Differential Algebraic Subgroups of $SL(2)$

William Sit, City College of New York

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Setting

- Let $\mathfrak{F}$ be a (partial differential) field of characteristic zero, with a set $\Delta = \{ \delta_1, \dots, \delta_m \}$ of derivations and let $\mathfrak{C}$ be the field of constants of $\mathfrak{F}$.
- Let $\mathfrak{U}$ be a universal differential extension of $\mathfrak{F}$ and let $\mathfrak{K}$ be the field of constants of $\mathfrak{U}$.
- $\mathfrak{K}$ and $\mathfrak{F}$ are linearly disjoint over $\mathfrak{C}$.
- Let $\mathfrak{G}$ be an extension of $\mathfrak{F}$, over which $\mathfrak{U}$ is still universal and let $\mathfrak{D}$ be its field of constants.
- An isomorphism of $\mathfrak{G}$ is a differential field isomorphism from $\mathfrak{G}$ onto its image $\mathfrak{G}^\sigma$ in $\mathfrak{U}$.
- The compositum of two Δ-subfields of $\mathfrak{U}$ is denoted by concatenation.

Specialization of Isomorphisms

- We say an isomorphism σ of $\mathfrak{G}$ specializes to another isomorphism σ' of $\mathfrak{G}$ if for every finite family $(\alpha_1, \ldots, \alpha_k) \in \mathfrak{G}^k$, $(\sigma\alpha_1, \ldots, \sigma\alpha_k)$ specializes (differentially) over $\mathfrak{G}$ to $(\sigma'\alpha_1, \ldots, \sigma'\alpha_k)$.
- Specialization $\sigma \to \sigma'$ is a reflexive and transitive relation.
- Generic specialization: $\sigma \leftrightarrow \sigma'$
- σ is isolated over $\mathfrak{F}$ if every specialization from any isomorphism to σ is generic.

Algebro-geometric Structures of Isomorphisms

- Let $\mathfrak{G}$ be a finitely generated extension of $\mathfrak{F}$, and let $\mathfrak{G} = \mathfrak{F}\langle \eta_1, \ldots, \eta_n \rangle$.
- σ specializes to σ' if and only if $(\sigma\eta_1, \ldots, \sigma\eta_n)$ (differentially) specializes to $(\sigma'\eta_1, \ldots, \sigma'\eta_n)$ over $\mathfrak{G}$.
- There exist finitely many isolated isomorphisms $\sigma_1, \ldots, \sigma_k$ of $\mathfrak{G}$ over $\mathfrak{F}$ such that every isomorphism of $\mathfrak{G}$ over $\mathfrak{F}$ is a specialization of one and only one of these. The number k is unique.
- The differential field of invariants of $\sigma_1, \ldots, \sigma_k$ is $\mathfrak{F}$.
- If $\mathfrak{F}^\circ$ is the algebraic closure of $\mathfrak{F}$ in $\mathfrak{G}$, then the differential field of invariants of the component of the identity (the specializations of say σ_1, where σ_1 specializes to the identity) is $\mathfrak{F}^\circ$.

Strong Isomorphisms

- Let σ be an isomorphism of $\mathcal{G}$ over $\mathfrak{D}$, and let $\mathfrak{D}_\sigma$ be the field of constants of $\mathcal{G}\mathcal{G}^\sigma$. The following conditions are equivalent:
  1. $\mathcal{G}^\sigma \subset \mathcal{G}\mathcal{K}$ and $\mathcal{G} \subset \mathcal{G}^\sigma\mathcal{K}$
  2. $\mathcal{G}\mathcal{K} = \mathcal{G}^\sigma\mathcal{K}$
  3. $\mathcal{G}\mathfrak{D}_\sigma = \mathcal{G}\mathcal{G}^\sigma = \mathcal{G}^\sigma\mathfrak{D}_\sigma$
- An isomorphism σ satisfying the conditions is said to be strong (e.g. automorphisms). Let $\mathsf{SI}(\mathcal{G})$ = set of strong isoms.
- σ strong implies $\operatorname{tr}\deg \mathcal{G}\mathcal{G}^\sigma/\mathcal{G} = \operatorname{tr}\deg \mathfrak{D}_\sigma/\mathfrak{D}$.
- $\mathsf{SI}(\mathcal{G}) \leftrightarrow \mathsf{Aut}(\mathcal{G}\mathcal{K}/\mathcal{K})$.
- Every specialization of a strong isomorphism is strong.

