Anomaly of Differential Dimension of Intersections

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1 Introduction

In [2, Chapter VII (p. 133)], Ritt shows an example of two irreducible differential variety of differential dimension 2 in $\mathcal{U}^3$ whose intersection consists of a single point: the origin. The details below follow Ritt's exposition using more modern language and notation. The first part is taken from my tutorial paper [3], with some change in notation. That paper provides the basic background needed. A generalization of Ritt's example to the partial case in $\mathcal{U}^n$ ($n \geqslant 3$), where an irreducible closed set in $\mathcal{U}^n$ of differential dimension $n-1$ intersects the hyperplane $y_n = 0$ in the single point $(0, \ldots, 0)$, is mentioned in Kolchin [1, p. 194, Ex. 3.].

In general, there is no easy way to show that $(0, 0, 0)$ is a solution of the general component of an irreducible differential polynomial in three variables, much less it is the only one. The first question is the (still unsolved) *Ritt Problem* and partial solutions involve deep results in differential algebra (for example, the Leading Coefficient Theorem, Prop. 6 on p. 191 and Theorem 4 on p. 172 of Kolchin [1], where the problem is treated with more generality). The second question, that (α, β, γ) is not a solution of the general component, is equally difficult and involves extending differentially the base field to include α, β, γ, translating the variables to $x-\alpha, y-\beta, z-\gamma$, and reducing it to the problem of showing that $(0, 0, 0)$ is not a solution. The only known results are Prop. 7 of p. 192 and Prop. 8 and some special cases, which may be traced to Levi's Lemma (p. 177) and the Domination Lemma (p. 181) in Kolchin's text. See also Levi's Lemma (p. 66) in Ritt [2].

2 The Component Theorems

We assume throughout that a prime differential ideal $\mathfrak{p}$ is given by its characteristic set $\mathbf{A}$ with respect to some orderly ranking. Then $\mathfrak{p} = [\mathbf{A}] : H^\infty$, where H is the product of initials and separants of elements of $\mathbf{A}$. In this section, we are mainly concerned with the components of a single differential polynomial A. We first look at the case when A is irreducible.

Theorem 2.1 (**General Component Theorem**) *Let A be a differential polynomial in $\mathcal{R} = \mathcal{F}\{ y_1, \cdots, y_n \}$. Suppose A is irreducible over $\mathcal{F}$, and let $\{ A \} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ be an irredundant decomposition of the radical differential ideal generated by A as the intersection of prime differential ideals $\mathfrak{p}_i$. Then among the components of $\{ A \}$, there is one, denoted by $\mathfrak{p}_{\mathcal{F}}(A)$, that does not contain any separant of A. Each of the other components of $\{ A \}$ contains every separant of A. Furthermore, for any separant S of A, $\mathfrak{p}_{\mathcal{F}}(A) = [A] \colon S^\infty = \{ A \} \colon S$ and an element of $\mathfrak{p}_{\mathcal{F}}(A)$ that is partially reduced with respect to A must be divisible by A.*

Definition 2.2 For any irreducible differential polynomial A, the component $\mathfrak{p}_{\mathfrak{F}}(A)$ is called the *general $\mathfrak{F}$-component* of A. All other components are called *singular components*. Geometrically, an element $\eta = (\eta_1, \ldots, \eta_n)$ where all η_i belong to some differential extension field of $\mathfrak{F}$ is a *solution* or *zero* of $A \in \mathfrak{R} = \mathfrak{F}\{\, y_1, \cdots, y_n \,\}$ if $A(\eta) = 0$. We say η is a *non-singular* solution if some separant (based on some ranking) of A does not vanish at η. Otherwise, we say η is a *singular* solution if every separant of A vanishes at η.

