Entrée to Malgrange ideas: General Involutivity Theorem

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Recall

Given a fiber bundle $\pi : E \to M$, we define the k-th order jet bundle $J^k\pi$ as the collection of equivalent classes of germs of sections of π, where two sections are identified if their image and all their derivatives, up to the k-th order, coincide. The class of $\phi \in \Gamma_p\pi$ is denoted by $j_p^k\phi$.
A local coordinate chart is given by the functions $(x^i, u^\alpha, u_I^\alpha)$, where:

- x^i are the coordinate functions of M.
- u^α are coordinate functions of the fibers of π.
- $I = (i_1, \ldots, i_n)$ is a multi-index, $|I| \leq k$, and $u_I^\alpha(j_p^k\phi) = \frac{\partial^{|I|}\phi^\alpha}{\partial x^I}$

Recall

- $\cdots \to J^k\pi \to \cdots \to J^1\pi \to E \to M$, and we define:
$$J\pi = \lim_{\leftarrow} J^k\pi$$
- $\mathscr{O}_M \to \mathscr{O}_E \to \mathscr{O}_{J^1\pi} \to \ldots \to \mathscr{O}_{J^k\pi} \to \ldots$, and we define:
$$\mathscr{O}_{J\pi} = \lim_{\rightarrow} \mathscr{O}_{J^k\pi}$$
- Locally, $\mathscr{O}_{J\pi}$ turns into a sheave of differential ring by setting
$$D_i F = \frac{\partial F}{\partial x^i} + \sum_{\alpha} u_i^{\alpha} \frac{\partial F}{\partial u^{\alpha}} + \sum_{\alpha, 1 \le |I| \le k} u_{I+\epsilon_i}^{\alpha} \frac{\partial^{|I|} F}{\partial u_I^{\alpha}}$$

Recall

With all this paraphernalia, we want to think about a system of differential equations as a coherent sheaf of ideals $\mathscr{I}$ on $\mathscr{O}_{J_\pi}$ that are locally differential ideals. The local solutions are given by differential morphisms of $\mathscr{O}_M$-algebras:

$$\Phi : \mathscr{O}_{J_\pi}/\mathscr{I} \longrightarrow \mathscr{O}_M(U)$$

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Arternatively, we can also think about a system of differential equations as a locally closed embedded sub-promanifold S of $J\pi$. We set

$$S^k = J^k\pi \cap S \qquad \mathscr{I}^k = \mathscr{O}_{J^k\pi} \cap \mathscr{I}$$

$\mathscr{I}^k$ and S^k are called k-th order differential equations.

Jet solutions

We turn our attention to the computational aspect of the differential equations. So we will not restrict our attention to an $\mathscr{I}^k$ and an S^k coming from objects in $\mathscr{O}_{J\pi}$ and $J\pi$. We simply start from elements defined on the k-th order jet.

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Not having a differential structure on $\mathscr{O}_{J^k\pi}$, we now consider k-th order jet solutions: these are simply (closed) points in S^k. Which one should think as a potential solution to the differential equation, in the sense that it is a k-th order Taylor expansion that may converge to a solution after extending it infinitely many times.

Extending a jet solution

Assume $F \in \mathscr{O}_E(U)[u_I^\alpha]$ ($U \subseteq E$ open) is zero on S^k, and

$$j_p^k \phi = (x^i, u^\alpha, u_I^\alpha)$$

is a k-th order jet solution (i.e. $F(j_p^k \phi) = 0$).

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is a k-th order jet solution (i.e. $F(j_p^k \phi) = 0$).If $\phi \in \Gamma(U)$ is a local solution of S^k, i.e. $j^k \phi(U) \subseteq S^k$ then, since F is constant (zero) in S^k,

$$D_i F(j_p^{k+1} \phi) = \frac{\partial}{\partial x^i} F \circ j^k \phi(p) = 0$$

Recall,

$$D_i F = \frac{\partial F}{\partial x^i} + \sum_\alpha u_i^\alpha \frac{\partial F}{\partial u^\alpha} + \sum_{\alpha, 1 \leq |I| \leq k} u_{I+\epsilon_i}^\alpha \frac{\partial^{|I|} F}{\partial u_I^\alpha}$$

