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The Wall of Complexity:

Summations Over Partitions of Integers

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What do the following problems have in common?

- Solving nonlinear ODEs/PDEs by recursively differentiating the PDEs to get their Taylor series
- Using the Lagrange Inversion Formula (LIF) to invert a hypergeometric function
- Finding the local extrema of a transcendental function
- Finding a smooth function which goes through a given finite set of monotonically increasing points such that the function is monotonically increasing on the entire interval containing the points. (This problem arose in a cell culture course in biotechnology.)
- Computing all infinitely many roots of a transcendental function, such as the pseudopolynomial $z^a - z + 2 = 0$ where *a* is an arbitrary complex number (My latest published paper in the Journal: *Mathematics in Computer Science*)
- Computing the Kostka numbers in the transition matrix from one basis of symmetric functions to another
- Computing terms in the Faa di Bruno formula
- Proving certain desired identities (like those in the book of 500 Combinatorial Identities by Henry Gould)

Answer: They all involve trying to express some large formula involving summations over partitions of some indexing integer *n*, usually in some kind of factored form, such as the familiar infinite product formula for the gamma function.

I will show with examples how far I could get before my hopes were dashed by the mysterious "wall of complexity".

Definition. Let $\mathbb{N}$ be the positive integers, $\mathbb{N}_0$ the nonnegative integers. Let $n \in \mathbb{N}$. Define $[n] \equiv \{k \in \mathbb{N} \ni 1 \le k \le n\}$ and $[n]_0 \equiv \{k \in \mathbb{N}_0 \ni 0 \le k \le n\}$.

Definition. Let $(u_0,...,u_{n-1}) \in \mathbb{N}_0^n$ or $(u_1,...,u_n) \in \mathbb{N}_0^n$ be n-tuples of nonnegative integers, with 0-indexing and 1-indexing, respectively. Then we denote and define the *weight* of this tuple as $|u| \equiv \sum_{i=0}^{n-1} i \cdot u_i$ or $|u| \equiv \sum_{i=1}^{n} i \cdot u_i$, respectively. And we denote and define the *length* of this tuple as $\ell(u) \equiv \sum_{i=0}^{n-1} u_i$ or $\ell(u) \equiv \sum_{i=1}^{n} u_i$ respectively.

For any integer M such that $|u| = M$ and an n-tuple with 1-indexing, u_i is the *multiplicity* of the integer i in the partition of M as a sum of u_1 1s, u_2 2s, etc.

Definition. B(x) is an infinitely differentiable function of the continuous variable x. Let $B_n \equiv \frac{d^n B}{dx^n}$ with $B_0 \equiv B$.

Definition. A(x) is the reciprocal of x. So A(x)*B(x) = 1.

Definition. For any $N \in \mathbb{N}$ and any finite set of nonnegative integers $u_1,...,u_N, M$

$$\binom{M}{u_1,...,u_N} \equiv \frac{M!}{(M - \sum_{i=1}^{N} u_i)! \cdot \prod_{i=1}^{N} u_i!} = \binom{M}{u_1,...,u_N, M - \sum_{i=1}^{N} u_i}. \qquad (1)$$

Problem. Let t be a continuous variable. Let x be an infinitely differentiable function of t. Without loss of generality let x(t=0)=0. Given the nonlinear autonomous first-order differential equation $\frac{dx}{dt} = B(x)$, express the Taylor series expansion of x in terms of t and the derivatives of B with respect to x. In other words, considering B, and A, to be functions directly *only* of x, not of t, indirectly functions of t via x=x(t), for each $n \in \mathbb{N}$ find a function $F_n(B,...,B_{n-1})$ of n and B and its derivatives such that $x_n = F_n(B,...,B_{n-1})$, which we can (at least in principle, before we concern ourselves with convergence) then use to "plug into" the Maclaurin series

$$x_n = \sum_{n=1}^{\infty} \frac{t^n}{n!} F_n(B(0), B_1(0),..., B_{n-1}(0)) \qquad (2)$$

Trivially, $F_n(B,...,B_{n-1})$ is a polynomial with integer coefficients in the Bs. In other words, there exists a function $d: \bigcup_{n\in\mathbb{N}} \mathbb{N}_0^n \to \mathbb{N}_0$ such that

$F_n(B,...,B_{n-1}) = \sum_{\substack{|u|=n-1 \\ \ell(u)=n}} d(u_0,...,u_{n-1}) \cdot \prod_{k=0}^{n-1} B_k^{u_k}$. These integer coefficients are functions of n.

But, as functions of n, are they fixed rational functions of n and "integer exponentials", i.e. $2^n, 3^n, ...$, of n?

More precisely, as n grows without bound, the number of arguments for the function $d(u_0,...,u_{n-1})$ grows without bound. It is better to think of d as a function of the infinitely many variables $\{u_m\}_{m\in\mathbb{N}_0}$, where by $d(u_0,...,u_{n-1})$ we mean we set $\forall m \geq n \Rightarrow u_m = 0$. Is it possible to write $d(u_0,...,u_{n-1})$ as an unknown function $f_n(v_1,...,v_n,w_1,...,w_{n-1})$ such that $d(u_0,...,u_{n-1}) = f_n(1,...,n,2^n,...,n^n)$? Is it possible to demand the stronger condition that there exists some $M, N \in \mathbb{N}$ such that $\forall n \geq M \Rightarrow f_n = f_n(v_1,...,v_N,w_1,...,w_{N-1})$?

Example. For any given n-tuple $(u_0,...,u_{n-1})$ the two conditions $|u| = n-1$ and $\ell(n) = n$ reduce the number of choices for the u_i s from n to n-2. If we further restrict the n-tuple by $u_i = 0$ for all but three $i \in [n-1]_0$ then $d(u_0,...,u_{n-1})$ becomes a function of only the single variable, n. Only when $n \geq 7$ do we have $u_i > 0$ for four or more $i \in [n-1]_0$.

$$\begin{aligned}
x_1 &= B_0 \\
x_2 &= B_0 B_1 \\
x_3 &= B_0 B_1^2 + B_0^2 B_2 \\
x_4 &= B_0 B_1^3 + 4B_0^2 B_1 B_2 + B_0^3 B_2 \\
x_5 &= B_0 B_1^4 + 11B_0^2 B_1^2 B_2 + 4B_0^3 B_2^2 + 7B_0^3 B_1 B_3 + B_0^4 B_4 \\
x_6 &= B_0 B_1^5 + 26B_0^2 B_1^3 B_2 + 34B_0^3 B_1 B_2^2 + 32B_0^3 B_1^2 B_3 + 15B_0^4 B_2 B_3 + 11B_0^4 B_1 B_4 + B_0^5 B_5
\end{aligned} \tag{3}$$

