$\mathcal{D}$-fields as a common formalism for difference and differential algebra

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The problem: informally

Many theories of rings with additional operators (for example of difference or differential rings) share common features, but in practice even the basic results and especially the more sophisticated structural results are proven independently.

Our goal is to produce a single, robust formalism in which these theories may be understood uniformly and precise comparisons may be made through the general language.

Examples

- Differential rings: (R, ∂) where $\partial(x+y) = \partial(x) + \partial(y)$ and $\partial(xy) = x\partial(y) + \partial(x)y$
- Difference rings: (R, σ) where $\sigma : R \to R$ is an endormorphism
- Partial difference-differential: $(R, \partial_1, \ldots, \partial_n, \sigma_1, \ldots, \sigma_m)$ where each ∂_i is a derivation and each σ_j is an endomorphism
- Hasse-Schmidt differential rings: $(R, \{\partial_n\}_{n=0}^{\infty})$ where $R \to R[[\epsilon]]$ given by $x \mapsto \sum_{n=0}^{\infty} \partial_n(x)\epsilon^n$ is a homomorphism.

Accidental non-examples

There are some other theories of operators which should be included, but because of technical restrictions are not yet covered by our general theorems.

- Hardouin's iterative q-difference operators
- Delon's λ-ring formalism: For K a field extending $\mathbb{F}_p[t]$ for which $t \notin K^p$ and $[K : K^p] = p$, define $\lambda_i : K \to K$ by $x = \sum_{i=0}^{n-1} \lambda_i(x)^p t^i$.

The goals, more precisely

Our goal is to find a common formalism of $\mathcal{D}$-rings so that the known theories of rings with operators may be obtained as specializations of $\mathcal{D}$-rings. in particular, the theory should achieve the following.

- There should be a good notion of $\mathcal{D}$-closed fields generalizing differentially closed fields (or, constrainedly closed fields) so that every $\mathcal{D}$-field embeds into a $\mathcal{D}$-closed field in which every consistent finite system of $\mathcal{D}$-equations and inequations has a solution.
- Part of the requirement for a good theory is that the class of $\mathcal{D}$-closed fields should be axiomatizable.
- There should be a good theory of $\mathcal{D}$-polynomial rings, $\mathcal{D}$-varieties, $\mathcal{D}$-schemes, *etc.*
- We should have a strong elimination theory so that the definable sets in $\mathcal{D}$-closed fields are easily described.

The goals, continued

- We should have uniform proofs of the refined classification theorems in the style of the Zilber trichotomy.
- Galois theory, at least in the sense of Picard-Vessiot and Kolchin's strongly normal differential Galois theory should be handled uniformly.
- Most intriguingly, explicit comparisons between theories of operators should be realized through formal specializations.

The difficulties

- There are real divergences between the theory of difference and differential algebra. For example, the theory of differentially closed fields of characteristic zero eliminates quantifiers and is totally transcendental. By contrast, the theory of a field with a generic automorphism does not eliminate quantifiers and is merely supersimple.
- The difference may be traced algebraically to the fact that for any differential field (K, ∂) of characteristic zero there is a unique structure of a differential field on the algebraic closure extending (K, ∂) but for a difference field (K, σ) there may be many non-isomorphic extensions of the difference field structure to K^{alg}.
- There are some natural theories of fields with operators (for example, fields with two or more commuting endomorphisms) for which it is known that the class of existentially closed structures is not axiomatizable.
- Differential algebra is already hard enough in positive characteristic. Could a general theory be any easier?

What will we do?

- While we have something to say about positive characteristic, for the most part we restrict to characteristic zero.
- We avoid the problems with pairs of commuting automorphisms by omitting compositional identities from our axioms.
- In a precise sense, the failure of quantifier elimination is always traceable to endomorphisms.
- Theories of $\mathcal{D}$-polynomial rings, $\mathcal{D}$-varieties, $\mathcal{D}$-schemes, $\mathcal{D}$-jets, *etc.* proceed smoothly.
- We prove the Zilber trichotomy for finite dimensional minimal types in complete generality.

$\mathcal{D}$-rings

Fix a commutative, unital ring A and an A-algebra $\mathcal{D}(A)$ which is finite and free as an A-module.

