields of positive characteristic, as the following example shows.

Example 7 *Let k be a field of characteristic 2, and consider the differential quasifield Hk. The elements $x^{[2]}$*

and $x^{[3]}$ *in* Hk *are certainly linearly independent over* $C_{Hk} \cong k$*, but a quick calculation using*

$$x^{[m]}x^{[n]} = \binom{m+n}{n} x^{[m+n]}$$

shows that

$$w(x^{[2]}, x^{[3]}) = 0.$$

In order to generalize this result about linear dependence over constants to the case of differential quasifields, we need to introduce the following.

Let (R, δ_R) be any differential ring, let $y = (y_1, \ldots, y_n) \in R^n$, and let $s = (s_1, \ldots, s_n) \in \mathbf{N}^n$.

The $s-$ **quasiwronskian** of y, denoted by $w_s(y)$, is defined by

$$w_s(y) = \det(\delta_R^{s_i}(y_j)).$$

So the (usual) Wronskian of y is the $(0, 1, \ldots, n-1)$-quasiwronskian of y.

Theorem 8 *Let E be a differential quasi-field of positive characteristic with field of constants C_E, suppose that E is partitioned, and let $y = (y_1, \ldots, y_n) \in E^n$.*

Then $\{y_1, \ldots, y_n\}$ *is linearly independent over* C_E *if and only if there exists* $s = (s_1, \ldots, s_n) \in \mathbf{N}^n$ *such that* $w_s(y)$ *is invertible in* E*.*

Proof: Assume first that $\{y_1, \ldots, y_n\}$ is linearly dependent over C_E, so that there exist $c_1, \ldots, c_n \in C_E$, not all zero, such that $\sum_{j=1}^n c_j y_j = 0$. Hence for any $s = (s_1, \ldots, s_n) \in \mathbf{N}^n$, $(c_1, \ldots, c_n)$ is a non-trivial solution in C_E^n to the system of linear equations

$$\sum_{j=1}^n \partial^{s_i}(y_j) x_j = 0, \quad i = 1, \ldots, n,$$

with coefficients in E in the unknowns $x_1, \ldots, x_n$. The determinant of the matrix of coefficients of the above system is the s-quasiwronskian $w_s(y)$ of $y = (y_1, \ldots, y_n)$, and since this system has a non-trivial solution, this determinant is not invertible in E, and hence is nilpotent in E.

Now assume that $\{y_1, \ldots, y_n\}$ is linearly independent over C_E. We proceed with a special case, namely when $E = Hk$ for a field k of positive characteristic.

Lemma 9 *Let k be a field of positive characteristic, $E = Hk$ the differential quasifield of Hurwitz series over k, and $(y_1, \ldots, y_n) \in E^n$. If $\{y_1, \ldots, y_n\} \subseteq E$ is linearly independent over k, then there exists some $(s_1, \ldots, s_n) \in \mathbf{N}^n$ such that* $\det(\partial^{s_i}(y_j))$ *is invertible in E.*

Proof: We proceed using induction on n. If $n = 1$, then y_1 is linearly independent over k if and only if $y_1 \neq 0$, so take $s_1 = \text{ord}(y_1)$. Then $\partial^{s_1}(y_1) = \det(\partial^{s_1}(y_1))$ is invertible in E, since

$$\pi_{s_1}(y_1) = \pi_0(\partial^{s_1}(y_1)) \neq 0.$$

Now suppose that $(y_1, \ldots, y_n) \subseteq E$ is linearly independent over k. We may also assume that $s_1 = \text{ord}(y_1) \leq \text{ord}(y_j)$ for $j = 2, \ldots, n$. Define

$$c_j = \pi_{s_1}(y_j)\pi_{s_1}(y_1)^{-1} \in k$$

for j = 2,. . . ,n, and define

$$z_1 = y_1 \quad \text{and} \quad z_j = y_j - c_j y_1$$

for j = 2,. . . ,n. A routine calculation shows that $\{z_1, \ldots, z_n\}$ is linearly independent over k. Furthermore, we see that $\text{ord}(z_j) > s_1$ for $j = 2, \ldots, n$, since

$$
\begin{aligned}
\pi_{s_1}(z_j) &= \pi_{s_1}(y_j - c_j y_1) \\
&= \pi_{s_1}(y_j) \\
&- \pi_{s_1}(y_j)\pi_{s_1}(y_1)^{-1}\pi_{s_1}(y_1) \\
&= 0
\end{aligned}
$$