Strongly Normal Extensions

- A strongly normal extension of $\mathfrak{F}$ is a Δ-finitely generated extension $\mathfrak{G}$ for which every isomorphism of $\mathfrak{G}$ over $\mathfrak{F}$ is strong.
- If $\mathfrak{G}$ is finitely generated over $\mathfrak{F}$ with $\mathfrak{D} = \mathfrak{C}$, and if $\mathfrak{G}^{\sigma_i} \subset \mathfrak{G}\mathfrak{K}$ for $1 \leqslant i \leqslant k$, then $\mathfrak{G}$ is strongly normal over $\mathfrak{F}$.
- Let $\mathfrak{G}$ be a strongly normal extension of $\mathfrak{F}$.
- $\mathfrak{D} = \mathfrak{C}$.
- $\mathfrak{G}$ is finitely generated as a field over $\mathfrak{F}$, and $\mathfrak{D}_\sigma = \mathfrak{C}_\sigma$ is finitely generated over $\mathfrak{C}$ for every isomorphism σ over $\mathfrak{F}$.
- The set $\mathrm{SI}(\mathfrak{G}/\mathfrak{F})$ of strong isomorphisms of $\mathfrak{G}$ over $\mathfrak{F}$, when identified as $\mathrm{Aut}(\mathfrak{G}\mathfrak{K}/\mathfrak{F}\mathfrak{K})$ has the structure of a $\mathfrak{C}$-group, now denoted by $G(\mathfrak{G}/\mathfrak{F})$.
- The field associated to a "point" σ is $\mathfrak{C}_\sigma$.

Examples: Primitives and Exponentials

- (primitive) Let $\mathfrak{G} = \mathfrak{F}\langle\alpha\rangle$ where $\delta\alpha \in \mathfrak{F}$ for all $\delta \in \Delta$. If $\mathfrak{D} = \mathfrak{C}$, then $\mathfrak{G}$ is strongly normal over $\mathfrak{F}$. We have $\delta(\sigma\alpha) = \sigma(\delta\alpha) = \delta\alpha$ and hence $c(\sigma) = \sigma\alpha - \alpha \in \mathfrak{K}$ and σ is strong, with $\mathfrak{C}_\sigma = \mathfrak{C}(c(\sigma))$ and $c : G(\mathfrak{G}/\mathfrak{F}) \to \mathfrak{K}$ is an injective $\mathfrak{C}$-homomorphism.
- (exponential) Let $\mathfrak{G} = \mathfrak{F}\langle\alpha\rangle$ where $\ell\Delta(\alpha) \in \mathfrak{F}^m$. If $\mathfrak{D} = \mathfrak{C}$, then $\mathfrak{G}$ is a strongly normal extension of $\mathfrak{F}$. We have

$$(\sigma\alpha)^{-1}\delta(\sigma\alpha) = \sigma(\alpha^{-1}\delta\alpha) = \alpha^{-1}\delta\alpha$$

and hence $c(\sigma) = \alpha^{-1}\sigma\alpha \in \mathfrak{K}^*$ and σ is strong, with $\mathfrak{C}_\sigma = \mathfrak{C}(c(\sigma))$ and $c : G(\mathfrak{G}/\mathfrak{F}) \to \mathfrak{K}^*$ is an injective $\mathfrak{C}$-homomorphism.