Example 2.3 The differential polynomial $A = (y')^2 - 4y$ is irreducible (here $y' = dy/dx$). Its only separant is $S_A = 2y'$. Differentiating A, we have $A' = 2y'y'' - 4y' = 2y'(y'' - 2)$. The prime decomposition is given by $\{\, A \,\} = \{\, A, y'' - 2 \,\} \cap \{\, A, y' \,\}$. Thus $\mathfrak{p}_1 = \{\, A, y'' - 2 \,\} = \{\, A \,\} : y'$ is the general component $\mathfrak{p}_{\mathfrak{F}}(A)$. The other (singular) component is $\mathfrak{p}_2 = \{\, A, y' \,\} = [y]$ which is the general component of $B = y$. It is clear that $\mathfrak{p}_{\mathfrak{F}}(A) \not\subseteq [y]$ and thus the decomposition is irredundant.

For an arbitrary differential polynomial, the next theorem ([1, Theorem 5, p. 185]) gives the structure of the decomposition, relating the components to the general components of irreducible differential polynomials. The theorem is well illustrated by Example 2.3 above.

Theorem 2.4 (**Component Theorem**) *Let F be a differential polynomial in $\mathfrak{R} = \mathfrak{F}\{\, y_1, \cdots, y_n \,\}$, $F \notin \mathfrak{F}$ and not necessarily irreducible. Then for any component $\mathfrak{p}$ of $\{\, F \,\}$ in $\mathfrak{R}$, there exists an irreducible differential polynomial $A \in \mathfrak{R}$ such that $\mathfrak{p} = \mathfrak{p}_{\mathfrak{F}}(A)$. If moreover F is irreducible, and if $\mathfrak{p} = \mathfrak{p}_{\mathfrak{F}}(A)$ is a singular component of $\{\, F \,\}$, then F involves a proper derivative of the leader of A relative to any ranking, and* $\operatorname{ord} A < \operatorname{ord} F$.

Example 2.5 The ordinary differential polynomial $A = (y')^2 - y^3$ is irreducible. The functions $\eta_c = (x + c)^{-2}/4$, for an arbitrary constant c, give a one-parameter family of non-singular solutions. Since the only separant is $S_A = 2y'$, the only singular solution is $\eta = 0$, which is also $\lim_{c \to \infty} (x + c)^{-2}/4$. Thus the singular solution is geometrically considered as "embedded in the general solution" (see accompanying figure). Algebraically, we have

$$A' = 2y'y'' - 3y^2y' = 2(y'' - \frac{3}{2}y^2)y'$$

and

$$\{\, A \,\} = \{\, A, y'' - \frac{3}{2}y^2 \,\} \cap [y] = \{\, A, y'' - \frac{3}{2}y^2 \,\}.$$

Hence $\{\, A \,\} : y'$ is the general and only component and

$$\mathfrak{p}_{\mathfrak{F}}(A) = \{\, A \,\} : y' \subset [y].$$

3 Preparation Equation and Congruence

We begin by describing the construction of a *preparation equation.* In the algebraic case, this is an equation involving a polynomial F, a subset $\mathbf{G}\colon G_1, \dots, G_r$ of polynomials triangular with respect to $u_1, \dots, u_r$ over some integral domain $\mathfrak{S}_0$, and a product L of corresponding initials. A preparation equation is very similar to the pseudo-remainder equation except that, instead of expressing LF as a *linear* combination of $G_1, \dots, G_r$ with coefficients $Q_1, \dots, Q_r$ plus a remainder R, in a preparation equation, we now express LF as a polynomial P in $G_1, \dots, G_r$. By carefully controlling the process to compute a preparation equation, we can ensure that the non-zero coefficients of the polynomial P have degrees in u_i strictly lower than $g_i = \deg_{u_i} G_i$. One may naively think this can be done by first applying the pseudo-division algorithm to obtain $Q_1, \dots, Q_r$ and R, and apply induction to the coefficients $Q_1, \dots, Q_r$. This idea does not work without modification, since in order to obtain the final relation, we must multiply, for example, R, by some product of initials, which may increase the degrees in the variables $u_1, \dots, u_r$ of the coefficients of the monomials in $G_1, \dots, G_r$ appearing so far.

