Dimension Quasi-polynomials of Inversive Difference Field Extensions with Weighted Translations

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A function $f : \mathbb{Z} \to \mathbb{Q}$ is called a (univariate) **quasi-polynomial** of period q if there exist q polynomials $g_i(x) \in \mathbb{Q}[x]$ $(0 \leq i \leq q-1)$ such that

$$f(n) = g_i(n) \ \text{whenever} \ n \in \mathbb{Z}, \ \text{and} \ n \equiv i \, (mod \ q).$$

An equivalent way of introducing quasi-polynomials is as follows.

A **rational periodic number** $U(n)$ is a function $U : \mathbb{Z} \to \mathbb{Q}$ with the property that there exists (a period) $q \in \mathbb{N}$ such that

$$U(n) = U(n') \ \text{whenever} \ n \equiv n' \, (mod \ q).$$

A rational periodic number can be represented by a list of q its possible values enclosed in square brackets:

$$U(n) = [a_0, \ldots, a_{q-1}]_n.$$

Example 1. $U(n) = \left[\frac{1}{2}, \frac{3}{4}, 1\right]_n$ is a periodic number with period 3 such that $U(n) = \frac{1}{2}$ if $n \equiv 0\,(mod\ 3)$,

$U(n) = \frac{3}{4}$ if $n \equiv 1\,(mod\ 3)$, and $U(n) = 1$ if $n \equiv 2\,(mod\ 3)$.

A (univariate) **quasi-polynomial** of degree d is a function $f : \mathbb{Z} \to \mathbb{Q}$ such that

$$f(n) = c_d(n)n^d + \cdots + c_1(n)n + c_0(n) \quad (n \in \mathbb{Z})$$

where $c_i(n)$'s are rational periodic numbers.

One of the main applications of the theory of quasi-polynomials is its application to the problem of counting integer points in polytopes.

Recall that a **rational polytope** in $\mathbb{R}^d$ is the convex hall of finitely many points (vertices) in $\mathbb{Q}^d$.
Equivalently, a rational polytope $P \subseteq \mathbb{R}^d$ is the set of solutions of a finite system of linear inequalities

$$A\mathbf{x} \leq \mathbf{b},$$

where A is an $m \times d$-matrix with integer entries (m is a positive integer) and $\mathbf{b} \in \mathbb{Z}^m$, provided that the solution set is bounded.

A **lattice polytope** is a polytope whose vertices have integer coordinates.

Let $P \subseteq \mathbb{R}^d$ be a rational polytope. (We assume that P has dimension d, that is, P is not contained in a proper affine subspace of $\mathbb{R}^d$.) Then a polytope

$$nP = \{n\mathbf{x} \mid \mathbf{x} \in P\}$$

($n \in \mathbb{N}$, $n \geq 1$) is called the nth dilate of P.

Clearly, if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are all vertices of P, then nP is the convex hall of $n\mathbf{v}_1, \ldots, n\mathbf{v}_k$.

In what follows, $L(P, n)$ denotes the number of integer points (that is, points with integer coordinates) in nP. In other words,

$$L(P, n) = \text{Card}(nP \cap \mathbb{Z}^d).$$

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Theorem 1 (Ehrhart, 1962)

Let $P \subseteq \mathbb{R}^d$ be a rational polytope. Then

(i) $L(P, n)$ is a degree d quasi-polynomial.

(ii) The coefficient of the leading term of this quasi-polynomial is independent of n and is equal to the Euclidean volume of P.

(iii) The period of $L(P, n)$ is a divisor of the least common multiple of the denominators of the vertices of nP.

(iv) If P is a lattice polynomial, then $L(P, n)$ is a polynomial of n with rational coefficients.

The main tools for computing Ehrhart quasi-polynomials are Alexander Barvinok's polynomial time algorithm and its modifications, see

A. I. Barvinok. Computing the Ehrhart polynomial of a convex lattice polytope, Discrete Comput. Geom. 12 (1994), 35–48.

A. I. Barvinok and J. E. Pommersheim, An algorithmic theory of lattice points in polyhedra, New Perspectives in Algebraic Combinatorics. Math. Sci. Res. Inst. Publ., vol. 38, Cambridge Univ. Press, 1999, 91–147.

