

Embedding Differential Algebraic Groups in Algebraic Groups

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Pillay proved that every differential algebraic group can be differentially embedded into an algebraic group. In these lecture notes I will survey the proof given by Kowalski and Pillay.

1 Model Theoretic Preliminaries

Quantifier Elimination

We work in \mathbb{K} a large rich differentially closed field. All other differential fields are assumed to be small subfields of \mathbb{K} .

Let $\mathcal{L} = \{+, -, \cdot, \delta, 0, 1\}$ be the language of differential rings. We let $\mathcal{L}^- = \{+, -, \cdot, 0, 1\}$, the language of rings. If k is a differential field, we can view k either as an \mathcal{L} -structure or an \mathcal{L}^- -structure.

Theorem 1.1 (Quantifier Elimination) *For any \mathcal{L} -formula $\phi(x_1, \dots, x_n)$, there is an \mathcal{L} -formula $\psi(x_1, \dots, x_n)$ without quantifiers such that if K is a differentially closed field and $a \in K^n$, then*

$$K \models \phi(a) \Leftrightarrow K \models \psi(a)$$

i.e., ϕ is true of a in K if and only if ψ is true of $a \in K$.

Recall that $X \subseteq \mathbb{K}^n$ is *definable* if there is a formula $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$ and $b \in \mathbb{K}^m$ such

$$X = \{a \in \mathbb{K}^n : \phi(a, b)\}.$$

We say X is definable over $k \subseteq \mathbb{K}$ if we can choose $b \in k$. Quantifier free definable sets over K are just finite Boolean combinations equations $p(X_1, \dots, X_n) = 0$ where $p \in K\{X_1, \dots, X_n\}$. Thus they are exactly the Kolchin constructible sets. This gives us a reformulation of quantifier elimination.

Corollary 1.2 *$X \subseteq \mathbb{K}^n$ is definable if and only if it is Kolchin constructible.*

It is easy to see that if $k \subseteq \mathbb{K}$ is a differential field, $\phi(\bar{v})$ is a quantifier free formula and $\bar{a} \in k^n$ then $\mathbb{K} \models \phi(\bar{a})$ if and only if $k \models \phi(\bar{a})$. Quantifier elimination tells us that for differentially closed subfields the same is true for all formulas.

Corollary 1.3 (Model Completeness) *If $K \subseteq \mathbb{K}$ is a differentially closed field, $\phi(\bar{v})$ is an \mathcal{L} -formula and $\bar{a} \in K$, then $K \models \phi(\bar{a})$ if and only if $\mathbb{K} \models \phi(\bar{a})$.*

There are similar results for algebraically closed fields. For \mathcal{L}^- -formula there is a quantifier free \mathcal{L}^- formula that is equivalent in every algebraically closed field.

Types

Note: We use \bar{a} to denote a finite sequence of elements.

Let $A \subseteq \mathbb{K}$ and $\bar{b} \in \mathbb{K}^n$. The *type* of \bar{b} over A is

$$\text{tp}(\bar{b}/A) = \{\phi(\bar{v}, \bar{a}) : \phi(\bar{v}, \bar{w}) \text{ is an } \mathcal{L} \text{ - formula, } \bar{a} \in A \text{ and } \mathbb{K} \models \phi(\bar{b}, \bar{a})\}.$$

Let $k = \mathbb{Q}\langle A \rangle$ be the differential field generated by A . It is easy to see that $\text{tp}(\bar{b}/A)$ determines $\text{tp}(\bar{b}, k)$ and visa versa. So we often only consider types over differential subfields.

Let

$$I(\bar{b}, k) = \{f \in k\{X_1, \dots, X_n\} : f(\bar{b}) = 0\}.$$

Then $I(\bar{b}, k)$ is a prime differential ideal in $k\{X_1, \dots, X_n\}$. Moreover, if $k \subseteq \mathbb{K}$ and $I \subseteq k\{\bar{X}\}$ is a prime differential ideal, let $l \supseteq k$ be the quotient field of $k\{\bar{X}\}/I$. Since \mathbb{K} is rich, there is a differential embedding of l into \mathbb{K} fixing k . Let \bar{b} be the image of \bar{X}/I . Then $I(\bar{b}/k) = I$. Thus every prime differential ideal arises this way. By quantifier elimination $\text{tp}(\bar{b}/k)$ determines and is determined by, $I(\bar{b}, k)$.