The algorithm for polynomials in case $r = 1$, which is the most important case, is given and proved in [3, Prop.12.1]. The general algorithm for polynomials to arbitrary r comes as a corollary.

In the differential case, a preparation equation for a differential polynomial F will involve an autoreduced set $\mathbf{A}$ and possibly proper derivatives of its elements, the collection of which forms $G_1, \dots, G_r$. Unfortunately, the corollary cannot be applied to the preparation process for differential polynomials, for the simple reason that we do not have *a priori* a set of inputs $G_1, \dots, G_r$ since the choice of the derivatives of $\mathbf{A}$ depends on F and $\mathbf{A}$. The process is similar to the Ritt-Kolchin algorithm to obtain a remainder of F. Whereas the Ritt-Kolchin algorithm completes all partial reductions involving proper derivatives of elements of $\mathbf{A}$ (differential reduction) before pseudo-reduction involving only elements of $\mathbf{A}$ (algebraic reduction), we now mix the two so that an algebraic reduction may be performed under certain circumstances even though F may not be partially reduced yet. In subsequent reductions, we have to be careful not to increase the degrees of $v_1, \dots, v_k$ in the partial results. We continue this process to obtain a polynomial P in the elements of $A \in \mathbf{A}$ and their derivatives, stopping only when all the coefficients in P are reduced. For a rigorous treatment, the reader is referred to [3, Section 12]. Here we state main result only.

Let $\mathbf{A}\colon A_1 < \cdots < A_p$ be an autoreduced subset of $\mathfrak{R} = \mathfrak{F}\{\, y_1, \cdots, y_n \,\}$ with leaders $v_1, \dots, v_p$. Let $Z = (z_1, \dots, z_p)$ be p differential indeterminates. Let ΘZ be the set of derivatives in Z. Let $E(\mathbf{A})$ be the set of derivatives u of the form θv_k with $\theta \in \Theta$, $1 \le k \le p$.

Definition 3.1 A *choice function* for $\mathbf{A}$ is a map $c\colon E(\mathbf{A}) \longrightarrow \Theta Z$ such that if we denote $c(u)$ by $\theta_u z_{k(u)}$, then $\theta_u v_{k(u)}$, the leader of $\theta_u A_{k(u)}$, is u.

A choice function preselects a $\theta_u \in \Theta$ and an $A_{k(u)} \in \mathbf{A}$ to represent a derivative u which is a derivative of the leader of some $A \in \mathbf{A}$. Our notation differs slightly from Kolchin [1], who uses the pair $(\theta_u, k(u))$ to denote $c(u)$. When $m = 1$ (ordinary case) or when $\mathbf{A}$ is a singleton, there exists only one choice function since the leaders of elements of $\mathbf{A}$ can have no common derivatives. When $m > 1$ and $\mathbf{A}$ consists of more than one element, the representation is quite arbitrary since there is no compatibility condition required between the choice function and differentiation.

Proposition 3.2 *Let $\mathbf{A}\colon A_1 < \cdots < A_p$ be a characteristic set of a prime differential ideal $\mathfrak{p}$ of $\mathfrak{R} = \mathfrak{F}\{ y_1, \cdots, y_n \}$, let $v_1, \ldots, v_p$ be the leaders of $\mathbf{A}$, let $Z = (z_1, \ldots, z_p)$ be differential indeterminates, let $u \mapsto (\theta_u, z_{k(u)})$, $u \in E(\mathbf{A})$, be a choice function for $\mathbf{A}$, and let $F \in \mathfrak{R}$. Then we can find an equation of the form*

$$LF = \sum_{j=1}^{s} C_j M_j(A_1, \cdots, A_p), \tag{1}$$

where L is a product of initials and separants, $C_1, \cdots, C_s \notin \mathfrak{p}$, and $M_1, \ldots, M_s$ are distinct differential monomials in Z with the property that every factor θz_k of an M_j has the form $\theta_u z_{k(u)}$ for some $u \in E(\mathbf{A})$.