For all $n \geq 7$ we have

$$\begin{aligned}
x_n &= B_0 B_1^{n-1} + (2^{n-1} - n)\cdot B_0^2 B_1^{n-3} B_2 + (\ ) \\
&+ \left(\frac{7\cdot 313}{2\cdot 3^2} - \frac{37}{2}n - \frac{1}{2}n^2 + \frac{3}{2^3}(n-2)\cdot 2^n - \frac{5n+2}{2\cdot 3^2}\cdot 3^n + \frac{19\cdot 61}{2^{12}\cdot 3^2}\cdot 4^n \right) B_0^4 B_1^{n-6} B_2 B_3 \\
&+ \cdots + \left(\frac{n^2 - 3n + 4}{2} \right) B_0^{n-2} B_1 B_{n-2} + B_0^{n-1} B_{n-1}
\end{aligned} \tag{4}$$

which we obtained by solving linear recursions and summation calculus on the exponents on the Bs. However, repeating this method does not help us find a general formula for $d(u_0,...,u_{n-1})$.

So, let us try a different approach. Let us show the intermediate step in the calculation above. We compute $x_n = \dfrac{d^{n-1}B(x(t))}{dt^{n-1}}$ by the chain rule of differentiation given by the Faa di Bruno formula

$$\begin{aligned}
x_1 &= B_0 \\
x_2 &= B_1 x_1 \\
x_3 &= B_2 x_1^2 + B_1 x_2 \\
x_4 &= B_3 x_1^3 + 3B_2 x_1 x_2 + B_1 x_3 \\
x_5 &= B_3 x_1^4 + 6B_3 x_1^2 x_2 + 3B_2 x_2^2 + 4B_2 x_1 x_3 + B_1 x_4
\end{aligned} \tag{5}$$

$$x_6 = B_4 x_1^5 + 10 B_4 x_1^3 x_2 + 15 B_3 x_1 x_2^2 + 6 B_3 x_1^2 x_3 + 10 B_2 x_2 x_3 + 5 B_2 x_1 x_4 + B_1 x_5$$

and then eliminate each x_j using the previous lines using Egorychev's generalization of the LIF, formula 48, page 258, to a system of n equations $\forall m \in [n] \Rightarrow F_m(B_0,...,B_{m-1},x_1,...,x_m) = 0$ and n dependent variables $\{x_m\}_{m=1}^n$. In our particular case, the number of independent variables equals the number of dependent variables. And our system is triangular, in that the m-th equation depends only upon the first m dependent and the first m independent variables. But, in Egorychev's formula, the number of independent variables can be anything. However, the number of dependent variables needs to equal the number of equations. Define

$\forall m \in [n] \Rightarrow \theta_m \equiv F_m(B,x) - x_m + B_{m-1}$ and $\Phi(x_1,...,x_n,B_0,...,B_{n-1}) \equiv x_n$ and $\theta^\beta \equiv \prod_{m=1}^n \theta_m^{\beta_m}$

and $\dfrac{\partial^{\ell(\beta)}}{\partial t^{\ell(\beta)}} \equiv \dfrac{\partial^{\sum_{m=1}^n \beta_m}}{\prod_{m=1}^n \partial t_m^{\beta_m}}$ and we use the dummy variable t_m in place of x_m in order to differentiate and specialize. Then

$$x_n = \sum_{\beta \ni |\beta| \geq 0} \frac{(-1)^{\ell(\beta)}}{\beta!} \frac{\partial^{\ell(\beta)}}{\partial t^{\ell(\beta)}} \left[\theta^\beta(B,t) \cdot t_n \cdot (-1)^n \right]_{\substack{\forall j \in [n] \\ t_j = B_{j-1}}} \tag{6}$$

$$= (-1)^n \sum_{\beta \ni \ell(\beta) \geq 0} \frac{(-1)^{\ell(\beta)}}{\prod_{k=1}^n \beta_k!} \frac{\partial^{\ell(\beta)}}{\partial t^{\ell(\beta)}} \left[t_n \cdot (t_1 - B_0)^{\beta_1} \cdot \prod_{m=2}^n (t_m - \sum_{k=1}^{m-1} \sum_{\substack{u \in \mathbb{N}_0^{m-1} \\ \ni |u| = m-1 \\ \ell(u) = k}} \frac{(m-1)!}{\prod_{l=1}^{m-1} u_l!} \cdot B_k \cdot \prod_{l=1}^{m-1} \left(\frac{t_l}{l!} \right)^{u_l}) \right]_{\substack{\forall j \in [n-1] \\ t_j = B_{j-1}}}$$

Identities that come directly from the Lagrange Inversion Formula (LIF)

Identity (7). $\displaystyle\sum_{\substack{v \in \mathbb{N}_0^n \\ |v| = n}} (-1)^{\ell(v)} \cdot \binom{n + \ell(v)}{v_1,...,v_n} \prod_{l=1}^n \left(\frac{1}{(l+1)!} \right)^{v_l} = 1$ (7)

Proof. Let $y = e^x \Leftrightarrow y - 1 = e^x - 1 = \sum_{n=1}^\infty \frac{x^n}{n!}$ and $x = \ln(y) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}(y-1)^n$ be inverse relations with nonconstant terms in each series equal to 0. By the LIF we know that

$x = \displaystyle\sum_{n=1}^\infty \frac{1}{n!} \cdot D_x^{n-1} \left\{ \left(\frac{x-0}{e^x - 1} \right)^n \right\} \Bigg|_{x=0} \cdot (y-1)^n = \sum_{n=1}^\infty \frac{1}{n!} \cdot D_x^{n-1} \left\{ \left(1 + \sum_{m=1}^\infty c_m \cdot x^m \right)^n \right\} \Bigg|_{x=0} \cdot (y-1)^n$ with

$c_m \equiv \dfrac{1}{(m+1)!}$. We know by the expansion of *any* binomial with negative integer exponent $(1+t)^{-n} = (-1)^{n-1}\sum_{k=0}^{\infty}\binom{n-1+k}{k,\ n-1}\cdot t^k$. Substituting $\sum_{m=1}^{\infty} c_m \cdot x^m$ for t and expanding yields $\forall\{c_m\}_{m\in\mathbb{N}} \Rightarrow \left(1+\sum_{m=1}^{\infty} c_m x^m\right)^{-n} = 1+(-1)^{n-1}\sum_{m=1}^{\infty} x^m \sum_{\substack{v\in\mathbb{N}_0^m \\ |v|=m}} (-1)^{\ell(v)}\binom{n-1+\ell(v)}{v_1,\ldots,v_m}\prod_{l=1}^{m} c_l^{v_l}$

So $\forall n\in\mathbb{N} \Rightarrow D_x^{n-1}\left\{\left(1+\sum_{m=1}^{\infty} c_m x^m\right)^{-n}\right\}\Bigg|_{x=0} = (-1)^{n-1}(n-1)!\sum_{\substack{v\in\mathbb{N}_0^{n-1} \\ |v|=n-1}} (-1)^{\ell(v)}\binom{n-1+\ell(v)}{v_1,\ldots,v_{n-1}}\prod_{l=1}^{n-1} c_l^{v_l}$.