We let $S_n(A)$ be the space of all types of elements $\bar{b} \in \mathbb{K}^n$ over A . Then there is a natural bijection between $S_n(A)$ and $\text{Spec}_\delta(k\{X_1, \dots, X_n\})$ the prime differential spectrum of $k\{X_1, \dots, X_n\}$. We can now use the Differential Basis Theorem to make a simple but powerful observation.

Proposition 1.4 $|S_n(A)| = \max(|A|, \aleph_0)$ for all $A \subseteq K$.

Theories with this property are called ω -stable and play an important role in model theory.

Similar results hold for algebraically closed fields. In algebraically closed fields type (in the language \mathcal{L}^-) are determined by prime ideals.

Morley Rank

For $X \subseteq \mathbb{K}^n$ and α an ordinal, we inductively define what it means for the *Morley rank* of X to be at least α :

$$\text{RM}(X) \geq 0 \text{ if } X \neq \emptyset;$$

If α is a limit ordinal, then $\text{RM}(X) \geq \alpha$ if and only if $\text{RM}(X) \geq \beta$ for all $\beta < \alpha$.

$\text{RM}(X) \geq \alpha + 1$ if there are definable disjoint $Y_1, Y_2, \dots, Y_n, \dots \subseteq X$ such that $\text{RM}(Y_i) \geq \alpha$ for all i .

We say $\text{RM}(X) = \alpha$ if $\text{RM}(X) \geq \alpha$ but $\text{RM}(X) \not\geq \alpha + 1$.

We say $\text{RM}(X) = \infty$ if $\text{RM}(X) \geq \alpha$ for all ordinals α .

For simplicity of statements we say $\text{RM}(\emptyset) = -1$.

If X is infinite and a_1, \dots, a_n, \dots are distinct elements of X , taking $Y_i = \{a_i\}$ shows that $\text{RM}(X) \geq 1$. Thus $\text{RM}(X) = 0$ if and only if X is finite.

Theorem 1.5 *A theory T is ω -stable if and only if $\text{RM}(X) < \infty$ for every definable X .*

In algebraically closed fields Morley rank has a very natural meaning. Let X be definable. By quantifier elimination X is constructible set. In this case it can be shown that $\text{RM}(X)$ is the dimension of the Zariski closure of X . In general, Morley rank gives a useful notion of dimension that makes sense in any ω -stable theory.

Here are some results for differentially closed fields.

Theorem 1.6 *i) $\text{RM}(K^n) = \omega n$ [Note: In ordinal arithmetic $\alpha\beta$ is β copies of α , so $n\omega = \omega < \omega n$.]*

ii) If $f(X) \in K\{X\}$ has order n , then and $Y = \{x : f(x) = 0\}$ then $\text{RM}(Y) \leq n$. Moreover if f is linear, then $\text{RM}(X) = n$. In particular, the field C of constants has rank exactly n .

One might hope that there for Kolchin closed subsets of \mathbb{K} of the form $f(X) = 0$, that Morley rank would be the order of f . This is not true. For example, Poizat showed that Morley of the set $X' = XX''$ is 1, as the only Kolchin closed subset is the field of constants. Painleve equations provide other examples of this phenomena.

There are relationships between the various notions of dimension. For example for Kolchin closed subsets of \mathbb{K} of the form $V(f) = \{x : f(x) = 0\}$,

$$\text{RM}(V(f)) \leq \text{Noetherian dim}(V(f)) \leq \text{ord}(f)$$

but the examples above can be used to show that either inequality can be strict. Pong has generalized these inequalities to definable sets in \mathbb{K}^n .

We have defined Morley rank for definable sets. We can easily extend the definition to formulas. The Morley rank of a formula $\phi(v_1, \dots, v_n, \bar{a})$ is equal to the Morley rank of the definable set $\{\bar{b} \in \mathbb{K}^n : \phi(\bar{b}, \bar{a})\}$.