A. I. Barvinok. Computing the Ehrhart quasi-polynomial of a rational simplex, Math. Comp. 75 (2006), no. 255, 1449–1466.

In some cases, Ehrhart quasi-polynomial can be found directly from the Ehrhart's theorem by evaluating the periodic numbers, which are coefficients of the quasi-polynomial.

Example 2. Consider a polytope

$$P = \{(x_1, x_2 \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 3,\ 2x_1 \leq 5\}.$$

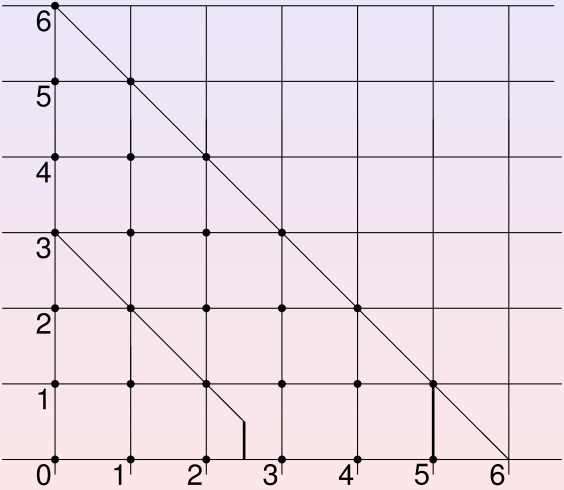
Then

$$nP = \{(x_1, x_2 \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 3n,\ 2x_1 \leq 5n\}$$

is a polytope with vertices $(0,0)$, $\left(\frac{5n}{2}, 0\right)$, $(0, 3n)$, and $\left(\frac{5n}{2}, \frac{n}{2}\right)$. By the Ehrhart Theorem,

$$L(P, n) = \alpha n^2 + [\beta_1, \beta_2]_n n + [\gamma_1, \gamma_2]_n.$$

The direct counting gives $L(P, 0) = 1$, $L(P, 1) = 9$, $L(P, 2) = 27$, $L(P, 3) = 52$, and $L(P, 4) = 88$. The following figure shows integer points in nP for $n = 0, 1, 2$.



Substituting $n = 0, 1, 2, 3, 4$ into the expression

$$L(P, n) = \alpha n^2 + [\beta_1, \beta_2]_n n + [\gamma_1, \gamma_2]_n$$

one obtains a system of linear equations

$$\begin{cases} \gamma_1 = 1, \\ \alpha + \beta_2 + \gamma_2 = 9, \\ 4\alpha + 2\beta_1 + \gamma_1 = 27, \\ 9\alpha + 3\beta_2 + \gamma_2 = 52, \\ 16\alpha + 4\beta_1 + \gamma_1 = 88. \end{cases}$$

That gives $\alpha = \frac{35}{8}$, $\beta_1 = \frac{17}{4}$, $\beta_2 = 4$, $\gamma_1 = 1$, and $\gamma_2 = \frac{5}{8}$. Thus,

$$L(P, n) = \frac{35}{8} n^2 + \left[\frac{17}{4}, 4\right]_n n + \left[1, \frac{5}{8}\right]_n .$$

In what follows, if $w = (w_1, \ldots, w_m)$ is am m-tuple of positive integers, then $\lambda_w^{(m)}(t)$ denotes the Ehrhart quasi-polynomial that describes the number of integer points in the conic polytope

$$\{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid \sum_{i=1}^{m} w_i x_i \leq t,\ x_j \geq 0\,(1 \leq j \leq m)\}.$$

It follows from the Ehrhart's Theorem that $\lambda_w^{(m)}(t)$ is a quasi-polynomial of degree m whose leading coefficient is

$$\frac{1}{m!\,w_1 \ldots w_m}.$$

A polynomial time algorithm for computing $\lambda_w^{(m)}(t)$ was obtained in the works of A. Barvinok.

Let K be an inversive difference field of zero characteristic with basic set of translations (automorphisms) $\sigma = \{\alpha_1, \dots, \alpha_m\}$ that are assigned positive integer weights $w_1, \dots, w_m$, respectively.

In what follows we set $\sigma^* = \{\alpha_1, \dots, \alpha_m, \alpha_1^{-1}, \dots, \alpha_m^{-1}\}$ and use prefix σ^*- instead of "inversive difference".