Substituting $\frac{1}{(m+1)!}$ for c_m and comparing each term with the known Taylor series for the logarithm centered at y=1 given above yields the identity.
Q.E.D.

Identity (8). $\sum_{\substack{v\in\mathbb{N}_0^n \\ |v|=n}} (-1)^{\ell(v)}\cdot\binom{n+\ell(v)}{v_1,\ldots,v_n}\prod_{l=1}^{n}\left(\frac{1}{l+1}\right)^{v_l} = \frac{(-1)^n}{n!}$ (8)

Proof. From

$$y-1 = \sum_{n=1}^{\infty}\frac{1}{n!}D_y^{n-1}\left\{\left(\frac{y-1}{\ln(y)-0}\right)^n\right\}\Bigg|_{y=1}\cdot x^n = \sum_{n=1}^{\infty}\frac{1}{n!}D_y^{n-1}\left\{\left(1+\sum_{m=1}^{\infty} c_m\cdot(y-1)^m\right)^{-n}\right\}\Bigg|_{y=1}\cdot x^n$$ with

$c_m = \frac{(-1)^m}{m+1}$ and using the expansion involving arbitrary c_m in the proof of Identity (7) we get Identity (8).
Q.E.D.

Identities (7) and (8) raise the question whether such summations over partitions of integers can always simplify into some form that factors over rational numbers or rational functions of gamma functions of rational numbers. While it may be unreasonable to ask this question in general, do such identities exist if we had $c_m = \frac{1}{m!\cdot(m+2)} = \frac{1}{\Gamma(m+1)\cdot(m+2)}$ or any rational function of m and $\Gamma(m)$?

One answer is known: when $\forall l\in\mathbb{N}\Rightarrow c_l = 1$. In fact, we may generalize to one parameter more than just n, by including an additional parameter, k, and summing over only those tuples of nonnegative integers with length k. Specifically,

Identity. $\sum_{\substack{v\in\mathbb{N}_0^n \\ |v|=n \\ \ell(v)=k}} \prod_{i=1}^{n}\frac{1}{v_i!} = \frac{n!}{(n+k)!}\sum_{\substack{v\in\mathbb{N}_0^n \\ |v|=n \\ \ell(v)=k}}\binom{n+\ell(v)}{v_1,\ldots,v_n} = \frac{1}{k!}\binom{n-1}{k-1}$ (9)

Again, a summation over tuples of given length and weight of this particular combination of factorials of the coordinates of those tuples results in a very simple factored form.
Proof.

Define generating functions $f(t) \equiv t^m$ and $g(x) \equiv (1-x)^{-1}$. Then $\frac{d^n}{dx^n} f(g(x)) = \frac{d^n}{dx^n}((1-x)^{-m}) = (-m)_n (-1)^n (1-x)^{-m-n}$ and $\frac{d^k}{dt^k} f(t) = (m)_k t^{m-k}$ and $\frac{d^i}{dx^i} g(x) = (1-x)^{-1-i} i!$. So $\left.\frac{d^k}{dt^k} f(t)\right|_{t=g(x)} = (m)_k (1-x)^{-m+k}$. Substituting these formula into both sides of the Faa di Bruno formula yields

$$(-m)_n (-1)^n (1-x)^{-m-n} = n! \sum_{k=1}^{n} (m)_k (1-x)^{-m+k} \sum_{\substack{v \in \mathbb{N}_0^n \\ |v|=n \\ \ell(v)=k}} \prod_{i=1}^{n} \frac{((1-x)^{-1-i})^{v_i}}{v_i!} \tag{10}$$

Define $\lambda_{n,k} \equiv \sum_{\substack{v \in \mathbb{N}_0^n \\ |v|=n \\ \ell(v)=k}} \prod_{i=1}^{n} \frac{1}{v_i!}$ (11)

Then (10) becomes, after factoring out $(1-x)^{-m}$

$$(-m)_n (-1)^n (1-x)^{-n} = n! \sum_{k=1}^{n} (m)_k (1-x)^k \sum_{\substack{v \in \mathbb{N}_0^n \\ |v|=n \\ \ell(v)=k}} (1-x)^{-\sum_{i=1}^{n} v_i - \sum_{i=1}^{n} i \cdot v_i} \prod_{i=1}^{n} \frac{1}{v_i!} \tag{12}$$

$$(-m)_n (-1)^n (1-x)^{-n} = n! \sum_{k=1}^{n} (m)_k (1-x)^k \sum_{\substack{v \in \mathbb{N}_0^n \\ |v|=n \\ \ell(v)=k}} (1-x)^{-n-k} \prod_{i=1}^{n} \frac{1}{v_i!} \tag{13}$$

So (12) and (13) imply $(-m)_n (-1)^n = n! \sum_{k=1}^{n} (m)_k \lambda_{n,k}$ for all $m, n \in \mathbb{N}$ or

$$\binom{m+n-1}{n} = \sum_{k=1}^{n} (m)_k \lambda_{n,k} \tag{14}$$

Since (14) depends upon a free variable, m, it follows that (14) is actually a system of equations. The definition of $\lambda_{n,k}$ immediately implies $k > n \Rightarrow \lambda_{n,k} = 0$ and $n < 0 \Rightarrow \lambda_{n,k} = 0$. These provide the discrete-variable boundary conditions necessary if one wishes to solve (14) by solving linear recurrences. The more proper way of solving the system like (14) is to apply Egorychev's Chapter III "Inversion and Classification of Linear Relations in Combinatorial Analysis". I honestly do not remember how I solved this system, other than by guessing. Substituting $\frac{1}{k!}\binom{n-1}{k-1}$ for $\lambda_{n,k}$ into (14) yields

$\binom{m+n-1}{n} = \sum_{k=1}^{n} (m)_k \frac{1}{k!}\binom{n-1}{k-1} = \sum_{k=1}^{n} \binom{m}{k}\binom{n-1}{k-1}$, whose truth *can* at least be confirmed by Egorychev 2.4.1c page 78, who references it in Knuth section 12.6 problem 1.
Q.E.D.