Since $\{z_2, \ldots, z_n\}$ is linearly independent over k, by induction there exists

$$
(s_2, \ldots, s_n) \in \mathbf{N}^{n-1}
$$

such that

$$
\det((\partial^{s_i}(y_j))_{2 \le i,j \le n})
$$

is invertible in Hk. Then since

$$\det(\partial^{s_i}(y_j)) = \det(\partial^{s_i}(z_j)),$$

and since

$$\partial^{s_1}(z_1) = \partial^{s_1}(y_1)$$

is invertible in Hk, we see by expanding $\det(\partial^{s_i}(z_j))$ along the first row that

$$\begin{aligned}\det(\partial^{s_i}(z_j)) &= \partial^{s_1}(z_1)\det((\partial^{s_i}(z_j))_{2\leq i,j\leq n}) \\ &+ \sum_{j=2}^{n}(-1)^{j+1}\partial^{s_1}(z_j)M_{(1,j)},\end{aligned}$$

where $M_{(1,j)}$ is the $(1,j)$-minor of $(\partial^{s_i}(z_j))$. Now since each $\partial^{s_1}(z_j)$ is nilpotent in

Hk for $j = 2, \ldots, n$, we see that

$$\sum_{j=2}^{n} (-1)^{j+1} \partial^{s_1}(z_j) M_{(1,j)}$$

is nilpotent in Hk, so that

$$\det(\partial^{s_i}(z_j)) = \det(\partial^{s_i}(y_j))$$

is invertible in Hk. $\square$

Continuing with the proof of Theorem 8, let $k = E/N_E$, where N_E is the nilradical of E, and consider the embedding $\eta_E : E \longrightarrow Hk$. By Proposition 4,

$\{y_1, \ldots, y_n\} \subseteq E$ is linearly independent over C_E if and only if

$$\{\eta(y_1), \ldots, \eta(y_n)\} \subseteq Hk$$

is linearly independent over $\eta(C_E)$. Since E is partitioned

$$\eta(C_E) \cong k$$

by Proposition 6, so Lemma 9 applies to show that there exists some

$$(s_1, \ldots, s_n) \in \mathbf{N}^n$$

such that $\det(\partial^{s_i}(\eta(y_j)))$ is invertible in Hk. But since

$$\det(\partial^{s_i}(\eta(y_j))) = \eta(\det(\partial^{s_i}(y_j))),$$

Proposition 2 tells us that $\det(\partial^{s_i}(y_j))$ is invertible in E, as desired. $\square$

Corollary 10 *Let k be a field of positive characteristic, and let K be any field extension of k. The finite set*

$$\{h_1, \ldots, h_n\} \subseteq Hk$$

is linearly independent over k if and only if

$$\{h_1, \ldots, h_n\} \subseteq HK$$

is linearly independent over K.

Proof: Clearly if $\{h_1, \ldots, h_n\}$ is linearly dependent over k, then $\{h_1, \ldots, h_n\}$ is linearly dependent over K. Now assume that $\{h_1, \ldots, h_n\}$ is linearly independent over k. By Theorem 8, there is some $s \in \mathbf{N}^n$ such that $w_s(h_1, \ldots, h_n)$ is invertible in k, and hence is invertible in K. It follows from Theorem 8 again that $\{h_1, \ldots, h_n\}$ is linearly independent over K. $\square$

Recall that if A, B, and C are rings and if $f : A \longrightarrow C$ and $g : B \longrightarrow C$ are ring morphisms, then the bilinear mapping

$A \times B \longrightarrow C$ given by $(a, b) \mapsto f(a)g(b)$ induces a ring morphism

$$\Phi : A \otimes B \longrightarrow C.$$

In addition if A, B and C are differential rings with derivations δ_A, δ_B and δ_C respectively and $f : A \longrightarrow C$ and $g : B \longrightarrow C$ are differential ring morphisms then $A \otimes B$ is a differential ring with derivation

$$\delta_{A \otimes B} = \delta_A \otimes \mathrm{id}_B + \mathrm{id}_A \otimes \delta_B$$

and $\Phi : A \otimes B \longrightarrow C$ is a differential ring morphism.