Examples: Picard-Vessiot and Weierstraussians

- Let $\alpha \in GL(n)$ be a matrix such that $\ell\Delta\alpha = (\Delta\alpha)\alpha^{-1}$ has entries in $\mathfrak{F}$. Then $\mathfrak{G} = \mathfrak{F}\langle\alpha\rangle$ is strongly normal over $\mathfrak{F}$. The matrix $c(\sigma) = \alpha^{-1}\sigma\alpha \in GL_{\mathfrak{K}}(n)$ and σ is strong, with $\mathfrak{C}_\sigma = \mathfrak{C}(c(\sigma))$ and $c : G(\mathfrak{G}/\mathfrak{F}) \to GL_{\mathfrak{K}}(n)$ is an injective $\mathfrak{C}$-homomorphism.
- Let $g_2, g_3 \in \mathfrak{C}$ be such that $g_2^3 - 27g_3^2 \neq 0$. Let $\mathfrak{G} = \mathfrak{F}\langle\alpha\rangle$, where α satisfies, for some $a_1, \ldots, a_m \in \mathfrak{F}$, the system:

  $$(\delta_i\alpha)^2 = a_i^2(4\alpha^3 - g_2\alpha - g_3), \qquad 1 \leqslant i \leqslant m.$$

  Then $\mathfrak{G}$ is strongly normal over $\mathfrak{F}$.
- Both primitive and exponential extensions are Picard-Vessiot, but if α is transcendental over $\mathfrak{F}$, and Weierstraussian, then $\mathfrak{G}$ is not a Picard-Vessiot extension of $\mathfrak{F}$.

Strongly Normal Subfields of a Simple Extension

- Let $t \in \mathfrak{U}$ be differentially transcendental over $\mathfrak{F}$. Let $\mathfrak{G} = \mathfrak{F}\langle t \rangle$.
- Problem: Find all intermediate differential subfields $\mathcal{E}$ between $\mathfrak{F}$ and $\mathfrak{G}$ such that $\mathfrak{G}$ is strongly normal over $\mathcal{E}$.
- Quick answer: $\mathcal{E}$ is generated basically over $\mathfrak{F}$ by:
  1. Schwarzian derivatives of t, $S\delta(t) = \dfrac{\delta^3 t}{\delta t} - \dfrac{3(\delta^2 t)^2}{2(\delta t)^2}$
  2. logarithmic derivatives of linear homogeneous differential polynomials in t with coefficients in $\mathfrak{F}$
  3. powers of linear homogeneous differential polynomials in t with coefficients in $\mathfrak{F}$
  4. polyhedral functions of t
  5. exceptions involving certain non-zero constants

Automorphisms of Simple Extensions

- Let $\mathcal{H}$ be a differential subfield of $\mathcal{U}$ over which $\mathcal{U}$ need not be universal but there is an $s \in \mathcal{U}$ differentially transcendental over $\mathcal{H}$. For example, $\mathcal{H} = \mathcal{F}\mathcal{K}$.
- Let $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(n)$. Say $x \in \mathbf{P}(\mathcal{H})$ if there exists $u \in \mathcal{U}^*$ such that $ux \in GL_{\mathcal{H}}(n)$.
- For any $p \in \mathcal{U}$ such that $-cp + a \neq 0$, define

$$p^x = \frac{dp - b}{-cp + a} \in \mathcal{U}.$$

- Define τ_x to be the automorphism in $\mathcal{H}\langle s\rangle/\mathcal{H}$ with $\tau_x(s) = s^x$. Then $\tau_{xx'} = \tau_x \circ \tau_{x'}$ and if we let $\mathbf{Z} = \{+I, -I\} \subset SL(2)$, we have an exact sequence of groups:

$$1 \to \mathbf{Z} \to \mathbf{P}(\mathcal{H}) \xrightarrow{\tau} \mathrm{Aut}\,(\mathcal{H}\langle s\rangle/\mathcal{H}) \to 1.$$