Let Γ denote the free commutative group generated by σ and for any transform $\gamma = \alpha_1^{k_1} \dots \alpha_m^{k_m} \in \Gamma$ ($k_i \in \mathbb{Z}$), let

$$\operatorname{ord} \gamma = \sum_{i=1}^{m} w_i |k_i|.$$

Furthermore, for any $r \in \mathbb{N}$, let

$$\Gamma(r) = \{\gamma \in \Gamma \mid \operatorname{ord} \gamma \leq r\}.$$

Theorem 2

Let $L = K\langle\eta_1, \dots, \eta_n\rangle^*$ *be a* σ^**-field extension of* K *generated by a finite set* $\eta = \{\eta_1, \dots, \eta_n\}$*. (*$L = K(\{\gamma\eta_j | \gamma \in \Gamma, 1 \le j \le n\})$ *as a field.) Then there exists a quasi-polynomial* $\Phi_{\eta|K}(t)$ *such that*

(i) $\Phi_{\eta|K}(r) = \text{tr.}\deg_K K(\cup_{i=1}^{n}\Gamma(r)\eta_i)$ *for all sufficiently large* $r \in \mathbb{N}$*.*

(ii) $\deg \Phi_{\eta|K} \le m = \text{Card}\,\sigma$*.*

(iii) $\Phi_{\eta|K}$ *is a linear combination with integer coefficients of Ehrhart quasi-polynomials of the form* $\lambda_w^{(m)}(t - a)$*,* $a \in \mathbb{Z}$*.*

(iv) The leading coefficient of $\Phi_{\eta|K}$ *is a constant that does not depend on the set of generators* η *of* L/K*. Furthermore, the coefficient of* t^m *in* $\Phi_{\eta|K}$ *is equal to* $\dfrac{2^m a}{w_1 \dots w_m m!}$ *where* a *is the difference transcendence degree of* L/K*.*

If all weights of the basic translations are 1, this statement gives the theorem on a dimension polynomial of a finitely generated inversive difference field extension (Theorem 4.2.12 of [A. B. Levin. *Difference Algebra*. Springer, 2008]).

The quasi-polynomial $\Phi_{\eta|K}$ is called the **difference dimension quasi-polynomial** associated with the extension L/K and the system of difference generators η.

Note that the existence of Ehrhart-type dimension quasi-polynomials associated with weighted filtrations of differential modules was established by C. Dönch in his dissertation [C. Dönch, *Standard Bases in Finitely Generated Difference-Skew-Differential Modules and Their Application to Dimension Polynomials*, Ph. D. Thesis, Johannes Kepler University Linz, RISC, 2012.]

In order to prove Theorem 2, we apply the technique of characteristic sets (using a ranking that respects the weighted order of transforms) and the following result that generalizes the theorem on the dimension polynomial of a subset of $\mathbb{Z}^m$ proved in Chapter 2 of [M. Kondrateva, A. Levin, A. Mikhalev, and A. Pankratev. *Differential and Difference Dimension Polynomials*. Kluwer Acad. Publ., 1998].

Let $\mathbb{N}$ and $\mathbb{Z}_-$ denote the sets of nonnegative and nonpositive integers, respectively. Then $\mathbb{Z}^m = \bigcup\limits_{1 \le j \le 2^m} \mathbb{Z}_j^{(m)}$ where $\mathbb{Z}_j^{(m)}$ are all distinct Cartesian products of m factors each of which is either $\mathbb{N}$ or $\mathbb{Z}_-$. A set $\mathbb{Z}_j^{(m)}$ $(1 \le j \le 2^m)$ an **orthant** of $\mathbb{Z}^m$.

We consider $\mathbb{Z}^m$ as a partially ordered set with respect to the ordering $\trianglelefteq$ defined as follows:

$$(x_1, \dots, x_m) \trianglelefteq (y_1, \dots, y_m)$$

if and only if $(x_1, \dots, x_m)$ and $(y_1, \dots, y_m)$ belong to the same orthant and $|x_i| \leq |y_i|$ for $i = 1, \dots, m$.

We fix positive integers $w_1, \dots, w_m$ and define the *order* of $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$ as $\operatorname{ord} \mathbf{x} = \sum_{i=1}^{m} w_i |x_i|$.