If p is a type, then $\text{RM}(p)$ is defined to be

$$\inf\{\text{RM}(\phi) : \phi \in p\}.$$

An easy calculation shows that if $X \subseteq \mathbb{K}^n$ is a set defined over A , then

$$\text{RM}(X) = \sup\{\text{RM}(\bar{b}/A) : \bar{b} \in X\}.$$

If X is defined by ϕ we say that p is a *generic type* for X if the rank of the formula defining X is $\text{RM}(p)$, i.e., p has maximal rank among the type containing X , or strictly speaking a formula defining X .

Independence

Morley rank gives rise to a general notion of independence in any ω -stable theory.

We say that A and B are independent over C and write $A \downarrow_C B$, if for all $\bar{a} \in A$, $\text{RM}(\bar{a}/B \cup C) = \text{RM}(\bar{a}/C)$.

In algebraically closed fields this is just the notion of *algebraic disjointness*, namely $A \downarrow_C B$ if and only if for all $\bar{a} \in \mathbb{Q}(A)$ if \bar{a} is algebraically dependent over $\mathbb{Q}(C)$ then it is already algebraically dependent over $\mathbb{Q}(B \cup C)$. This generalizes to differentially closed fields.

Proposition 1.7 *In differentially closed fields $A \downarrow_C B$ if and only if $\mathbb{Q}\langle A \rangle$ and $\mathbb{Q}\langle B \cup C \rangle$ are algebraically disjoint over $\mathbb{Q}\langle C \rangle$.*

2 ω -stable Groups

We consider groups (G, \cdot, \dots) , possibly with extra structure, that are ω -stable. One way that such groups arise is by taking a group definable in an ω -stable structure, i.e., we could take a definable $X \subseteq \mathbb{K}^n$ and a definable $f : X \times X \rightarrow X$ (i.e. a function where the graph is definable) such that (X, f) is a group.

Theorem 2.1 *In an ω -stable group there are no proper infinite chains of definable subgroups.*

The proof is easy for ω -stable groups. If $\text{RM}(X) = \alpha$, there is a maximal n such that X can be partitioned into n disjoint definable sets of rank α . This is called the *Morley degree* of X . If G is a definable proper subgroup of H , then either the Morley rank goes down or the rank is constant and the Morley degree goes down. This can't happen infinitely often.

Corollary 2.2 *If G is an ω -stable group, there is definable subgroup $G^0 \subseteq G$ such that G^0 is the minimal definable subgroup of finite index.*

We call G^0 the *connected component* of G . We say that G is *connected* if $G = G^0$.

Generic types play an important role in the study of ω -stable groups.

Theorem 2.3 *Let G be a connected group defined over A . Then there is a unique generic type.*

If G is not connected, then there will be distinct generic types for each coset of G^0 .

We say that a definable subset $U \subseteq G$ is generic if $\text{RM}(U) = \text{RM}(G)$, or equivalently if a formula defining U is in the generic type.

If $A \subseteq G$ we say that b is *generic over A* if b realizes a generic type over A . We say that a_1, \dots, a_n are independent generics over A if each a_i is generic over $A \cup \{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n\}$.

We state some useful properties of generics

- Lemma 2.4** *i) If $U \subseteq G$ is generic, then $G = U \cdot U$.
ii) If a is generic over A and $b \in A$, then ab is generic over A .
iii) If $U \subseteq G$ is generic, there are $a_1, \dots, a_n \in G$ such that*

$$G = U \cup a_1 U \cup \dots \cup a_n U.$$

The following Lemma is a useful property of generic types that follows from Definability of Types in ω -stable theories.

Lemma 2.5 *Let G be a connected ω -stable group defined over A and $\bar{b} \in A$. Then for any formula $\phi(x, y, \bar{b})$, the set $\{g : \text{for all } y \text{ generic over } g \phi(g, y, \bar{b})\}$ is definable.*

Differential Algebraic Groups as ω -stable groups

We take a Weil-style definition of abstract differential varieties. We assume they have finite covers U_1, \dots, U_n where each U_i is homeomorphic to a Kolchin closed set V_i and the transition maps are differential morphisms. We can then think of these as objects interpretable in \mathbb{K} by taking the set X to be the disjoint union of the V_i . We have a definable equivalence relation on X determined by the transition maps. We can then identify our abstract variety with X/E . We now use Elimination of Imaginaries for differential closed field. Poizat proved that whenever $X \subseteq \mathbb{K}^n$ is definable and E is a definable equivalence relation on X then there is a definable function $f : X \rightarrow \mathbb{K}^m$ for some m such that

$$xEy \Leftrightarrow f(x) = f(y).$$

We can then identify our abstract variety “definably” with the image of f . If we started with a differential algebraic group, then the image would naturally be a definable group.