Differential Algebraic Groups

- A differential algebraic group is a differentially closed subset G of $\mathcal{U}^n$ such that G is a group and the group laws are everywhere defined differential rational maps.
- G is defined over $\mathcal{F}$ if the differentially closed set G and the group laws are defined over $\mathcal{F}$ (that is, given by differentially polynomial or rational functions with coefficients in $\mathcal{F}$).
- (Cassidy) The components of G have the same differential dimension, are mutually disjoint, and are cosets of the component G° containing the identity, which is a connected, normal differential algebraic subgroup of G.
- G is linear if $G \subset GL(n)$. Viewed as a differential algebraic subgroup of $SL(n+1)$, the group laws are polynomials maps.
- (Cassidy) Every differential algebraic group whose group laws are given by differential polynomial functions is isomorphic to a linear differential algebraic group.

Galois Groups and $SL(2)$

- $G(\mathcal{G}/\mathcal{E})$ is identified with a subgroup of $\mathrm{Aut}(\mathcal{GK}/\mathfrak{FK}) = \mathrm{Aut}(\mathfrak{FK}\langle t\rangle/\mathfrak{FK})$.
- Any $\sigma \in \mathrm{Aut}\,(\mathfrak{FK}\langle t\rangle/\mathfrak{FK})$ can be represented by a matrix in $\mathbf{P}(\mathfrak{FK}) \subset SL(2)$, or a matrix in $GL_{\mathfrak{FK}}(2)$.
- There is a unique differential algebraic subgroup $H(\mathcal{E})$ of $SL(2)$, $\mathbf{Z} \subset H(\mathcal{E}) \subset \mathbf{P}(\mathfrak{FK})$, such that the sequence is exact:

$$1 \to \mathbf{Z} \to H(\mathcal{E}) \xrightarrow{\tau} G(\mathcal{G}/\mathcal{E}) \to 1.$$

  Moreover, the fixed field of $H(\mathcal{E})$ is $\mathcal{E}$.
- (Sketch) $H(\mathcal{E})$ is the set of $x \in SL(2)$ satisfying the system obtained after clearing denominators of the following:

$$A_\eta(y^x)B_\eta(y) = A_\eta(y)B_\eta((y^x), \quad \eta \in \mathcal{E}$$

  where $\eta = A_\eta(t)/B_\eta(t)$ and y is a differential indeterminate over $\mathcal{U}$. This just says σ_x fixes every $\eta \in \mathcal{E}$ when $y \mapsto t$.

Strategy to Compute the Subfields $\mathcal{E}$

- Goal: Compute all subfields $\mathcal{E}$ such that $\mathcal{G}$ is strongly normal over $\mathcal{E}$.
- (Strategy):
  1. Compute all differential algebraic subgroups H of $SL(2)$.
  2. Identify those satisfying $\mathbf{Z} \subset H \subset \mathbf{P}(\mathcal{FK})$.
  3. For each such subgroup H, compute its fixed field $\mathcal{E}(H)$.
  4. Prove that $\mathcal{G}$ is strongly normal over each $\mathcal{E}(H)$.
  5. Compute its Galois group $G(\mathcal{G}/\mathcal{E}(H))$.
  6. Show that its Galois group is H.
- Suffices to do this up to conjugation over $\mathcal{F}$, which means we need to first classify all differential algebraic subgroups of $SL(2)$.
- The Zariski closure G of a differential algebraic subgroup H defined over $\mathcal{F}$ is an algebraic subgroup, also defined over $\mathcal{F}$.