Furthermore, for any $A \subseteq \mathbb{Z}^m$, we set

$$V_A = \{\mathbf{v} \in \mathbb{Z}^m \,|\, \mathbf{a} \not\trianglelefteq \mathbf{v} \text{ for any } \mathbf{a} \in A\}$$

and $A(r) = \{\mathbf{a} \in A \,|\, \operatorname{ord} \mathbf{a} \leq r\} \quad (r \in \mathbb{N})$.

Theorem 3

With the above conventions, for any set $A \subseteq \mathbb{Z}^m$, there exists a quasi-polynomial $\omega_A(t)$ such that

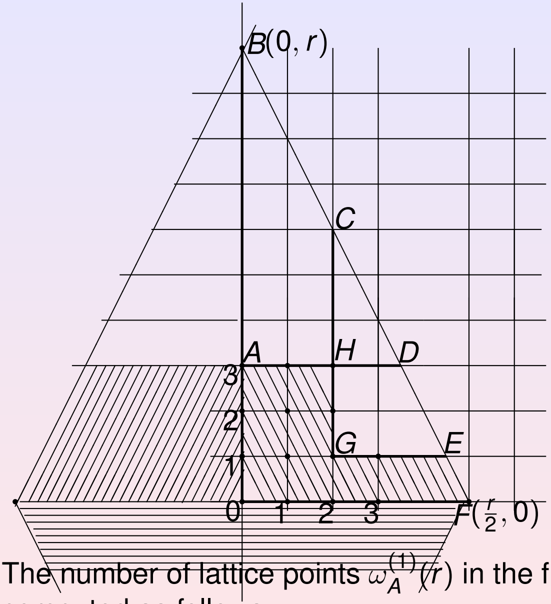
(i) $\omega_A(r) = \text{Card } V_A(r)$ *for all sufficiently large* $r \in \mathbb{N}$.

(ii) $\deg \omega_A \leq m$.

(iii) $\omega_A = 0$ *if and only if* $(0, \ldots, 0) \in A$.

(iv) $\omega_A(t)$ *is a linear combination with integer coefficients of Ehrhart quasi-polynomials of the form* $\lambda_w^{(m)}(t - a)$, $a \in \mathbb{Z}$, *associated with conic rational polytopes.*

Example 3. Let $A = \{((2, 1), (0, 3)\} \subset \mathbb{N}^2$ and let $w_1 = 2$, $w_2 = 1$. Then the set V_A consists of all lattice points (i. e., points with integer coordinates) that lie in the shadowed region in the following figure.



The number of lattice points $\omega_A^{(1)}(r)$ in the first quadrant can be computed as follows.

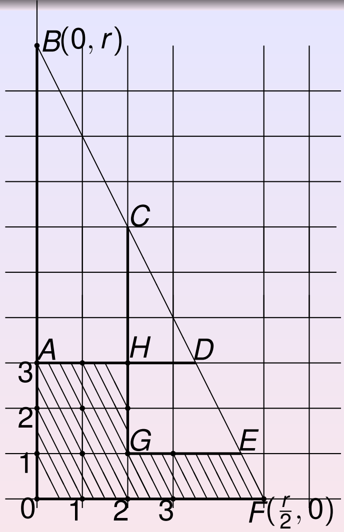
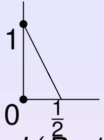


Fig. 1

$BF: \; 2x + y = r. \qquad C(2, r-4), \;\; D(\frac{r-3}{2}, 3), \;\; E(\frac{r-1}{2}, 1).$

$P_1 := \triangle BOF$; $P_2 := \triangle BAD$; $P_3 := \triangle CGE$; $P_4 := \triangle CHD$.

The conic polytope P_1 is the rth dilate of 

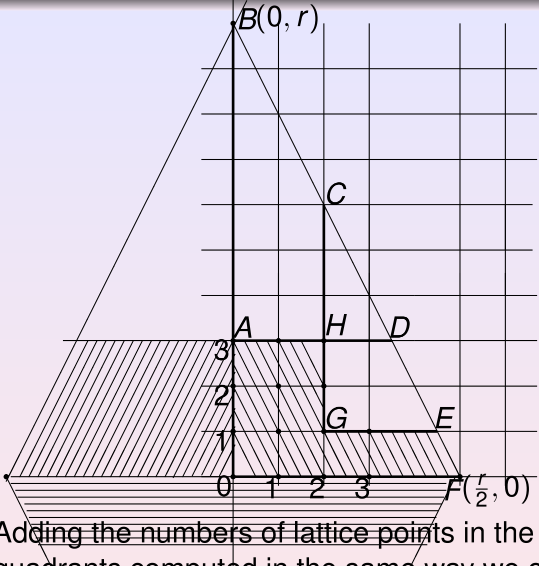
The direct computation shows that $L(P_1, 0) = 1$, $L(P_1, 1) = 2$, $L(P_1, 2) = 4$, and $L(P_1, 3) = 6$. As in Example 2 we obtain that