The gamma function has a known factored form, with all nonpositive poles, formula 29 page 199 in Ahlfors

$$\Gamma(z) = e^{-\gamma \cdot z} z^{-1} \prod_{n=1}^{\infty} \left(\left(1 + z \cdot n^{-1}\right)^{-1} e^{z \cdot n^{-1}} \right) \tag{15}$$

with γ the Euler-Mascheroni constant.

Example. Consider another function with a known Taylor series inverse whose terms in the Taylor series factor over rational numbers.

$$\text{Let } z \equiv \sin(w) = \sum_{\substack{n \in \mathbb{N} \\ n \equiv 1 \bmod 4}} \frac{w^n}{n!} - \sum_{\substack{n \in \mathbb{N} \\ n \equiv 3 \bmod 4}} \frac{w^n}{n!} \text{ and} \tag{16}$$

$$w = \arcsin(z) = z + \sum_{\substack{n \in \mathbb{N} \\ n \equiv 1 \bmod 2}} \frac{\prod_{i=1}^{(n-1)/2} (2i-1)}{\prod_{i=1}^{(n-1)/2} (2i)} \frac{z^n}{n} \tag{17}$$

which one obtains by methods of basic calculus. As one can see, the coefficients of the terms in the Maclaurin series of arcsin(z) factor over the integers. I immediately tried variations of these functions of the form $z = \sum_{j=1}^{N} c_j \cdot \sum_{\substack{n \in \mathbb{N} \\ n \equiv j \bmod N}} \frac{w^n}{n!}$ for some integers $\{c_j\}_{j=1}^{N}$ and some $N \in \mathbb{N}$, to see if I could invert them using the LIF, but I had no luck. In fact, I had no luck deriving the formula for arcsin(z) from the power series for z=sin(w) and the LIF. All I could do was then conclude a new identity. But, can we *use* these identities to derive more complicated identities and factorizations?

Definition. This notation is sometimes used to denote the coefficient of the n-th power of a variable in a power series: $\left[z^n\right] \sum_m c_m z^m \equiv c_n$. Then the LIF is sometimes written in the following manner $\left[z^n\right] H(f(z)) = \frac{1}{n} \left[w^{n-1}\right] \left(H'(w) \cdot \left(\frac{w}{g(w)} \right)^n \right)$ (18)

if you are given $z = g(w)$ and wish to find $w = f(z)$, or, more generally, $H(f(z))$ for any differentiable function H.

So given g(w) = the Maclaurin series of sin(w) and plugging it into the LIF, and taking H(w)=w, and matching up the coefficients in the power series we obtain from the LIF with the coefficients in (15) we can derive a new identity

$$\frac{\prod_{i=1}^{(n-1)/2} (2i-1)}{\prod_{i=1}^{(n-1)/2} (2i)} \frac{z^n}{n} = \sum_{u \in \mathbb{N}_0^{n-1}} \binom{n-1+\ell(u)}{n-1, u_2, u_4, \ldots} \cdot \prod_{\substack{m \in \mathbb{N} \\ m \equiv 0 \bmod 4}} \left(\frac{-1}{(m+1)!} \right)^{u_m} \cdot \prod_{\substack{m \in \mathbb{N} \\ m \equiv 2 \bmod 4}} \left(\frac{1}{(m+1)!} \right)^{u_m} \tag{19}$$

where $m \equiv 1 \bmod 2 \Rightarrow u_m = 0$.

The Girard-Waring Formula

Another example of summations over partitions of integers are the various variations of the Girard-Waring Formula to express one type of symmetric polynomial in terms of the symmetric bases over rational numbers of another type of symmetric polynomial. We will concern ourselves only with the powersum and the elementary symmetric functions.

Definition. Let $x \equiv \{x_i\}_{i\in\mathbb{N}}$ be a countable number of indeterminates.

Definition. For each $k \in \mathbb{N}_0$ let $e_k = e_k(x)$ be the k-th elementary symmetric function in the x, which is the sum of all products of k distinct members of x, and $e_0 \equiv 1$.

Definition. For each $k \in \mathbb{N}$ the k-th powersum is $p_k \equiv \sum_{i\in\mathbb{N}} x_i^k$.

When x is finite, N of them, then $p_0 \equiv N$ and $\forall k \in \mathbb{N} \ni k > N \Rightarrow e_k = 0$, and there are nontrivial algebraic relations among any N+1 of the p_k.

Theorem. (2.4) page 20 Macdonald. The ring Γ of all homogeneous symmetric polynomials in the x over the integers of total degree T has as a basis over $\mathbb{Z}$ all monomials $\{\prod_{k=1}^{n} e_k^{u_k}\}$ such that $|u| = T$.

Theorem. (2.10 page 24) Macdonald. The ring Γ of all homogeneous symmetric polynomials in the x over the integers of total degree T has as a basis over $\mathbb{Q}$ all monomials $\{\prod_{k=1}^{n} p_k^{u_k}\}$ such that $|u| = T$.

Girard-Waring Formula $\forall n \in \mathbb{N} \Rightarrow p_n = \sum_{\substack{u\in\mathbb{N}_0^n \\ |u|=n}} (-1)^{\ell(u)} \frac{(l(u)-1)!}{u!} e^u$ (20)

where $e^u \equiv \prod_{k=1}^{n} e_k^{u_k}$ and $u! \equiv \prod_{k=1}^{n} u_k!$.

Inverse Girard-Waring Formula $\forall n \in \mathbb{N} \Rightarrow e_n = \sum_{\substack{u\in\mathbb{N}_0^n \\ |u|=n}} (-1)^{\ell(u)} \frac{1}{u!} \prod_{k=1}^{n} \left(\frac{p_k}{k}\right)^{u_k}$ (21)

Interestingly, this inverse formula (21) even works for finitely many, N, of the x: it will yield zero for all $n > N$, providing infinitely many nontrivial algebraic relations among more than N of the p_k.

Again, note how the denominators and numerators of the coefficients of the monomials e^u and $p^u \equiv \prod_{k=1}^{n} p_k^{u_k}$ in (20) and (21) are factorials, i.e. they factor over integers.

But, do we have such a simple formula for the *product* of two or more p_n in terms of e^u?