Proposition 11 *Let K be any field extension of k. Then the differential k-algebra homomorphism*

$$\Phi : K \otimes_k Hk \longrightarrow HK,$$

defined by

$$\Phi(a \otimes (b_n)) = (ab_n)$$

for any $a \in K$ and $(b_n) \in Hk$, is injective.

Proof: Here we must show that if

$$\sum_{i=1}^{n} a_i \otimes h_i \neq 0$$

with $a_i \in K$ and $h_i \in Hk$ then

$$\sum_{i=1}^{n} a_i h_i \neq 0.$$

Reduce to the case where $\{h_1, \ldots, h_n\}$ is linearly independent over k. Now, if we were to have

$$\sum_{i=1}^{n} a_i h_i = 0,$$

then since $\{h_1, \ldots, h_n\}$ is linearly independent over k, by Corollary 10 $\{h_1, \ldots, h_n\}$ is also linearly independent over K. Thus we must have

$$a_i = 0$$

for each i, but this implies that

$$\sum_{i=1}^{n} a_i \otimes h_i = 0.$$

□

Theorem 12 *Let E be a partitioned differential quasifield and let A be a differential ring that is obtained from E by extension of scalars. Then A is a partitioned differential quasifield.*

Proof: Let E have field of constants C_E, nilradical N_E and let $C_E \subset K$ be a field extension. We have

$$A = K \otimes_{C_E} E$$

and since $E = C_E \oplus N_E$, it follows that

$$A = (K \otimes_{C_E} C_E) \oplus (K \otimes_{C_E} N_E)$$

as an abelian group. Now

$$K \cong K \otimes_{C_E} C_E,$$

so identify $K \otimes_{C_E} C_E$ with K. Set

$$N = K \otimes_{C_E} N_E.$$

Clearly N is the nilradical of A and

$$A = K \oplus N.$$

If $x \in A$ and $x \notin N$ then $x = r + n$ for $0 \neq r \in K$ and n a nilpotent. If $n^m = 0$, then letting

$$y = \frac{1}{r}(1 + \sum_{i=1}^{m-1} \left(\frac{-n}{r}\right)^i)$$

it is easy to verify that $xy = 1$. Thus A is a differential ring such that each element is nilpotent or is invertible. We must show that for every nilpotent $n \in A$, there is some $l \in \mathbf{N}$ such that $\delta_A^l(n)$ is invertible in A, where δ_A is the derivation on A. Since E is partitioned,

$$E/N_E \cong C_E,$$

and so we have an embedding

$$\eta : E \longrightarrow HC_E.$$

Since K is a C_E-vector space, K is flat as a C_E-module, and so

$$id_K \otimes \eta : A \longrightarrow K \otimes_{C_E} HC_E$$

is an embedding. Also

$$\Phi : K \otimes_{C_E} HC_E \longrightarrow HK$$

is an embedding by Proposition 11, so the composition

$$\vartheta = \Phi \circ id_K \otimes \eta, \quad \vartheta : A \longrightarrow HK$$

is an embedding. If $n \in A$ is a nonzero nilpotent, then $\vartheta(n)$ is a nonzero nilpotent in HK. Thus there is some $l \in \mathbf{N}$

such that $\partial_K^l \vartheta(n)$ is invertible, so from what we have already shown $\delta_A^l(n)$ is invertible. We conclude that A is a quasifield. Since $A = K \oplus N$, A is partitioned. $\square$