Pillay proved the following converse

Theorem 2.6 (Pillay) *If G is a group definable in a differentially closed field, G is definably isomorphic to a differential algebraic group.*

Pillay’s proof is an analog of the corresponding result for algebraic groups and algebraically closed field.

We will also need the following general theorem of Hrushovski.

Theorem 2.7 *Suppose we are in an ω -stable theory and have a type p over a model and a definable function f such that if a and b are independent realizations of p then so is $f(a, b)$ and if a, b, c are independent realizations then $f(a, f(b, c)) = f(f(a, b), c)$. Then there is a definable group G with generic type p such that the multiplication on independent generics is given by f .*

In algebraically (differentially) closed fields this tells us that if we have an irreducible (differential) algebraic variety V and a (differential) rational function from $V \times V$ to V that is generically associative, then we can construct an irreducible (differential) algebraic group (differentially) birational to V with the multiplication generically determined by f .

The following result of Benoist connects the abstract notions about ω -stable groups with more familiar differential algebraic ones.

Theorem 2.8 (Benoist) *If G is a differential algebraic group, then G is connected if and only if G is irreducible. If G is irreducible, the unique generic type of G is the δ -generic, i.e., the type corresponding to the differential ideal of G .*

3 Reducing to Generic Data

Suppose we have the following data:

- K is a differentially closed field;
- G is a connected differential algebraic group defined over K and p is the generic type of G over K ;
- f is a differential rational function defined over K for realizations of p ;
- q is a type over K in the language of pure fields such that if a realizes p , then $f(a)$ realizes q ;
- g is a rational function defined over K such that if a, b, c are independent realizations of q , then

$$g(g(a, b), c) = g(a, g(b, c));$$

- if a and b are independent realizations of p , then $f(a \cdot b) = g(f(a), f(b))$.

We claim that this data suffice to construct an embedding of G into an algebraic group. We first construct H . This is Hrushovski's version of Weil's group-chunk yoga. In any ω -stable structure when we have a type q over a model and a definable function that is associative on independent realizations of the type, then there is a definable group H_1 with generic q such that the function agrees with multiplication for independent generics. In algebraically closed fields any definable group is definably isomorphic to an algebraic group H .

Thus we have an algebraic group and a partial differential rational map $f : G \rightarrow H$ such that if a is a differential generic of G , then $f(a)$ is a generic of H and

$$f(a \cdot b) = f(a) \cdot f(b).$$

We need to argue that f extends to an embedding of G into H .

Lemma 3.1 *There is $\hat{f} : G \rightarrow H$ a definable embedding with $\hat{f} \supseteq f$.*

Proof Let

$$U = \{x \in G : \text{for all } y \text{ generic over } x \ f(x \cdot y) = f(x)f(y) \text{ and } f(y \cdot x) = f(y)f(x)\}.$$

By 2.5 U is definable, and if x realizes p then $x \in U$. Thus U is a generic subset of G . By 2.4 every element of G is a product of two elements of U . For $x \in G$, let $a, b \in U$ with $x = ab$. Let $\widehat{f}(x) = f(a) \cdot f(b)$.

We must first show that \widehat{f} is well defined. We will use repeated the following fact:

Claim Suppose $a, b \in U$ and c is generic and independent from a, b over K . Then $f((ab)c) = f(a)f(b)f(c)$.

Since $a \in U$ and, by 2.4 bc is generic over a . Thus

$$f(a(bc)) = f(a)f(bc) = f(a)f(b)f(c).$$

Similarly $f(c(ab)) = f(c)f(a)f(b)$

Suppose $x = ab = a_1b_1$ where $a, b, a_1, b_1 \in U$. Let c be generic and independent from a, b, a_1, b_1 over K .

Then $f((ab)c) = f(a)f(b)f(c)$ and $f((a_1b_1)c) = f(a_1)f(b_1)f(c)$, Thus $f(a)f(b) = f(a_1)f(b_1)$. Thus \widehat{f} is well defined.