The Kostka Numbers

Definition. Formally, the Kostka number $K_{\lambda\mu}$ is the number of tableaux of shape λ and μ. (6.4) p101 in Macdonald. We will deal only with the powersums p_λ and the elementary symmetric functions e_μ and the transition matrix $M(p,e)$ between them in Macdonald's notation: $p_\lambda = \sum_\mu M_{\lambda\mu}(p,e)\cdot e_\mu$.

Can we use the LIF to compute infinitely many roots of a transcendental function?

Consider the trinomial $z^N - z + w = 0$ (22)
where N is a positive integer greater than 1. This particular trinomial, with the other nonzero coefficient on the first power of z, besides z^N, has been studied by many authors. Lawrence M Glasser gives the formula for N-1 roots as follows:

$$\forall k \in [N-1] \Rightarrow x_k = e^{\frac{2\pi\sqrt{-1}}{N-1}\cdot k} - \frac{w}{(N-1)}\cdot\sum_{n=0}^{\infty} w^n \cdot e^{\frac{2\pi\sqrt{-1}}{N-1}\cdot k\cdot n}\cdot\frac{\Gamma\left(\frac{N\cdot n}{N-1}+1\right)}{\Gamma\left(\frac{n}{N-1}+1\right)\cdot\Gamma(n+2)} \qquad (23)$$

Curiously, Glasser does not give the N-th root x_N but we can derive it from the fact that for N>2 the first elementary symmetric function of the roots, which equals minus the coefficient of z^{N-1}, equals 0. Hence, $x_N = -\sum_{k=1}^{N} x_k$. We can use the ratio test to test for the radius of convergence of this series. The ratio of the absolute value of the (n+1)-th term over the n-th term in (14) equals

$$\left|\frac{w^{n+1}}{w^n}\right|\cdot\left|e^{\frac{2\pi\sqrt{-1}}{N-1}\cdot k\cdot((n+1)-n)}\right|\cdot\frac{\Gamma\left(\frac{n}{N-1}+1\right)}{\Gamma\left(\frac{n}{N-1}+\frac{1}{N-1}+1\right)}\frac{\Gamma\left(\frac{N\cdot n}{N-1}+1+\frac{N}{N-1}\right)}{\Gamma\left(\frac{N\cdot n}{N-1}+1\right)}\cdot\frac{\Gamma(n+2)}{\Gamma(n+3)}$$

$$=|w|\cdot\frac{\Gamma\left(\frac{n}{N-1}+1\right)}{\Gamma\left(\frac{n}{N-1}+1+\frac{1}{N-1}\right)}\cdot\frac{\Gamma\left(\frac{N\cdot n}{N-1}+1+\frac{N}{N-1}\right)}{\Gamma\left(\frac{N\cdot n}{N-1}+1\right)}\cdot\frac{1}{n+2} \qquad (24)$$

Definition. Define the real number $s \equiv \frac{1}{N-1}$. So for N>1, 0<s<1.

$$\frac{\Gamma\left(\frac{n}{N-1}+1\right)}{\Gamma\left(\frac{n}{N-1}+1+\frac{1}{N-1}\right)} = \frac{\Gamma(s\cdot n+1)}{\Gamma(s\cdot n+1+s)} \qquad (25)$$

Definition. Define the real number $r \equiv \dfrac{N}{N-1}$ so 1 < r < 2 for N>2. Also s = r – 1.

So the second ratio of gammas in (24)

$$\frac{\Gamma\left(\frac{N\cdot n}{N-1}+1+\frac{N}{N-1}\right)}{\Gamma\left(\frac{N\cdot n}{N-1}+1\right)}=\frac{\Gamma(r\cdot n+1+r)}{\Gamma(r\cdot n+1)}=\frac{\Gamma(r\cdot n+r)}{\Gamma(r\cdot n)}\cdot\frac{r\cdot n+r}{r\cdot n}=\frac{\Gamma(r\cdot n+r)}{\Gamma(r\cdot n)}\frac{n+1}{n} \quad (26)$$

So (26) has a slight bound

$$0<\frac{\Gamma(r\cdot n+r)}{\Gamma(r\cdot n)}\frac{n+1}{n}=\frac{\Gamma(r\cdot n+r-1)}{\Gamma(r\cdot n)}\cdot(r\cdot(n+1)-1)\frac{n+1}{n}<\frac{\Gamma(r\cdot n+s)}{\Gamma(r\cdot n)}\cdot r\cdot\frac{(n+1)^2}{n} \quad (27)$$

Definition. We say two real-valued functions, f(x) and g(x), are asymptotic as x goes to infinity, if $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=1$. We write this as "$f(x)\sim g(x)$ as $x\to\infty$".

Apply Stirling's asymptotic formula for large real arguments of the gamma function:

$$\Gamma(x)\sim\sqrt{2\pi(x-1)}\cdot\left(\frac{x-1}{e}\right)^{x-1} \text{ as } x\to\infty \quad (28)$$

So

$$\frac{\Gamma(x+s)}{\Gamma(x)}\sim\frac{\sqrt{2\pi(x-1+s)}}{\sqrt{2\pi(x-1)}}\cdot\frac{\left(\frac{x-1+s}{e}\right)^{x-1+s}}{\left(\frac{x-1}{e}\right)^{x-1}}=\sqrt{\frac{x-1+s}{x-1}}\cdot\left(1+\frac{s}{x-1}\right)^{x-1}\cdot\left(\frac{x-1+s}{e}\right)^{s} \quad (29)$$

as $x\to\infty$. Since $\lim\limits_{x\to\infty}\sqrt{\dfrac{x-1+s}{x-1}}=1$ and $\lim\limits_{x\to\infty}\left(1+\dfrac{s}{x-1}\right)^{x-1}=e^s$ and $\left(\dfrac{x-1+s}{e}\right)^s\sim e^{-s}\cdot x^s$ as $x\to\infty$. Substituting these limits into (29) yields

$$\frac{\Gamma(x+s)}{\Gamma(x)}\sim 1\cdot e^s\cdot x^s\cdot e^{-s}=x^s \text{ as } x\to\infty. \quad (30)$$

From (30) it follows that in (25) $\dfrac{\Gamma(s\cdot n+1)}{\Gamma(s\cdot n+1+s)}\sim(s\cdot n+1)^{-s}$ as $n\to\infty$. (31)

Replacing x with r*n into (30) back into (27) back into (26) back into (24), and multiplying in the asymptotic limit (31) yields the result that the limit of the ratio of the terms in the power series (23) in w is bounded below by 0 and above by