For an A-algebra R, we define $\mathcal{D}(R) := R \otimes_A \mathcal{D}(A)$. If we fix an isomorphism of A-modules $\psi : \mathcal{D}(A) \to A^n$, then ψ induces an isomorphism of R modules $\psi^R : \mathcal{D}(R) \to R^n$.

For the time being, we shall assume that there is a map $\pi : \mathcal{D}(A) \to A$ of A-algebras, which again induces a map $\pi^R : \mathcal{D}(R) \to R$. We presume that ψ is chosen so that in co-ordinates, π is given by $(x_0, \ldots, x_{n-1}) \mapsto x_0$.

A $\mathcal{D}$-ring is then an A-algebra R together with a map $E : R \to \mathcal{D}(R)$ for which $\pi^R \circ E = \mathrm{id}_R$.

$\mathcal{D}$-rings as rings with operators

If (R, E) is a $\mathcal{D}$-ring for $(A, \mathcal{D}, \psi)$, then identifying $\mathcal{D}(R)$ with R^n via ψ^R we may write $E(x) = (\partial_0(x), \partial_1(x), \ldots, \partial_{n-1}(x))$ where $\partial_i : R \to R$ is additive.

Conversely, given a sequence of maps $\partial_i : R \to R$ (for $0 \leq i < n$), the map $R \to \mathcal{D}(R)$ given by $x \mapsto (\psi^R)^{-1}(\partial_0(x), \ldots, \partial_{n-1}(x))$ gives R the structure of a $\mathcal{D}$-ring just in case

- $\partial_0(x) = x$
- $\partial_i(x + y) = \partial_i(x) + \partial_i(y)$
- $\partial_i(xy) = \sum c_{i,j,k} \partial_j(x) \partial_k(y)$ where $c_{i,j,k} \in A$ is defined by $e_j e_k = \sum c_{i,j,k} e_i$ for e_j the image under ψ^{-1} of j^{th} standard basis vector.

Examples

- $A = \mathbb{Z}$, $\mathcal{D}(A) = \mathbb{Z}[\epsilon]/(\epsilon^2)$, a $\mathcal{D}$-ring is a differential ring
- $A = \mathbb{Z}$, $\mathcal{D}(A) = \mathbb{Z} \times \mathbb{Z}$, a $\mathcal{D}$-ring is a difference ring
- $A = \mathbb{Z}[e]$, $\mathcal{D}(A) = \mathbb{Z}[e, \epsilon]/(\epsilon^2 - e\epsilon)$, a $\mathcal{D}$-ring is a $\mathbb{Z}[e]$-module R together with an additive map $D : R \to R$ satisfying $D(xy) = xDy + yDx + eDxDy$.
- $A = \mathbb{Z}$, $\mathcal{D}(A) = \mathbb{Z}[\epsilon]/(\epsilon^3)$, a $\mathcal{D}$-ring is a ring R with two additive maps ∂_1 and ∂_2 where ∂_1 is a derivation and $\partial_2(xy) = x\partial_2(y) + y\partial_2(x) + \partial_1(x)\partial_1(y)$.
- Not quite an example: $A = \mathbb{F}_p[t]$, $\mathcal{D}(A) = \mathbb{F}_p[\sqrt[p]{t}]$, then a $\mathcal{D}$-ring is an A-algebra R together with a sequence of maps $\lambda_i : R \to R$ so that the map $E : x \mapsto \sum_{i=0}^{p-1} \lambda_i(x) \otimes \sqrt[p]{t}^i$ is a homomorphism. The trouble here is that the natural map $\mathcal{D}(A) \to A$ is given by $x \mapsto x^p$ so that $\pi \circ E$ is the Frobenius, not the identity.

Further generalizations

- The association $R \mapsto \mathcal{D}(R)$ is a functor $\mathrm{Alg}_A \to \mathrm{Alg}_A$. It would make sense to consider more general (ie not necessarily representable) functors and to define $\mathcal{D}$-rings as A-algebras R together with maps $E : R \to \mathcal{D}(R)$.
- The requirement that $\pi \circ E = \mathrm{id}$ is strong and rules out some interesting examples. It makes sense to drop it, but our proof of the Zilber trichotomy would fail without it.
- We have implicitly taken E to be a map of A-algebras, but it might be better to fix $E_A : A \to \mathcal{D}(A)$ (which may differ from the standard A-algebra structure on $\mathcal{D}(A)$) and then ask for E to extend this second A-algebra structure.
- One should encode compositional identities through the theory of comonads, and we do exactly this in our paper on generalized Hasse-Schmidt differential rings.