$$L(P_1, r) = \frac{1}{4}r^2 + r + \left[1, \frac{3}{4}\right]_r .$$

As it is seen from Fig. 1, the conic polytopes P_2, P_3 and P_4 are similar to P_1. If N_i denotes the number of lattice points in P_i $(i = 1, 2, 3, 4)$, then

$$\omega_A^{(1)}(r) = N_1 - N_2 - N_3 + N_4, \text{ that is,}$$

$$\omega_A^{(1)}(r) = L(P_1, r) - L(P_1, r-3) - L(P_1, r-5) + L(P_1, r-7) = \frac{1}{2}r + \left[5, \frac{9}{2}\right]_r$$



Adding the numbers of lattice points in the 2nd, 3rd, and 4th quadrants computed in the same way we obtain that

$$\omega_A(r) = \frac{1}{2}r^2 + 2r + \left[3, \frac{5}{2}\right]_r.$$

Sketch of the proof of Theorem 2

Let K be a σ^*-field, $\sigma = \{\alpha_1, \dots, \alpha_m\}$, Γ the free commutative group generated by σ, and $\mathbb{Z}^m = \bigcup_{1 \le j \le 2^m} \mathbb{Z}_j^{(m)}$ a representation of $\mathbb{Z}^m$ as the union of orthants.

As before, we assume that each α_i has a positive weight w_i and if $\gamma = \alpha_1^{k_1} \dots \alpha_m^{k_m} \in \Gamma$, then $\operatorname{ord} \gamma = \sum_{i=1}^{n} w_i |k_i|$.

Let $K\{y_1, \dots, y_n\}^*$ be an algebra of inversive difference (also called σ^*-) polynomials in σ^*-indetermiantes $y_1, \dots, y_n$ over K. (This is a polynomial ring in the set of indeterminates $\{\gamma y_i \,|\, \gamma \in \Gamma,\ 1 \le i \le n\}$ treated as an inversive difference K-algebra where $\alpha(\gamma y_i) = (\alpha\gamma) y_i$ for any $\alpha \in \sigma^*$.

let Y denote the set $\{\gamma y_i | \gamma \in \Gamma, 1 \leq i \leq s\}$ whose elements are called *terms* (here and below we often write γy_i for $\gamma(y_i)$). By the *order* of a term $u = \gamma y_j$ we mean the order of the element $\gamma \in \Gamma$.

Setting $Y_j = \{\gamma y_i | \gamma \in \Gamma_j, 1 \leq i \leq n\}$ $(j = 1, \ldots, 2^m)$ we obtain a representation of the set of terms as a union $Y = \bigcup_{j=1}^{2^m} Y_j$.

A term $v \in Y$ is called a **transform** of a term $u \in Y$ if and only if u and v belong to the same set Y_j $(1 \leq j \leq 2^m)$ and $v = \gamma u$ for some $\gamma \in \Gamma_j$. If $\gamma \neq 1$, v is said to be a proper transform of u.

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A well-ordering $\leq$ of the set of terms Y is called a **ranking** if
(i) If $u \in Y_j$ and $\gamma \in \Gamma_j$ ($1 \leq j \leq 2^n$), then $u \leq \gamma u$.
(ii) If $u, v \in Y_j$, $u \leq v$ and $\gamma \in \Gamma_j$, then $\gamma u \leq \gamma v$.

A ranking is called **orderly** if for any $u, v \in Y$, the inequality $\operatorname{ord} u < ordv$ implies that $u < v$. As an example of an orderly ranking one can consider the *standard ranking*:
$u = \alpha_1^{k_1} \dots \alpha_m^{k_m} y_i \leq v = \alpha_1^{l_1} \dots \alpha_m^{l_m} y_j$ if and only if the $(2m+2)$-tuple $(\sum_{\nu=1}^{m} |k_\nu|, |k_1|, \dots, |k_m|, k_1, \dots, k_m, i)$ is less than or equal to $(\sum_{\nu=1}^{m} |l_\nu|, |l_1|, \dots, |l_m|, l_1, \dots, l_m, j)$ with respect to the lexicographic order on $\mathbb{Z}^{2m+2}$.