Infinite Algebraic Subgroups of $SL(2)$

- Up to conjugation, the infinite algebraic subgroups of $SL(2)$ defined over a field K are classified into five types.
  1. $\dim G = 3$; then $G = SL(2)$.
  2. $\dim G = 2$; then G is connected and conjugate over K to $ST(2)$, the special triangular subgroup.
  3. $\dim G = 1$ and G° is unipotent; then G is conjugate over K to $U^n(2)$ for some positive integer n.
  4. $\dim G = 1$, G is connected and diagonalizable; then G is conjugate over K to $SO^e(2)$ for some $e \in K^*$.
  5. $\dim G = 1$, $G \neq G^\circ$, and G° is diagonalizable; then G is conjugate over K to $SO^e(2)^\dagger$ for some $e \in K^*$.
- Here, H and H', subsets of $SL(2)$, are conjugate over K if there exists $z \in \mathbf{P}(K)$ such that $zHz^{-1} = H'$.
- $U^n(2)$: $\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, \alpha^n = 1 \right\}$; $SO^e(2), SO^e(2)^\dagger$: $\left\{ \begin{pmatrix} \alpha & \pm e\gamma \\ \gamma & \pm\alpha \end{pmatrix}, \alpha^2 - e\gamma^2 = \pm 1 \right\}$.

Finite Subgroups of $SL(2)$

- Let G be a finite subgroup of $SL(2)$ or order n. Then there are five possibilities:
  1. G is cyclic.
  2. $G/\mathbf{Z} \subset G$ and $G/\mathbf{Z}$ is dihedral.
  3. $G/\mathbf{Z} \subset G$ and $G/\mathbf{Z}$ is isomorphic to A_4; tetrahedral.
  4. $G/\mathbf{Z} \subset G$ and $G/\mathbf{Z}$ is isomorphic to S_4; octahedral.
  5. $G/\mathbf{Z} \subset G$ and $G/\mathbf{Z}$ is isomorphic to A_5; icosahedral.
- If K contains appropriate roots of unity, and if $G \subset \mathbf{P}(K)$, then G is defined over K and is conjugate over K to one of the groups: $C(n)$, $D(4m, e)$, $D(8, e, f, g)$, T, OC, DI.

Lie Vector Spaces and Fields of Constants

- Let $\mathfrak{D} = \mathcal{U}\Delta$, the vector space over $\mathcal{U}$ with basis Δ. Let $\mathfrak{E}$ be a subspace of dimension k, and let $\mathcal{K}(\mathfrak{E}) = \cap_{E \in \mathfrak{E}} \mathrm{Ker}\, E$ be its field of constants, which is clearly algebraically closed.

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- $\mathfrak{E}$ is Lie subspace if the following equivalent conditions hold:
  1. $D \in \mathfrak{D}$ and $D(\mathcal{K}(\mathfrak{E})) = 0$ implies $D \in \mathfrak{E}$.
  2. $\mathfrak{E}$ has a commuting basis $E_1, \ldots, E_k$, where $[E_i, E_j] = 0$ for all i, j.

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- $E = \sum_{i=1}^{k} a_i E_i \in \mathfrak{E}$ is rational over $\mathfrak{F}$ relative to a basis $E_1, \ldots, E_k$ if $a_i \in \mathfrak{F}$ for all i. $\mathfrak{E}$ is defined over $\mathfrak{F}$ if it has a basis where E_i is rational over $\mathfrak{F}$ relative to Δ.

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- If $\mathfrak{E}$ is defined over $\mathfrak{F}$, then rationality of $E \in \mathfrak{E}$ relative to a rational basis is rationality relative to Δ. The subset of $\mathfrak{E}$ consisting of all rational operators over $\mathfrak{F}$ is denoted by $\mathfrak{E}_{\mathfrak{F}}$.

(Cassidy) Zariski Dense Subgroups of $SL(n)$

- Let $\mathfrak{E}$ be Lie subspace of $\mathfrak{D}$, defined over $\mathfrak{F}$ and let $s \in SL(n)$ be such that $\ell E(s) = (Es)s^{-1} \in sl_{\mathfrak{F}}(n)$ for all $E \in \mathfrak{E}_{\mathfrak{F}}$. Let $SL(\mathfrak{E}, s) = sSL_{\mathcal{K}(\mathfrak{E})}(n)s^{-1}$.