Definition 3.3 When $\mathbf{A}$ is a characteristic set of a prime differential ideal $\mathfrak{p}$, an equation of the form Equation (1), where the coefficients $C_1, \ldots, C_s$ are not in $\mathfrak{p}$ (but not necessarily reduced) is called a *preparation equation* of F with respect to $\mathbf{A}$ (and the choice function c).

The proof is constructive. We will refer to the algorithm as the *Ritt-Kolchin*[1] *preparation algorithm* and the preparation equation thus obtained as the *Ritt-Kolchin preparation equation*. We summarize the properties of the Ritt-Kolchin preparation equation below:

- $C_1, \ldots, C_s \in \mathfrak{R}$ are (non-zero and) reduced with respect to $\mathbf{A}$.
- If θz_k and $\theta' z_{k'}$ are two distinct factors present in at least one M_j, then $\theta v_k \neq \theta' v_{k'}$.
- If $F \notin \mathfrak{F}$, and u_F is its leader, then $\theta v_k \leq u_F$ for every factor θz_k of some M_j.
- The rank of C_j is less than or equal to the rank of F for $1 \leq j \leq s$, and
- If $M_i = 1$ is among the monomials $M_1, \ldots, M_s$, then C_i is a remainder of F.

If q is the lowest degree of the differential monomials $N_1, \ldots, N_\ell$ among $M_1, \ldots, M_s$, we may rewrite (1) in the form of a congruence (with a change of notation)

$$LF \equiv \sum_{\lambda=1}^{\ell} D_\lambda N_\lambda(A_1, \cdots, A_p) \qquad (\mathrm{mod}\, [A_1, \cdots, A_p]^{q+1}). \tag{2}$$

[1] The author has not done a historical research to justify this credit. This attribution is only a matter of convenience.

Definition 3.4 We call (2) a *preparation congruence* of F with respect to $\mathbf{A}$ and c.

Give $F \in \mathfrak{R}$ and a prime differential ideal $\mathfrak{p} \subset \mathfrak{R}$, a preparation congruence, just like a preparation equation, is not unique and depends on the ranking, the characteristic set, and the choice function. By a result of Hillman [1, Lemma 7, p. 184)], the lowest integer q and the set of differential monomials $N_1, \ldots, N_\ell$ of degree q, are unique and independent of the ranking, the characteristic set, the choice function, and the preparation congruence.

Remark 3.5 When there is only one differential polynomial $A \in \mathbf{A}$ which is linear in its leader v, there is an efficient method to obtain a preparation equation: write $A = I_A v + B$ and simply substitute the leader v if it appears in F by $(A - B)/I_A$ and clear denominators.

Example 3.6 Ritt, Part I. Let $\mathfrak{R} = \mathfrak{F}\{\, x, y, z \,\}$ be an ordinary differential polynomial ring with differential indeterminates x, y, z over an ordinary differential field $\mathfrak{F}$ with derivation δ. Assume an orderly ranking with $x > y > z$. Let

$$F = x^5 - y^5 + z(x\delta y - y\delta x)^2.$$

Let ζ be a fifth root of unity. For $1 \leqslant j \leqslant 5$, let $A_j = x - \zeta^{j-1} y$. For every j, A_j is irreducible (as is F), the leader of A_j is x, and the leader of F is δx. The general component of F is the prime differential ideal $[F] : S^\infty$, where S is any separant of F, for example:

$$S = \frac{\partial F}{\partial \delta x} = -2zy(x\delta y - y\delta x).$$

Now differentiate A_j to get $\delta A_j = \delta x - \zeta^{j-1}\delta y$, use this and the definition of A_j to substitute for x and δx in $x\delta y - y\delta x$, and get $x\delta y - y\delta x = \delta y\, A_j - y\, \delta A_j$. Let B_j be defined by $A_j B_j = x^5 - y^5$. Then

$$F = B_j A_j + z(\delta y\, A_j - y\, \delta A_j)^2 \tag{3}$$

and, (respectively,

$$F \equiv B_j A_j \pmod{[A_j]^2} \tag{4}$$

) is a preparation equation (respectively, congruence) with respect to the autoreduced set consisting of A_j alone (the reader should verify that none of the "coefficients" B_j, $z(\delta y)^2$, $zy\delta y$, and zy^2 in the preparation equation belong to $[A_j]$).