In what follows, we fix an orderly ranking on Y. If $A \in K\{y_1, \dots, y_n\}^*$, then the greatest (with respect to the ranking $\leq$) term from Y that appears in A is called the **leader** of A; it is denoted by u_A.

If $d = \deg_u A$, then the σ^*-polynomial A can be written as $A = I_d u^d + I_{d-1} u^{d-1} + \cdots + I_0$ where $I_k (0 \leq k \leq d)$ do not contain u.

I_d is called the **initial** of A; it is denoted by I_A.

If A and B are two σ^*-polynomials, then A is said to have rank less than B (we write $A < B$) if either $A \in K, B \notin K$ or $A, B \in K\{y_1, \dots, y_s\}^* \setminus K$ and $u_A < u_B$, or $u_A = u_B = u$ and $deg_u A < deg_u B$. If $u_A = u_B = u$ and $\deg_u A = \deg_u B$, we say that A and B are of the same rank and write $rk\, A = rk\, B$.

If $A, B \in K\{y_1, \ldots, y_n\}^*$, then A is said to be **reduced** with respect to B if A does not contain any power of a transform γu_B whose exponent is greater than or equal to $\deg_{u_B} B$. If $S \subseteq K\{y_1, \ldots, y_n\}^* \setminus K$, then a σ^*-polynomial A, is said to be reduced with respect to S if it is reduced with respect to every element of S.

A set $\mathcal{A} \subseteq K\{y_1, \ldots, y_n\}^*$ is said to be **autoreduced** if either it is empty or $S \bigcap K = \emptyset$ and every element of S is reduced with respect to all other elements of $\mathcal{A}$.
It is easy to show that distinct elements of an autoreduced set have distinct leaders and every autoreduced set is finite.

Proposition 1. *Let $\mathcal{A} = \{A_1, \ldots, A_p\}$ be an autoreduced set in $K\{y_1, \ldots, y_n\}^*$. Let $I(\mathcal{A}) = \{B \in K\{y_1, \ldots, y_n\}^* \,|\,$ either $B = 1$ or B is a product of finitely many σ^*-polynomials of the form $\gamma(I_{A_i})$ $(\gamma \in \Gamma, i = 1, \ldots, p)\}$. Then for any $C \in K\{y_1, \ldots, y_n\}^*$, there exist $J \in I(\mathcal{A})$ and C_0 such that C_0 is reduced with respect to $\mathcal{A}$ and $JC \equiv C_0 (mod\, [\mathcal{A}]^*)$ (that is, $JC - C_0 \in [\mathcal{A}]^*$).*

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Let $\mathcal{A} = \{A_1, \dots, A_p\}$ and $\mathcal{B} = \{B_1, \dots, B_q\}$ be two autoreduced sets whose elements are written in the order of increasing rank. We say that $\mathcal{A}$ has lower rank than $\mathcal{B}$ if one of the following conditions holds:
(i) *there exists* $k \in \mathbb{N}$, $1 \le k \le \min\{p, q\}$, *such that* $rk\, A_i = rk\, B_i$ *for* $i = 1, \dots, k-1$ *and* $A_k < B_k$*;*
(ii) $p > q$ *and* $rk\, A_i = rk\, B_i$ *for* $i = 1, \dots, q$.

Proposition 2. (i) Every nonempty set of autoreduced sets contains an autoreduced set of lowest rank.
(ii) Let J be an ideal of $K\{y_1, \dots, y_n\}^*$ and $\mathcal{A}$ an autoreduced subset of J of lowest rank in J (such an autoreduced set is called a **characteristic set** of J). Then J does not contain nonzero σ^*-polynomials reduced with respect to $\mathcal{A}$. In particular, if $A \in \mathcal{A}$, then $I_A \notin J$.