First-order theory of $\mathcal{D}$-fields

Fix A, $\mathcal{D}$, and ψ. The theory of $\mathcal{D}$-fields is axiomatized in the language $\mathcal{L}(+, \cdot, -, \{a\}_{a \in A}, \{\partial_i\}_{i=0}^{n-1})$ by the natural formal sentences asserting about a model K that

- K is an A-algebra
- K is a field
- The map $E : x \mapsto (\partial_0(x), \ldots, \partial_{n-1}(x)) \in K^n \cong \mathcal{D}(K)$ makes K into a $\mathcal{D}$-ring

$\mathcal{D}$-closed fields

By a $\mathcal{D}$-equation over a $\mathcal{D}$-field (K, E) we mean an expression of the form $t(x_1, \ldots, x_n) = 0$ where t is a term in the language $\mathcal{L}(+, \cdot, -, \{a\}_{a \in K}, \{\partial_i\}_{i=0}^{n-1})$. (It would be better to think in terms of $\mathcal{D}$-polynomials or prolongations; these are coming.)

By a $\mathcal{D}$-closed field we mean a $\mathcal{D}$-field (K, E) such that for any finite system of $\mathcal{D}$-equations and $\mathcal{D}$-inequations Ξ, if there is a $\mathcal{D}$-field (L, E) extending (K, E) in which Ξ as a solution, then Ξ already has a solution in (K, E).

Proposition

If (K, E) is a $\mathcal{D}$-field, then there is a $\mathcal{D}$-closed field (L, E) extending (K, E).

Axiomatization of $\mathcal{D}$-closed fields

Theorem

The class of $\mathcal{D}$-closed fields of characteristic zero is axiomatized by a first-order theory $\mathcal{D}$-CF_0.

Theorem

If A is a field of characteristic $p > 0$ and there is some $\epsilon \in \mathcal{D}(A)$ with $\epsilon^p \neq 0 = \epsilon^{p+1}$, then the class of $\mathcal{D}$-closed fields is not axiomatizable.

What are the issues?

We need to present a method to determine whether a given system of $\mathcal{D}$-equations and inequations could have a solution in some extension $\mathcal{D}$-field. In particular, this method should require the checking of only finitely many conditions on the coefficients.

In the positive characteristic case, we prove that no axiomatization of the $\mathcal{D}$-closed fields exists by showing that in some cases the question of whether or not a certain element of the field has a p^{th} root is equivalent the satisfaction of a (properly) infinite system of equations.

Towards an axiomatization of $\mathcal{D}$-CF_0: separable (hence, algebraic) closedness

To start, a $\mathcal{D}$-closed field must be separably closed since for every $\mathcal{D}$-field (K, E), there is an extension of E to the separable closure of K making K^{sep} into a $\mathcal{D}$-field.

Why? The key point is that $\mathcal{D}(K)$ may be expressed as a finite product of local K-algebras. By a Hensel's Lemma argument, one sees that if B is a local artinian K-algebra, $f \in K[x]$ is a separable polynomial, and $E : K \to B$ is a map of rings, then there is a unique map $\tilde{E} : K[x]/(f) \to B \otimes_K K[x]/(f)$ extending E.

Towards an axiomatization of $\mathcal{D}$-CF_0: differential and difference fields as guides

Blum's axiomatization of (ordinary) differentially closed fields of characteristic zero simply requires that for each pair of one-variable differential polynomials f and g with $\mathrm{ord}(f) > \mathrm{ord}(g)$ there is a solution to $f(x) = 0$ and $g(x) \neq 0$.

For partial differential fields, an axiomatization in terms of differential ideals is possible, though far less simple. The original axiomatization for difference closed fields also passes through an analysis of difference ideals, but how to generalize this is not transparent.