$\lim\limits_{n\to\infty}|w|\cdot r\cdot\dfrac{(n+1)^2}{n\cdot(n+2)}\cdot(r\cdot n)^s(s\cdot n+1)^{-s}=|w|\cdot r^{s+1}\cdot\lim\limits_{n\to\infty}\left(\dfrac{n}{s\cdot n+1}\right)^s$ where we factored out the limit $\lim\limits_{n\to\infty}\dfrac{(n+1)^2}{n\cdot(n+2)}=1$. The relations s > 0 and

$0<\dfrac{n}{s\cdot n+1}<\dfrac{n}{s\cdot n}=s^{-1}\Rightarrow 0<\left(\dfrac{n}{s\cdot n+1}\right)^s<s^{-s}$. So indeed the limit of the absolute value of the ratio of the terms in (23) is bounded strictly above by

$|w|\cdot r^r s^{-s} = |w|\cdot\left(\dfrac{N}{N-1}\right)^{\frac{N}{N-1}} (N-1)^{1/(N-1)} = |w|\cdot\dfrac{N^{\frac{N}{N-1}}}{N-1}$. From this we conclude that the series (23) is absolutely convergent for each value of $k\in[N-1]$, hence for all N roots, with a radius of convergence given by $|w|\le\dfrac{N-1}{\sqrt[N-1]{N^N}}$

Q.E.D.

Cayley's Differential Resolvent for Trinomials

Arthur Cayley chose a slightly different form for the trinomial (22) in his paper to compute the world's first linear differential resolvent of one of the first polynomials over a differential field with a single derivation. Giuseppe Belardinelli years later also independently rediscovered a variation of Cayley's formula for a resolvent of a slight variation on Cayley's trinomial. More importantly, while Cayley's proof was done by clever ad hoc recognition of the patterns of differentiation using the operator $x\cdot\dfrac{d}{dx}$, Belardinelli actually boldly solved (22) using the LIF for the one root which has z=0 when w=0, and then repeatedly differentiated and found a linear relation over the ring $\mathbb{Z}[x,\alpha]$ among the derivatives. Belardinelli's LIF solution to the trinomial is essentially Glasser's.

Let a be a constant. Let x be the independent variable. Let z be the dependent variable, considered as a function of x. Let n and p be positive integers with p<n. Let q=n-p. Let α be any indeterminate constant with respect to x. Then the trinomial

$$z^n + a\cdot z^p - x = 0 \qquad (32)$$

has the n-th order linear ordinary differential resolvent
(repeated as Lemma 104 page 170 in my dissertation)

$$(-1)^q n^{2n}\left(x^2\sqrt[n]{x^{-p}}\,\frac{d}{dx}\right)^n(\sqrt[n]{x^{-\alpha}}z^\alpha)$$
$$= a^n\cdot\prod_{i=0}^{p-1}(-n\cdot p\cdot x\cdot\frac{d}{dx}+q\cdot\alpha+n\cdot q\cdot i)\cdot\prod_{j=0}^{q-1}(-n\cdot q\cdot x\cdot\frac{d}{dx}-q\cdot\alpha+n\cdot q\cdot j)(\sqrt[n]{x^{-\alpha}}z^\alpha) \qquad (33)$$

Notice that Cayley gives his resolvent in a factored operator form. It factors over the rationals. It is understood for instance

$$(5x\frac{d}{dx}+6)(5x\frac{d}{dx}+7) = (5x\frac{d}{dx})(5x\frac{d}{dx}) + (5x\frac{d}{dx})(7) + 6\cdot(5x\frac{d}{dx}+7)$$
$$= 25\cdot x^2\cdot\frac{d^2}{dx^2}+25\cdot x\cdot\frac{d}{dx}+35\cdot x\cdot\frac{d}{dx}+30\cdot x\cdot\frac{d}{dx}+42 = 25\cdot x^2\cdot\frac{d^2}{dx^2}+90\cdot x\cdot\frac{d}{dx}+42$$

Hence

$$(5x\frac{d}{dx}+6)(5x\frac{d}{dx}+7)(\sqrt[n]{x^{-\alpha}}z^{\alpha}) = 25\cdot x^2\cdot\frac{d^2}{dx^2}(\sqrt[n]{x^{-\alpha}}z^{\alpha}) + 90\cdot x\cdot\frac{d}{dx}(\sqrt[n]{x^{-\alpha}}z^{\alpha}) + 42\cdot(\sqrt[n]{x^{-\alpha}}z^{\alpha})$$

$$= 25\cdot\left(\frac{-\alpha}{n}\right)\left(\frac{-\alpha}{n}-1\right)\cdot\sqrt[n]{x^{-\alpha}}\cdot z^{\alpha} + 50\cdot x\cdot\left(\frac{-\alpha}{n}\right)\sqrt[n]{x^{-\alpha}}\cdot\frac{dz^{\alpha}}{dx} + 25\cdot x^2\cdot\sqrt[n]{x^{-\alpha}}\cdot\frac{d^2z^{\alpha}}{dx^2}$$

$$+90\cdot\left(\frac{-\alpha}{n}\right)\cdot\sqrt[n]{x^{-\alpha}}\cdot z^{\alpha} + 90\cdot x\cdot\sqrt[n]{x^{-\alpha}}\cdot\frac{dz^{\alpha}}{dx} =$$

$$\sqrt[n]{x^{-\alpha}}\cdot\left\{25\cdot x^2\cdot\frac{d^2z^{\alpha}}{dx^2} + \left(90-\frac{50\alpha}{n}\right)\cdot x\cdot\frac{dz^{\alpha}}{dx} - \left(\frac{\alpha}{n}\right)\left(65-\frac{25\alpha}{n}\right)\cdot z^{\alpha}\right\}$$

The Binomial Equation $z^w = C$

Reminder: in expanding a resolvent, the linear resolvent is linear in z^{α} and its derivatives. We do not apply the chain rule to z^{α}, so we do not pull down the power α on z. In my dissertation Theorem 105 page 171 I expanded out Cayley's resolvent (33) to get the coefficients of all the derivatives of z^{α}. All non-integer and α powers of the x factor out, as they necessarily must, by definition of a differential resolvent over the base field of coefficients of the polynomial (32).

As one would expect, the explicit terms in the expanded form involve multiple summations over falling factorials in α, p and n. The summations seem not to be able to split over rationals.