Extending a jet solution

So a necessary condition for a k-th order jet solution $j_p^k\phi$ to converge to a local solution is that there is a point in $S^{k+1} = pr_1 S^k \in J^{k+1}\pi$ above $j_p^k\phi$. Where:

$$pr_1 S^k := \{\text{zeroes in } J^{k+1}\pi \text{ of } F \text{ and } D_i F | \; F \text{ vanishes in } S^k\}$$

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So a necessary condition for a k-th order jet solution $j^k_p\phi$ to converge to a local solution is that there is a point in $S^{k+1} = pr_1 S^k \in J^{k+1}\pi$ above $j^k_p\phi$. Where:

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In other words, we solve for $\{u^\alpha_{I+\epsilon_i}\}_{|I|=k, i\in\{1,\ldots,n\}}$ the system:

$$\sum_{\alpha,|I|=k} \frac{\partial^{|I|} F}{\partial u^\alpha_I}(x^i, u^\alpha, u^\alpha_I) u^\alpha_{I+\epsilon_i} = -(\frac{\partial F}{\partial x^i} + \sum_{\alpha, 0\leq |I| < k} u^\alpha_{I+\epsilon_i} \frac{\partial^{|I|} F}{\partial u^\alpha_I})(x^i, u^\alpha, u^\alpha_I)$$

Extending a jet solution

Set $\delta_j F = \delta_j F(x^i, u^\alpha, u_I^\alpha)$ as the linear form on the $\mathbb{C}$-vector space with coordinates $\{u^\alpha_{I+\epsilon_i}\}_{|I|=k, i\in\{1,\dots,n\}}$ defined by:

$$\delta_j F : (u^\alpha_{I+\epsilon_i}) \longmapsto \sum_{\alpha,|I|=k} \frac{\partial^{|I|} F}{\partial u_I^\alpha}(x^i, u^\alpha, u_I^\alpha) u^\alpha_{I+\epsilon_j}$$

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Set $v_j = v_j(x^i, u^\alpha, u_I^\alpha)$

$$v_j = (\frac{\partial F}{\partial x^j} + \sum_{\alpha, 0\le |I| < k} u_{I+\epsilon_j}^\alpha \frac{\partial^{|I|} F}{\partial u_I^\alpha})(x^i, u^\alpha, u_I^\alpha)$$

Extending a jet solution

So the question is now: is there a vector $(u^{\alpha}_{I+\epsilon_i})$ such that:

$$(\delta_j F(u^{\alpha}_{I+\epsilon_i})) = -(v_j)?$$

A necessary and sufficient is that the linear functionals that vanishes at the image of $(\delta_j F)$ vanishes at (v_j). In other words:

$$\sum_j \lambda_j \delta_j F = 0 \quad \Rightarrow \quad \sum_j \lambda_j v_j = 0$$

Extending a jet solution

$\sum_j \lambda_j \delta_j F = 0$ means:

- $\sum_j \lambda_j D_j F$ is constant in the fiber of $S^{k+1} \to S^k$ above $(x^i, u^\alpha, u^\alpha_I) = j^k_p \phi$

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- $\sum_j \lambda_j \delta_j F(x^i, u^\alpha, u_I^\alpha) \equiv 0$ implies that $\sum_j \lambda_j D_j F$ can be seen as a function over S^k.

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Introducing some notation we will interpret $\sum_j \lambda_j \delta_j F = 0$ as $\sum_j \lambda_j \delta_j F$ is a cycle and $\sum_j \lambda_j v_j$ will correspond to the value of a function at this cycle, vanishing at the borders.

Example: The curvature

We take M an open set of $\mathbb{C}^n$, and $\pi : E \to M$ an m-dimensional vector bundle over M. We set the connection defined by $(\nabla_{\frac{\partial}{\partial x^i}} u)^\alpha = \sum_\beta \Gamma^\alpha_{i\beta} u^\beta$:

$$D_i \begin{pmatrix} u^1 \\ \vdots \\ u^m \end{pmatrix} = \begin{pmatrix} u^1_i \\ \vdots \\ u^m_i \end{pmatrix} = A_i \begin{pmatrix} u^1 \\ \vdots \\ u^m \end{pmatrix}$$

where A_i is the matrix with $(A_i)^\alpha_\beta = \Gamma^\alpha_{i\beta}$ holomorphic over M. So S^1 is defined by the ideal $\mathscr{I}$ generated by the

$$F^\alpha_i(x^i, u^\alpha, u^\alpha_i) \quad = \quad u^\alpha_i - \sum_\beta \Gamma^\alpha_{i\beta}(x^1, \ldots, x^n) u^\beta$$