The geometric axioms for the theory of difference closed fields (called ACFA) assert about the difference field (K, σ) that σ is an automorphism and that for any irreducible affine algebraic variety X and irreducible subvariety $Y \subseteq X \times X^\sigma$ which projects dominantly in both directions, there is a point $a \in X(K)$ with $(a, \sigma(a)) \in Y(K)$.

The geometric axioms for differential fields

Pierce and Pillay transposed the geometric axioms for difference closed fields to differential fields.

For (K, ∂) an ordinary differential field and X a variety over K, $\tau_\partial X \to X$ is the prolongation or ∂-twisted tangent bundle of X. In coordinates, if $X \subseteq \mathring{A}^n_K$ is defined by the ideal $I \subseteq K[x_1, \ldots, x_n]$, then $\tau_\partial X$ is defined by the ideal
$(\{f \ : \ f \in I\}, \{f^\partial + df_x \cdot \dot{x} \ : \ f \in I\}) \subseteq K[x_1, \ldots, x_n; \dot{x}_1, \ldots, \dot{x}_n]$.

Theorem

A differential field (K, ∂) of characteristic zero is differentially closed if and only if it is algebraically closed and for every irreducible affine algebraic variety X over K and every irreducible subvariety $Y \subseteq \tau_\partial X$ which projects dominantly to X, there is a point $a \in X(K)$ with $\nabla(a) := (a, \partial(a)) \in Y(K)$.

Axiomatization of $\mathcal{D}$-CF_0: geometric axioms

To generalize the geometric axiomatization we need:

- to construct $\mathcal{D}$-prolongation spaces, $\tau_{\mathcal{D}} X \to X$, in analogy to $\tau_{\partial} X \to X$ and $X \times X^{\sigma} \to X$,
- to identify the analogue of the condition that $\sigma : K \to K$ is an automorphism, and
- to deal with an annoying issue concerning variation in the number of prime ideals in $\mathcal{D}(K)$.

To sweep the last issue under the rug, from now on we will assume that A is a field and that for each prime ideal $\mathfrak{p} \subseteq \mathcal{D}(A)$ we have $A \cong \mathcal{D}(A)/\mathfrak{p}$. Some easy tricks permit such a reduction.

The other two issues are more fundamental.

$\mathcal{D}$-prolongation spaces

For a $\mathcal{D}$-ring R, there are two R-algebra structures on $\mathcal{D}(R)$, one coming from the standard R-algebra structure on $\mathcal{D}(R) = R \otimes_A \mathcal{D}(A)$ and the other coming from the map $E : R \to \mathcal{D}(R)$.

If X is a scheme over R, then via E we may form the base change $X \otimes_E \mathcal{D}(R)$ to a scheme over $\mathcal{D}(R)$. On the other hand, considering the corresponding functor of points, for Y a scheme over $\mathcal{D}(R)$, we have a restriction of scalars functor $\mathrm{R}(Y) : \mathrm{Alg}_R \to \mathrm{Set}$ defined by $\mathrm{R}(Y)(S) := Y(S \otimes_R \mathcal{D}(R))$ where in this case the tensor product is taken with respect to the standard structure on $\mathcal{D}(R)$. If $\mathrm{R}(Y)$ is represented by a scheme over R, then we write $\mathrm{R}(Y)$ for this scheme as well.

We define $\tau_{\mathcal{D}}(X) := \mathrm{R}(X \otimes_{R,E} \mathcal{D}(R))$ and the map $\pi : \tau_{\mathcal{D}}(X) \to X$ comes from the map of rings $\pi^R : \mathcal{D}(R) \to R$.

Examples

- If $\mathcal{D}(A) = A[\epsilon]/(\epsilon^2)$ and (K, ∂) is a differential field (equivalently, a $\mathcal{D}$-field), then for X a scheme over K, $\tau_{\mathcal{D}} X = \tau_{\partial} X$.
- If $\mathcal{D}(A) = A \times A$ and (K, σ) is a difference field, then for X a scheme over K, $\tau_{\mathcal{D}} X = X \times X^{\sigma}$.
- If $\mathcal{D}(A) = A[\epsilon]/(\epsilon^{n+1})$, (K, E) is a $\mathcal{D}$-field, and X is a scheme over K, then $\tau_{\mathcal{D}} X$ is a twisted version of the n^{th} arc bundle of X.