But what happens when the total degree N or n is no longer a positive integer in the trinomial of either Glasser or Cayley? Suppose this power is an arbitrary complex number. Let us return to the original question in this section: can we use the LIF to compute infinitely many roots of a transcendental function? The simplest nontrivial example of such a transcendental function all of whose countably infinitely many roots are already known would be:

$$z^w = C \qquad (34)$$

for any non-real complex number $C \in \mathbb{C}$, where we think of z as a complex-valued (multi)function of the complex variable w. From now on, write $C = e^{A+B\cdot i}$ with $i \equiv \sqrt{-1}$. Let w = u + i*v. For any integer $k \in \mathbb{Z}$, let z_k denote the root given by

$$z_k \equiv \exp\left(\frac{A\cdot u+(B+2\pi\cdot k)\cdot v}{u^2+v^2}\right)\cdot\exp\left(\frac{(B+2\pi\cdot k)\cdot u - A\cdot v}{u^2+v^2}\cdot i\right). \qquad (35)$$

Suppose w is in the first quadrant of $\mathbb{C}$. So u>0 and v>0. Then, as k increases without bound, the magnitude of z_k increases without bound while it "spins around" the origin, spiraling out to infinity. As k decreases without bound towards minus infinity, the magnitude of z_k decreases towards 0 while spiraling in to the origin of $\mathbb{C}$. This implies "$z = C^{\frac{1}{w}}$" has an essential singularity at w=0, as z takes infinitely many complex values of *unbounded size* inside any disk containing w=0: no matter how small one makes the real and imaginary parts of w, one can always take a sufficiently large k.

This is equivalent to z taking on *all possible* complex values inside any disk in the w-plane containing w=0. To see that this is true, write z in polar form: $z = e^{x+i\cdot(y+2\pi j)}$ for any integer j. Then

$$w = \frac{\ln C}{\ln z} = \frac{A + B\cdot i}{x+(y+2\pi j)\cdot i} = \frac{[A\cdot x + B\cdot(y+2\pi j)] + i\cdot[B\cdot x - A\cdot(y+2\pi j)]}{x^2+(y+2\pi j)^2} \qquad (36)$$

Now, since x and y may be chosen arbitrarily, along with integer j, and since they determine z, we choose x and y first. (Remember: A and B are fixed.) Then, given any $\varepsilon > 0$, to find a w satisfying (36) such that $|w| < \varepsilon$ we simply choose j sufficiently large to make the absolute value of formula (36) smaller than ε.

The important point is that we have a formula, (35), for all infinitely many roots of this binomial. It's just the Laurent series in w centered at w = infinity. We could, if we wanted, expand (35) as the ordinary power series of an exponential in its arguments and obtain a power series solution of the binomial. This power series would *not* be a power series in u,v, since they appear in the denominator. And it would require further expansion - collecting like powers of A,B and k – to get it into the proper form as a power series in A, B, k. But, it *would* be a power series in the arguments of the exponential shown in (35) which would converge.

So, it is natural to ask if we can compute other power series solutions, obtained at Taylor (or Laurent) series for *all* countably infinitely many roots of other transcendental functions. And can we use the Lagrange Inversion Formula to help us do it?

The Trinomial Equation $z^w + z - 2 = 0$

Let's set C = 2 in the previous section, and let's add another term, linear in z. This trinomial is a slight variation on (22), without a minus sign on z. It is this version studied in my paper in Mathematics in Computer Science. In it is not obvious at all whether z as a function of w has any essential singularities or not, thanks to the linear term. While for any w there may exist a point z at infinity solution to this equation, it soon becomes clear that no matter which direction w approaches 0, z can approach only *one* finite value: z=1, since $z_0 \notin \{1,\infty\}$ would imply $z_0^0 + z_0 - 2 = 1 + z_0 - 2 = z_0 - 1 \neq 0$. Therefore, z(w) does not have an essential singularity at w=0.

However, as soon as w moves away from 0, it's clear there are infinitely many different first derivatives: $\frac{dz}{dw} = -\frac{\ln z}{\frac{1}{2-z}+\frac{w}{z}} \Rightarrow \left.\frac{dz}{dw}\right|_{w=0} = -\frac{\ln 1}{\frac{1}{2-1}+\frac{0}{1}} = 2\pi\cdot k$ for possible integers k. Nevertheless, as in the binomial case, we might expect some sort of series solution for every function z=z(w) that satisfies this equation. And, since z(w) does not have an essential singularity – or a singularity of any kind at w=0 – we expect that there are power series in w solutions. And, as in the case when w = N, a positive integer as we proved above, we expect all of them to have nonzero radii of convergence.

To compute the Taylor series for z(w) around any point, w, we must of course compute all infinitely many derivatives of z with respect to w, given an implicit function of both variables $G(w,z) \equiv z^w + z - 2$. It would help greatly to compute $\frac{d^n z}{dw^n}$ in terms of the partial derivatives of G with respect to w and z.

Definition 2.13 $G_{\pi,\rho} \equiv \dfrac{\partial^{\pi+\rho}G}{\partial z^{\pi}\partial w^{\rho}}$.

It is trivial to show by recursion that $\dfrac{d^n z}{dw^n} \in G_{1,0}^{-(2n-1)} \cdot \mathbb{Z}[\{G_{\pi,\rho}\}_{(\pi,\rho)\in\mathbb{N}_0\times\mathbb{N}_0}]$ where the denominator $G_{1,0} \equiv \dfrac{\partial G}{\partial z}$ is the *separant* of G, choosing z as the dependent and w as the independent variable. In Section 1 of my paper, suppose we are given $z = g(w) = \sum_{n=1}^{\infty} b_n \cdot w^n$ and the Lagrange Inversion Formula finds $w = f(z) = \sum_{n=1}^{\infty} c_n \cdot z^n$, in other words, the LIF gives us a formula for $\left.\dfrac{1}{n!}\dfrac{d^n f}{dz^n}\right|_{z=0} = c_n = L_n(b_1,...,b_n) = L_n(\left.\dfrac{1}{1!}\dfrac{d^1 g}{dw^1}\right|_{w=0},...,\left.\dfrac{1}{n!}\dfrac{d^n g}{dw^n}\right|_{w=0})$ as a polynomial in $\{b_1^{-1}, b_1, b_2, ..., b_n\}$. We can always shift the center of either power series to what we want. However, the LIF itself, nor in any of its proofs, in particular the analytic proofs, asserts that $\dfrac{1}{n!}\dfrac{d^n w}{dz^n} = \dfrac{1}{n!}\dfrac{d^n f}{dz^n} = L_n(\dfrac{1}{1!}\dfrac{d^1 g}{dw^1},...,\dfrac{1}{n!}\dfrac{d^n g}{dw^n}) = L_n(\dfrac{1}{1!}\dfrac{d^1 z}{dw^1},...,\dfrac{1}{n!}\dfrac{d^n z}{dw^n})$ where the n-th derivative of w with respect to z is a rational function in the 1st through n-th derivatives of z with respect to w. Franz Kamber did in fact derive a formula for L_n, which is rational in its arguments, using determinants, which is not dependent on a particular form – such as a power series centered at a particular point, or an implicit function G(z,w)=0 – for w in terms of z or vice versa. My Theorem 5.6 does prove that $\dfrac{d^n z}{dw^n}$ is a rational function of the $G_{\pi,\rho}$, if given G(z,w)=0, and derives a formula (5.1) for it, and derives the formula non-analytically – i.e. purely discretely – by recursion on the exponents of the $G_{\pi,\rho}$, and does so *without* any restriction on the separant, other than the separant being nonzero. All other proofs, including Egorychev's, require the separant to equal 1.