Suppose z is generic. Then z^{-1} is also generic. We claim that $f(z^{-1}) = f(z)^{-1}$. Let a be generic and independent from z over K . By the claim

$$f(c) = f((zz^{-1})c) = f(z)f(z^{-1})f(c).$$

Thus $f(z^{-1}) = f(z)^{-1}$.

We next argue that f is a group homomorphism. Suppose $x = ab$ and $y = cd$ where $a, b, c, d \in U$. Let z be generic independent from a, b, c, d over K .

Then

$$\begin{aligned} \widehat{f}(xy) &= \widehat{f}(xzz^{-1}y) \\ &= \widehat{f}((ab)z)(z^{-1}(cd)) \\ &= f((ab)z)f(z^{-1}(cd)) \\ &= f(a)f(b)f(z)f(z^{-1})f(c)f(d) \\ &= f(a)f(b)f(c)f(d) \\ &= \widehat{f}(x)\widehat{f}(y) \end{aligned}$$

Finally we argue that \widehat{f} is one-to-one. Suppose $\widehat{f}(a) = \widehat{f}(b)$. Choose c generic over a and b . Then $\widehat{f}(a) = \widehat{f}((ac)c^{-1}) = f(ac)f(c^{-1})$. The same is true for $f(b)$. It follows that $f(ac) = f(bc)$. Since f is one-to-one, $ac = bc$ and $a = b$.

Lemma 3.2 \widehat{f} is a differential rational map.

Proof Since \widehat{f} is definable. There is a Kolchin open $U \subseteq G$ such that $\widehat{f}|_U$ is differential rational and generic. Finitely many translates of U cover G . Let

$$G = U \cup a_1U \cup \dots \cup a_nU$$

For $x \in a_iU$

$$\widehat{f}(x) = \widehat{f}(a_i)\widehat{f}(a_i^{-1}x).$$

Since $a_i^{-1}x \in U$, the map

$$x \mapsto a^{-1}x \mapsto \widehat{f}(a_i^{-1}x) \mapsto \widehat{f}(a_i)\widehat{f}(a_i^{-1}x) = \widehat{f}(x)$$

is differential rational on a_iU . Thus U, a_1U, \dots, a_nU is a finite open cover of G such that \widehat{f} is differential rational on each set in the cover.

4 The Finite Dimensional Case

Suppose G is a differential algebraic group of finite Morley rank defined over a differentially closed fields K . Equivalently, if $K\langle G \rangle$ is the differential function field of G , then $K\langle G \rangle$ has finite transcendence degree over K .

Let p be the generic type of G over K and let a realize p . There is an n such that $K\langle a \rangle = K(a, a', \dots, a^{(n)})$. Let $f(a) = (a, a', \dots, a^{(n)})$. Clearly, f is differential rational and $K\langle a \rangle = K(f(a))$. Let q be the type in the language of pure fields of $f(a)$ over K . Then q is the generic type of the algebraic variety

$$V = \{h(x) = 0 : h \in K[\overline{X}], h(f(a)) = 0\}$$

and f maps realizations of p to realizations of q .

Suppose $b \in G$ is generic over $K\langle a \rangle$. Then

$$K\langle a, b \rangle = K\langle a \rangle\langle b \rangle = K\langle a \rangle(f(b)) = K(f(a), f(b)).$$

In particular, since $a \cdot b \in K\langle a, b \rangle$, $f(a \cdot b) \in K(f(a), f(b))$. Thus there is a differential rational g such that $f(a \cdot b) = g(f(a), f(b))$. Thus for a, b independent realizations of p , $f(a \cdot b) = g(f(a), f(b))$.

If a, b, c are independent realizations of p it is easy to see that

$$g(f(a), g(f(b), f(c))) = g(g(f(a), f(b)), f(c)).$$

Since $f(a), f(b), f(c)$ are independent realizations of q , it follows that for any x, y, z independent realizations of q ,

$$g(x, g(y, z)) = g(g(x, y), z).$$

Thus g is generically associative on realizations of q .

We now have all of the data required for the construction in §3. Thus there is a differential rational embedding of G into an algebraic group.

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