Example: The curvature

$$\begin{aligned}
F_i^\alpha(x^i, u^\alpha, u_i^\alpha) &= u_i^\alpha - \sum_\beta \Gamma_{i\beta}^\alpha(x^1, \ldots, x^n) u^\beta \\
\delta_j F_i^\alpha(x^i, u^\alpha, u_i^\alpha) &= u_{ij}^\alpha \\
v_{i,j}^\alpha(x^i, u^\alpha, u_i^\alpha) &= -\sum_\beta u^\beta \frac{\partial}{\partial x^j} \Gamma_{i\beta}^\alpha + \Gamma_{i\beta}^\alpha u_j^\beta \pmod{F_i^\alpha}
\end{aligned}$$

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&= -\sum_\beta (u^\beta \frac{\partial}{\partial x^j} \Gamma_{i\beta}^\alpha + \sum_\gamma \Gamma_{i\beta}^\alpha \Gamma_{j\gamma}^\beta u^\gamma) \pmod{F_i^\alpha}
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v_{i,j}^\alpha(x^i, u^\alpha, u_i^\alpha) &= -\sum_\beta u^\beta \frac{\partial}{\partial x^j}\Gamma_{i\beta}^\alpha + \Gamma_{i\beta}^\alpha u_j^\beta \pmod{F_i^\alpha} \\
&= -\sum_\beta (u^\beta \frac{\partial}{\partial x^j}\Gamma_{i\beta}^\alpha + \sum_\gamma \Gamma_{i\beta}^\alpha \Gamma_{j\gamma}^\beta u^\gamma) \pmod{F_i^\alpha} \\
&= -\sum_\gamma u^\gamma (\frac{\partial}{\partial x^j}\Gamma_{i\gamma}^\alpha + \sum_\beta \Gamma_{i\beta}^\alpha \Gamma_{j\gamma}^\beta) \pmod{F_i^\alpha}
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$$\delta_j F_i^\alpha - \delta_i F_j^\alpha = u_{ij}^\alpha - u_{ji}^\alpha = 0$$

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\begin{aligned}
\delta_j F_i^\alpha - \delta_i F_j^\alpha &= u_{ij}^\alpha - u_{ji}^\alpha = 0 \\
D_j F_i^\alpha - D_i F_j^\alpha \quad (\operatorname{mod} F_i^\alpha) &= v_{i,j}^\alpha - v_{j,i}^\alpha (\operatorname{mod} F_i^\alpha) \\
&= \sum_\gamma u^\gamma \left(\frac{\partial}{\partial x^j} \Gamma_{i\gamma}^\alpha - \frac{\partial}{\partial x^i} \Gamma_{j\gamma}^\alpha \right. \\
&+ \left. \sum_\beta \Gamma_{i\beta}^\alpha \Gamma_{j\gamma}^\beta - \Gamma_{j\beta}^\alpha \Gamma_{i\gamma}^\beta \right) \quad (\operatorname{mod} F_j^\alpha)
\end{aligned}
$$

In other words we can prolong the solution to the second order if:

$$
\frac{\partial}{\partial x^i} A_j - \frac{\partial}{\partial x^j} A^i - [A_i, A_j] = 0
$$

Another example

This time $\pi : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ is the first projection. And we take S^1 defined the ideal generated by:

$$F_1(x^i, u, u_i) = u_1, \quad F_2(x^i, u, u_i) = x^1 u_2 + x^2 u_3 + \ldots + x^{n-1} u_n$$

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so that:

$$\delta_j x^{j-1} F_1 = x^{j-1} u_{1,j}, \quad \delta_1 F_2 = x^1 u_{1,2} + x^2 u_{1,3} + \ldots + x^{n-1} u_{1,n}$$

Whence $\delta_1 F_2 - \sum_j \delta_j x^{j-1} F_1 = 0$;

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Whence $\delta_1 F_2 - \sum_j \delta_j x^{j-1} F_1 = 0$; but:

$$\begin{aligned} D_1 F_2 - \sum_j D_j x^{j-1} F_1 &= u_2 + x^1 u_{1,2} + \ldots + x^{n-1} u_{1,n} \\ &- (x^1 u_{1,2} + \ldots + x^{n-1} u_{1,n}) \\ &= u_2 \end{aligned}$$

Bibliography

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