Symmetries of meromorphic connections over Riemann Surfaces

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Example

Consider:

$$y''' + \frac{3(3z^2-1)}{z(z-1)(z+1)}y'' + \frac{221z^4-206z^2+5}{12z^2(z-1)^2(z+1)^2}y' + \frac{374z^6-673z^4+254z^2+5}{54z^3(z-1)^3(z+1)^3}y = 0$$

The equation is invariant under the automorphism $z \mapsto -z$; and, setting $z^2 = x$ the equation transforms into:

$$y''' + \frac{3(2x-1)}{x(x-1)}y'' + \frac{329x^2+41-350}{48x^2(x-1)^2}y' + \frac{374x^3-673x^2+254x+5}{432x^3(x-1)^3}y = 0$$

The interplay between the Galois Groups and this symmetry is given by:

$$1 \longrightarrow G_{54} \longrightarrow F_{36}^{SL_3(\mathbb{C})} \longrightarrow \langle z \mapsto -z \rangle \longrightarrow 1$$

Example

Consider:

$$y''' + \frac{21(z^2 - z + 1)}{25z^2(z-1)^2}y' + \frac{21(-2z^3 + 3z^2 - 5z + 2)}{50z^3(z-1)^3}y = 0$$

The equation is invariant under the automorphism $z \mapsto 1 - z$; and, setting $z(z-1) = x$ the equation transforms into:

$$y''' + \frac{6}{4x-1}y'' + \frac{21(x-1)}{25x^2(4x-1)}y' - \frac{21(x-2)}{50x^3(4x-1)}y = 0$$

The interplay between the Galois Groups and this symmetry is given by:

$$1 \longrightarrow A_5 \longrightarrow I_h \longrightarrow \langle z \mapsto 1 - z \rangle \longrightarrow 1$$

Picard-Vessiot Extensions and Galois Group

We take a finite field extension K of $\mathbb{C}(z)$; and, we extend the differentiation $\frac{\partial}{\partial z} = \cdot'$. We consider a matrix differential equation $X' = AX$ were $A = (a^i_j)$ is an $n \times n$ matrix with coefficients in K.

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where I is a maximal differential ideal.
- A representation of the Galois Group in $GL_n(\mathbb{C})$ is given by
$$G = \{(g^i_j) \in GL_n(\mathbb{C}) | (g^i_j)(I) = (I)\}$$
where $(g^i_j) : X^i_j \mapsto X^i_k g^k_j$.

Galois Correspondence

Proposition: Let E be a Picard-Vessiot Extension for $X' = AX$, and $f \in E$. Then

$$f \in K \iff (g^i_j)(f) = f \quad \forall (g^i_j) \in G$$

Setting $I = \ker(K[X^i_j, \frac{1}{\det}] \to E)$.

$$\begin{array}{ccc} K[X^i_j, \frac{1}{\det}] & \xrightarrow{\phi} & E \\ \uparrow & & \uparrow \\ \mathbb{C}[X^i_j]^G & \xrightarrow[\phi\restriction]{} & K \end{array}$$

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$$\begin{array}{ccc} K[X^i_j, \frac{1}{\det}]/I & \longrightarrow & E \\ \uparrow & & \uparrow \\ \mathbb{C}[X^i_j]^G / \ker(\phi\restriction) & \longrightarrow & K \end{array}$$

Compoint's Theorem

Theorem: Assume G is reductive and unimodular.

- Let $P_1, \ldots, P_r$ be a basis of G-invariants in $\mathbb{C}[X_j^i]$.

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Then I is (algebraically) generated over $K[X_j^i, \frac{1}{\det}]$ by $P_1 - f_1, \ldots, P_r - f_r$.

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In other words, if G is unimodular and reductive, the ideal I in

$$\begin{array}{ccc} K[X_j^i, \frac{1}{\det}] & \xrightarrow{\phi} & E \\ \uparrow & & \uparrow \\ \mathbb{C}[X_j^i]^G & \xrightarrow[\phi\restriction]{} & K \end{array}$$

is uniquely determined by the map

$$\phi\restriction : \mathbb{C}[X_j^i]^G \to K$$

Fano Group

Take the homogeneous part of $\ker(\phi\restriction)$, i.e.

$$\ker(\phi\restriction)^h = \{P \in \mathbb{C}[X^i_j]^G \mid \phi(P) = 0 \text{ and } P \text{ homogeneous}\}$$

G. Fano considered the group of automorphisms of $\mathbb{P}^{n^2-1} = \mathrm{Proj}(\mathbb{C}[X^i_j])$ mapping the projective variety define by $\ker(\phi\restriction)^h$ into itself.

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Alas, the group is too big! So we slightly modify the setting.

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$V(\ker(\phi\restriction)^h) \cap D_{\det}$ is a ruled surface.

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$V(\ker(\phi\restriction)^h) \cap D_{\det}$ is a ruled surface.
- On the other hand $V(\ker(\phi\restriction))$ defines a curve in this surface.

Next time

We will see how the symmetries are going to be given by the automorphisms of the surface

$$V(\ker(\phi\restriction)^h) \cap D_{\det} \subseteq \mathrm{Spec}(\mathbb{C}[X^i_j]^G)$$

sending the curve $V(\ker(\phi\restriction))$ into itself.