It follows that $F, S \in [A_j]$. Then

$$\{F\} = [F] : S^\infty \cap \{F, S\} = \mathfrak{p}_{\mathfrak{F}}(F) \cap [A_1] \cap \cdots \cap [A_5].$$

Since $S \notin \mathfrak{p}_{\mathfrak{F}}(F)$, these prime differential ideals are all distinct and the last five are irredundant. But what about the containment relationship between $\mathfrak{p}_{\mathfrak{F}}(F)$ and $[A_j]$?

4 Low Power Theorem

To decide whether these are the components as depicted by the Component Theorem, we need to use a preparation congruence and apply the Low Power Theorem.

In the case when $\mathbf{A}$ consists of a single irreducible differential polynomial A, the choice function is unique. We state below only this simplified version of the Low Power Theorem for general m (partial case) and n (arbitrary number of differential indeterminates), followed by the most commonly illustrated version when $n = 1$ and $A = y \in \mathfrak{F}\{\, y \,\}$.

Theorem 4.1 (**Low Power Theorem**) *Let A and F be differential polynomials in $\mathfrak{R} = \mathfrak{F}\{\, y_1, \cdots, y_n \,\}$, with A irreducible and $F \neq 0$. Let*

$$LF \equiv \sum_{\lambda=1}^{\ell} D_\lambda N_\lambda(A) \qquad (\mathrm{mod}\, [A]^{q+1})$$

be a preparation congruence of F with respect to A. A necessary and sufficient condition that $\mathfrak{p}_{\mathfrak{F}}(A)$ be a component of $\{\, F \,\}$ is that $q \neq 0, \ell = 1$, and $N_1 = z^q$.

Corollary 4.2 *A necessary and sufficient condition for $[y]$ to be a component of a differential polynomial $F \in \mathfrak{F}\{\, y \,\}$ is that F contains a term in y alone which is of lower degree than any other terms of F.*

Example 4.3 Ritt, Part II. By the Low Power Theorem, using the preparation congruence (4), every $[A_j]$ is a (singular) component of F. These together with the general component of F, shows that $\{\, F \,\}$ has 6 components.

5 Levi's Lemma

Levi's lemma ([1, Lemma 3, p. 177], [2, Lemma, p. 66]) is a technical lemma on differential elimination. It is used to prove the Low Power Theorem. It is intimately related to preparation congruence.

Lemma 5.1 *Let $r \geqslant 1$ and let r non-negative integers $g_1, \ldots, g_r$ be given. Let*

$$\Gamma = \{z_1, \ldots, z_r, (u_{\rho\gamma})_{1 \leqslant \rho \leqslant r, 0 \leqslant \gamma \leqslant g_\rho})\}$$

be a set of differential indeterminates and let Σ be the differential polynomial ring $\mathbb{Q}\{\Gamma\}$. Let $G_1, \ldots, G_r \in \Sigma$ have the form

$$G_\rho = u_{\rho 0} z_\rho^{q_\rho} + \sum_{\gamma=1}^{g_\rho} u_{\rho\gamma} M_{\rho\gamma} \qquad (1 \leqslant \rho \leqslant r),$$

where (for each ρ), $q_\rho \in \mathbb{N}$ and $M_{\rho 1}, \ldots, M_{\rho g_\rho}$ are differential monomials in $z_1, \ldots, z_r$ of degree greater than q_ρ. Then there exist a monomial