Whence the automorphisms

With our reduction to the case that A is a field and $\mathcal{D}(A)/\mathfrak{p} \cong A$ for each prime ideal $\mathfrak{p} \subseteq \mathcal{D}(A)$, because $\mathcal{D}(A)$ is artinian, we may express $\mathcal{D}(A) = \prod \mathcal{D}_j(A)$ where each $\mathcal{D}_j(A)$ is a local A-algebra with $\mathcal{D}_j(A)/\mathfrak{m} \cong A$. Let $\pi_j : \mathcal{D}(A) \to \mathcal{D}_j(A) \to \mathcal{D}_j(A)/\mathfrak{m} \cong A$ be the composite of the projection to the j^{th} factor of this product with the reduction map. (We may reorder so that $\pi = \pi_0$.)

For any $\mathcal{D}$-ring (R, E), the map $\sigma_i := \pi_i^R \circ E : R \to R$ is an endomorphism (and $\sigma_0 = \mathrm{id}_R$).

Axiomatization of $\mathcal{D}$-CF_0

A $\mathcal{D}$-field (K, E) is $\mathcal{D}$-closed if and only if

- K is algebraically closed,
- for each j, the map $\sigma_j : K \to K$ is an automorphism, and
- for each irreducible affine variety X over K and irreducible subvariety $Y \subseteq \tau_{\mathcal{D}} X$ for which each projection $Y \to X^{\sigma_j}$ is dominant, there is a point $a \in X(K)$ with $\nabla(a) = (\partial_0(a), \partial_1(a), \ldots, \partial_{n-1}(a)) \in Y(K)$.

Quantifier elimination and completions

- If $\mathcal{D}(A)$ is a local ring, then $\mathcal{D}$-CF_0 eliminates quantifiers and is a stable theory.
- In general, every formula is equivalent to one of the form $(\exists u)\phi(x_1, \ldots, x_n; u)$ where ϕ is quantifier-free and explicitly implies that u satisfies a non-zero polynomial equation with coefficients which are $\mathcal{D}$-polynomials in $x_1, \ldots, x_n$. Moreover, in general, $\mathcal{D}$-CF_0 is a simple theory (in the Kim-Pillay sense).
- The completions of $\mathcal{D}$-CF_0 are determined by the isomorphism type of the $\mathcal{D}$-ring structure on $\mathbb{Q}^{\mathrm{alg}}$.

Zilber trichotomy

The Zilber trichotomy holds for minimal types of finite dimension in $\mathcal{D}$-CF_0.

Here, for an extension of $\mathcal{D}$-fields $K \subseteq L$ and a finite tuple a from L, we define the dimension of a over K to be the transcendence degree over K of the $\mathcal{D}$-field generated by a.

What I mean by the Zilber trichotomy and minimal types is already well-known to you or should be the subject of a second talk.

The proof follows the Campana-Fujiki-Pillay-Ziegler jet space method, but finding the right definition of jet spaces for $\mathcal{D}$-varieties is is itself a non-trivial problem. One must first develop a theory of $\mathcal{D}$-varieties or $\mathcal{D}$-schemes. What would seem to be the natural definitions of jet spaces either through appropriate dual spaces or sequences of algebraic jet spaces fail for intrinsic reasons; one is infinite dimensional and the other cannot have points in a $\mathcal{D}$-field in any reasonable sense. Nevertheless, a sufficient theory of $\mathcal{D}$-jets may be developed to prove the Zilber trichotomy.

Questions

- Are there interesting, new instances of $\mathcal{D}$-fields? For example, if $\mathcal{D}(R) = R[\epsilon]/(\epsilon^3)$, then the theory $\mathcal{D}$-CF_0 is stable and has quantifier elimination. Are there interesting examples of $\mathcal{D}$-fields other than those which come from ordinary differential fields?
- If we impose compositional identities, say through our theory of iterative $\mathcal{D}$-rings, then the class of $\mathcal{D}$-closed fields need not be axiomatizable. On the other hand, it is exactly such iterativity which permits a good theory of Hasse-Schmidt differential fields of positive characteristic. What conditions on the iteration rules permit the axiomatization of the class of $\mathcal{D}$-closed fields?