I show in Lemma 3.1 of my paper that my formula (5.1) and the LIF specialize to the same formula upon specialize my formula to a particular center (w(0),z(0)) and specializing the separant in the LIF to 1.

Theorem 5.6 $\dfrac{d^n z}{dw^n} = n! \cdot \sum_{u\in U_n} (-1)^{u_{1,0}} \cdot \dfrac{(-1-u_{1,0})!}{\prod_{\substack{(\pi,\rho)\\ \neq(1,0)}} (\pi!\rho!)^{u_{\pi,\rho}} u_{\pi,\rho}!} \prod_{\substack{(\pi,\rho)\\ \in I_n}} G_{\pi,\rho}^{u_{\pi,\rho}}$ where

Definition 2.17 U_n is the finite set of all maps $u: \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{Z}$ with finite support contained in I_n, $(\pi,\rho) \neq (1,0) \Rightarrow u_{\pi,\rho} \in \mathbb{N}_0$, $-u_{1,0} \in [2n-1]$.

Definition 2.10 I_n is the finite set of pairs of indices $(\pi,\rho) \in \mathbb{N}_0 \times \mathbb{N}_0$ such that $\pi + \rho = n$. So, in particular, $(0,0) \notin I_n$.

Application of Theorem 5.6 to computing infinitely many roots of a transcendental function

So, back to our question on page 9, let us use my version Theorem 5.6 of the LIF to compute all the roots of $G(z,w) \equiv z^{w} + z - 2 = 0$ for some arbitrary complex w. In my paper, I chose $w = \sqrt{-1}$, mainly because it is within distance 1 from w=0, and often power series of these types have a radius of convergence equal to 1, but, I never did anything with this information. In this section, I have to change the index π to θ since I will need to make use of the number pi. Let us center our power series at w=0, z=1.

So $G_{1,0} = w \cdot z^{w-1} + 1$. So $G_{1,0}\big|_{\substack{w=0 \\ z=1}} = 1 \neq 0$ for all roots. Differentiating with respect to z further, we find that for $\theta \in \mathbb{N} \Rightarrow G_{\theta,0} = (w)_{\theta} z^{w-\theta}$. So far, there have been no distinction among any of the roots. But when we differentiate with respect to w, we get $G_{\theta,\rho} = \sum_{j=0}^{\rho} \binom{\rho}{j} \frac{\partial^{\rho-j}(w)_{\theta}}{\partial w^{\rho-j}} \cdot (\ln(z))^{j} z^{w-\theta}$. Now, let us denote the k-th root by z_k, which are distinguished by the fact that $\ln(z_k(w=0)) = \ln(1) + 2\pi\sqrt{-1}\cdot k = 2\pi\sqrt{-1}\cdot k$. Therefore $G_{\theta,\rho}\big|_{\substack{w=0 \\ z=1}} = \sum_{j=0}^{\rho} \binom{\rho}{j} \frac{\partial^{\rho-j}(w)_{\theta}}{\partial w^{\rho-j}}\bigg|_{\substack{w=0 \\ z=1}} \cdot (2\pi\sqrt{-1}\cdot k)^{j}$.

Definition. The Stirling number of the first kind $s_{\theta,j}$ is the coefficient for the change of basis of polynomials in a single variable, w, of degree less than or equal to θ from powers to falling factorials: $\forall \theta \in \mathbb{N}_0 \Rightarrow (w)_{\theta} = \sum_{j=0}^{\theta} s_{\theta,j} w^{j}$ and $\forall j, \forall \theta < 0 \Rightarrow s_{\theta,j} = 0$ and $\forall \theta, \forall j < 0 \Rightarrow s_{\theta,j} = 0$.

Then $G_{\theta,\rho}\big|_{\substack{w=0 \\ z=1}} = \sum_{j=0}^{\rho} \frac{\rho!}{j!} s_{\theta,\rho-j} \cdot (2\pi\sqrt{-1}\cdot k)^{j}$. Inserting this into Theorem 5.6 yields

$$\frac{d^n z_k}{dw^n}\bigg|_{\substack{w=0 \\ z_k=1}} = n! \cdot \sum_{u \in U_n} (-1)^{u_{1,0}} \cdot \frac{(-1-u_{1,0})!}{\prod_{\substack{(\theta,\rho) \\ \neq(1,0)}} (\theta!\rho!)^{u_{\theta,\rho}} u_{\theta,\rho}!} \prod_{\substack{(\theta,\rho) \\ \in I_n}} \left(\sum_{j=0}^{\rho} \frac{\rho!}{j!} s_{\theta,\rho-j} \cdot (2\pi\sqrt{-1}\cdot k)^{j} \right)^{u_{\theta,\rho}}$$

Substituting this into the Taylor series $z_k(w) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n z_k}{dw^n}\bigg|_{\substack{w=0 \\ z_k=1}} \cdot w^n$ yields

$$z_k(w) = 1 + \sum_{n=1}^{\infty} \sum_{u \in U_n} (-1)^{u_{1,0}} \cdot \frac{(-1-u_{1,0})!}{\prod_{\substack{(\theta,\rho) \\ \neq(1,0)}} (\theta!\rho!)^{u_{\theta,\rho}} u_{\theta,\rho}!} \prod_{\substack{(\theta,\rho) \\ \in I_n}} \left(\sum_{j=0}^{\rho} \frac{\rho!}{j!} s_{\theta,\rho-j} \cdot (2\pi\sqrt{-1}\cdot k)^{j} \right)^{u_{\theta,\rho}} \cdot w^n .$$

What is the radius of convergence of this series? Can we apply the same methods as we did above – namely, the ratio test and variations of Stirling's asymptotic formula – to determine this radius? If the radius is larger than 1, then we may substitute $\sqrt{-1}$ for w, for example, and obtain all infinitely many roots of $z^{\sqrt{-1}} + z - 2 = 0$.

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