$$U = \prod_{\rho=1}^{r} u_{\rho 0}^{d_\rho}$$

and a differential polynomial $Z \in \Sigma$ with $Z \in [z_1, \ldots, z_r]$ and with Z homogeneous in $(\theta u_{\rho\gamma})_{\theta\in\Theta, 0\leqslant\gamma\leqslant g_\rho}$ of degree d_ρ $(1 \leqslant \rho \leqslant r)$ and with the degree of Z in $(\theta u_{\rho 0})_{\theta\in\Theta, 1\leqslant\rho\leqslant r}$ strictly smaller than $d_1 + \cdots d_r$, such that

$$z_\rho(U + Z) \in \{G_1, \ldots, G_r\} \qquad (1 \leqslant \rho \leqslant r).$$

Example 5.2 Ritt, Part III. We now show that the variety V defined by the general component $\mathfrak{p}_{\mathcal{F}}(F)$ intersects the hyperplane $z = 0$ at precisely the origin $(0, 0, 0)$.

We first show that $(0, 0, 0) \in V$. Let $G, H \in \mathcal{F}[X_1, \ldots, X_5]$ be *polynomials* defined by

$$G = X_1^5 - X_2^5 + X_3(X_1X_4 - X_2X_5)^2, \qquad H = X_1^5 - X_2^5.$$

Let $\mathcal{U}$ be a differential extension of $\mathcal{F}$ for which there exists a differential zero $(\alpha, \beta, \gamma) \in \mathcal{U}^3$ of F such that $S(\alpha, \beta, \gamma) \neq 0$. For example, we may take $\mathcal{U}$ to be a universal extension over $\mathcal{F}$. Then $G(\alpha, \beta, \gamma, \beta', \alpha') = 0$. Note that $S(\alpha, \beta, \gamma) \neq 0$ if and only if $H(\alpha, \beta) \neq 0$. Let $\mathcal{G} = \mathcal{F}\langle\alpha, \beta, \gamma\rangle$. Let $\lambda \in \mathcal{G}$ be any non-zero constant. We have $G(\lambda\alpha, \lambda\beta, \lambda\gamma, \lambda\beta', \lambda\alpha') = 0$ and $H(\lambda\alpha, \lambda\beta) \neq 0$ by homogeneity. So $F(\lambda\alpha, \lambda\beta, \lambda\gamma) = 0$ and $S(\lambda\alpha, \lambda\beta, \lambda\gamma) \neq 0$. For any differential polynomial $T \in \mathfrak{p}_{\mathcal{F}}(F) \subset \mathcal{F}\{x, y, z\}$, we have $S^eT \in [F]$ and hence $T(\lambda\alpha, \lambda\beta, \lambda\gamma) = 0$. It follows that the univariate polynomial $T(X\alpha, X\beta, X\gamma) \in \mathcal{G}[X]$, where X is a *constant* (algebraic) indeterminate over $\mathcal{G}$, must be identically zero for every $T \in \mathfrak{p}_{\mathcal{F}}(F)$ and hence $(\lambda\alpha, \lambda\beta, \lambda\gamma) \in V$ for all constants λ and in particular, $(0, 0, 0) \in V$.

Now let $(\alpha, \beta, 0) \in V$ and we will show that $\alpha = \beta = 0$. Since $F(\alpha, \beta, 0) = 0$ and $S(\alpha, \beta, 0) = 0$, we have $G(\alpha, \beta, 0, \beta', \alpha') = 0$ and $H(\alpha, \beta) = 0$. Thus for some i, $(1 \leqslant i \leqslant 5)$, $A_i(\alpha, \beta) = 0$.