Let P be the defining σ^*-ideal of the extension $L = K\langle\eta_1, \dots, \eta_n\rangle^*$, $P = \mathrm{Ker}(K\{y_1, \dots, y_n\}^* \to L)$, $\quad y_i \mapsto \eta_i$. Let $\mathcal{A} = \{A_1, \dots, A_d\}$ be a characteristic set of P and let u_i denote the leader of A_i ($1 \le i \le d$) and for every $j = 1, \dots, n$, let

$E_j = \{(k_1, \dots, k_m) \in \mathbb{N}^m | \alpha_1^{k_1} \dots \alpha_m^{k_m} y_j$ is a leader of a σ^*-polynomial in $\mathcal{A}\}$.

Let $V = \{u \in Y \mid u$ is not a transform of any u_i and for every $r \in \mathbb{N}$, let $V(r) = \{u \in V \mid \mathrm{ord}\, u \le r\}$. By Proposition 2, the ideal P does not contain non-zero σ^*-polynomials reduced with respect to $\mathcal{A}$. It follows that for every $r \in \mathbb{N}$, the set $V_\eta(r) = \{v(\eta) \mid v \in V(r)\}$ is algebraically independent over K.

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If $A_i \in \mathcal{A}$, then $A(\eta) = 0$, hence $u_i(\eta)$ is algebraic over the field $K(\{\gamma\eta_j \mid \gamma y_j < u_A\,(\gamma \in \Gamma,\, 1 \le j \le n)\})$. Therefore, for every $r \in \mathbb{N}$, the field $L_r = K(\{\gamma\eta_j \mid \gamma \in \Gamma(r),\, 1 \le j \le n\})$ is an algebraic extension of the field $K(\{v(\eta) \mid v \in V(r)\})$. It follows that $V_\eta(r)$ is a transcendence basis of L_r over K and $\operatorname{tr.deg}_K L_r = \operatorname{Card}\, V_\eta(r)$. For every $j = 1, \dots, n$.
The number of terms $\alpha_1^{k_1} \dots \alpha_m^{k_m} y_j$ in $V(r)$ is equal to the number of m-tuples $k = (k_1, \dots, k_m) \in \mathbb{Z}^m$ such that $\operatorname{ord}_w k \le r$ and k does not exceed any m-tuple in E_j with respect to the product order $\trianglelefteq$ on $\mathbb{Z}^m$. By Theorem 3, this number is expressed by a quasi-polynomial of degree at most m.

Therefore, for all sufficiently large $r \in \mathbb{N}$,

$$\operatorname{tr.deg}_K L_r = \text{Card } V_\eta(r) = \sum_{j=1}^{n} \omega_{E_j}(r)$$

where $\omega_{E_j}(t)$ is the dimension quasi-polynomial of the set $E_j \subseteq \mathbb{Z}^m$.

Thus, the quasi-polynomial

$$\Phi_{\eta|K}(t) = \sum_{i=1}^{n} \omega_{E_j}(t)$$

satisfies the conditions of Theorem 2.

Theorem 2 allows one to assign a quasi-polynomial to a system of algebraic difference equations with weighted basic translations

$$f_i(y_1, \dots, y_n) = 0 \qquad (i = 1, \dots, p) \tag{1}$$

($f_i, \in R = K\{y_1, \dots, y_n\}^*$ for $i = 1, \dots, p$) such that the reflexive difference ideal P of R generated by $f_1, \dots, f_p$ is prime (e. g. to a system of linear difference equations).

In this case, one can consider the quotient field $L = q.f.(R/P)$ as a finitely generated inversive (σ^*-) field extension of K: $L = K\langle \eta_1, \dots, \eta_n \rangle^*$ where η_i is the canonical image of y_i in R/P. The corresponding dimension quasi-polynomial $\Phi_{\eta|K}(t)$ is called the **difference dimension quasi-polynomial** of system (1).

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System of the form (1) arise, in particular, as finite difference approximations of systems of PDEs with weighted derivatives, see, for example,

[Shananin, N. A. Unique continuation of solutions of differential equations with weighted derivatives, *Mat. Sb.*, 191 (2000), no. 3, 113–142] and

[Shananin, N. A. Partial quasianalyticity of distribution solutions of weakly nonlinear differential equations with weights assigned to derivatives. *Math. Notes 68* (2000), no.3–4, pp. 519–527].

In this case the difference dimension quasi-polynomial can be viewed as the Einstein's strength of the system of partial difference equations with weighted translations.

Thanks!