Referring to Eq. (3), let $\mathcal{R}' = \mathbb{Q}\{z_1, u_{1,0}, u_{1,1}, u_{1,2}, u_{1,3}\}$ be a differential polynomial ring and consider the differential polynomial $G_1 \in \mathcal{R}'$ defined by:

$$G_1 = u_{1,0}z_1 + u_{1,1}z_1^2 + u_{1,2}z_1\delta z_1 + u_{1,3}(\delta z_1)^2. \tag{5}$$

By Levi's Lemma (Kolchin, Lemma 3, p. 177; Ritt, Lemma, p. 66), there exists a power $u_{1,0}^{d_1}$, where $d_1 \geqslant 1$, and a differential polynomial $Z \in \mathcal{R}'$ with $Z \in [z_1]$ where Z is homogeneous of degree d_1 in $u_{1,1}, u_{1,2}, u_{1,3}$ and their derivatives while the degree of Z in $u_{1,0}$ and its derivatives is strictly smaller than d_1, such that

$$z_1(u_{1,0}^{d_1} + Z) \in \{G_1\}. \tag{6}$$

Substituting into Eq. (6)

$$z_1 \to A_i, \quad u_{1,0} \to B_i, \quad u_{1,1} \to z(\delta y)^2, \quad u_{1,2} \to -2zy\delta y, \quad u_{1,3} \to zy^2$$

we obtain

$$A_i \cdot \big(B_i^{d_1} + Z(A_i, B_i, z(\delta y)^2, -2zy\delta y, zy^2)\big) \in \{F\} \subset \mathfrak{p}_{\mathcal{F}}(F).$$

Letting

$$T = B_i^{d_1} + Z\big(A_i, B_i, z(\delta y)^2, -2zy\delta y, zy^2\big)$$

we obtain $A_i T \in \{F\} \subset \mathfrak{p}_{\mathcal{F}}(F)$. Since $A_i \notin \mathfrak{p}_{\mathcal{F}}(F)$, it follows that $T \in \mathfrak{p}_{\mathcal{F}}(F)$. By the properties of Z, $Z \in [z_1]$ and so $T \equiv B_i^{d_1} \bmod [A_i]$. Since $(\alpha, \beta, 0)$ is a zero of $\mathfrak{p}_{\mathcal{F}}(F)$ and of $[A_i]$, $T(\alpha, \beta, 0) = B_i(\alpha, \beta)^{d_1} = 0$. Now $d_1 \geqslant 1$ implies that for some $j \neq i$, $A_j(\alpha, \beta) = 0$ and hence $\alpha = \beta = 0$.

Remark 5.3 Replacing F with a differential polynomial of the same form but of lower degree would not work, say for $F_3 = x^3 - y^3 + z(x\delta y - y\delta x)$. The separants of F_3 are $S_1 = -yz, S_2 = zx$, and $S_3 = x\delta y - y\delta x$. Let $A_j = x - \omega^{j-1}y$ where ω is a primitive third root of unity. We have

$$\{F_3\} = \mathfrak{p}_{\mathcal{F}}(F_3) \cap \{F_3, S_3\} = \mathfrak{p}_{\mathcal{F}}(F_3) \cap [A_1] \cap [A_2] \cap [A_3].$$

If $[A_j]$ were a singular component of $\{F_3\}$, the separants would have to be in $[A_j]$ by the Component Theorems. However, when we write $F_3 = B_j A_j + z(y\delta A_j - (\delta y)A_j)$, this preparation equation (and congruence) would have three terms of lowest degree, namely degree one, in A_j and its derivatives, and hence by the Low Power Theorem, $[A_j]$ is not a component of $\{F_3\}$. Since the preparation equations are also not in the form suitable to apply Levi's Lemma, the proof that $(0, 0, 0)$ is the *only* zero on the intersection of F_3 and $z = 0$ would not work.

Remark 5.4 In Kolchin's generalization [1, Ex. 3, p. 194], the polynomial F (Kolchin used A for F) need not be homogeneous, but the degree of the product of linear terms, here represented as $\prod_{j=1}^{5} A_j$) must be strictly larger than twice that of the coefficient of z (here represented as $(x\delta y - y\delta x)^2$. The generalization yields an example of an irreducible closed set in $\boldsymbol{\mathcal{U}}^n$ of differential dimension $n-1$ that intersects the hyperplane $y_n = 0$ at the single point, the origin.

References

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