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- $SL(\mathfrak{E}, s)$ is conjugate over $\mathfrak{F}$ to $SL(\mathfrak{E}', s')$ if and only if $\mathfrak{E} = \mathfrak{E}'$ and there exists $z \in \mathbf{P}(\mathfrak{F})$ such that $s'^{-1}zs \in SL_{\mathcal{K}(\mathfrak{E})}(n)$.

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- Example. Let $\mathfrak{F} = \mathbb{Q}(i \sin x, i \cos x)$, $\delta = d/dx$, $i = \sqrt{-1}$. Then $\mathfrak{C} = \mathbb{Q}$. Fix elements $\alpha, \gamma \in \mathfrak{U}$ such that

$$\alpha^2 - \gamma^2 = \sin x, \quad 2\alpha\gamma = -\cos x, \quad \alpha^2 + \gamma^2 = -1.$$

Let $s = \begin{pmatrix} \alpha & \gamma \\ \gamma & -\alpha \end{pmatrix}$. Then $\ell\delta s = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$ and $H = SL(\mathfrak{D}, s)$ is a Zariski dense differential algebraic subgroup, contained in $\mathbf{P}(\mathfrak{F}\mathfrak{K})$, but not conjugate over $\mathfrak{F}$ to $SL_{\mathfrak{K}}(2)$.

Differential Algebraic Subgroups of $\mathbf{G}_{\mathrm{a}}^n$

- (Cassidy) Every linear differential ideal $\mathfrak{p}$ of $\mathcal{U}\{y_1, \ldots, y_n\}$ defines a differential algebraic subgroup B of $\mathbf{G}_{\mathrm{a}}^n$, which is connected, is a vector subspace of $\mathcal{U}^n$ over $\mathcal{K}$, and is defined over $\mathfrak{F}$ if and only if $\mathfrak{p}$ is. Moreover, these are all.

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- The set $\mathfrak{K}(B)$ of all $a \in \mathfrak{U}$ such that $aB \subseteq B$ is the field of constants $\mathfrak{K}(\mathfrak{E})$ of a unique Lie subspace $\mathfrak{E}$ of $\mathfrak{D}$, which is defined over $\mathfrak{F}$. Note that $\mathfrak{K}(0) = \mathfrak{U}$ when $B = 0$ or $\mathfrak{E} = 0$.

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- Relative to any orderly ranking, $\mathfrak{p}$ has a canonical, linear, homogeneous characteristic set $\mathrm{B} \subset \mathfrak{F}\{y_1, \ldots, y_n\}$ such that

$$L(ay_1, \ldots, ay_n) = aL(y_1, \ldots, y_n), \qquad (L \in \mathrm{B}, a \in \mathfrak{K}(B)).$$

Indeed, for every $L \in \mathfrak{p} \cap \mathfrak{F}\{y_1, \ldots, y_n\}_1$.

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  Indeed, for every $L \in \mathfrak{p} \cap \mathfrak{F}\{y_1, \ldots, y_n\}_1$.
- An element $\beta \in \mathfrak{U}^n$ is said to be rational over $\mathfrak{F}$ mod B, that is, the coset $\beta + B \in \mathbf{G}_{\mathrm{a}}^n/B$ is rational over $\mathfrak{F}$, if $L(\beta) \in \mathfrak{F}$ for every linear homogeneous differential polynomial $L \in \mathfrak{p}$; or equivalently, if $L(\beta) \in \mathfrak{F}$ for every $L \in \mathrm{B}$. If $B = 0$, $\beta \in \mathfrak{F}$.

Differential Algebraic Subgroups of $\mathbf{G}_{\mathrm{m}}$

- The logarithmic derivative map $\ell\Delta : \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{a}}^m$ given by $\ell\Delta x = (\frac{\delta_1 x}{x}, \ldots, \frac{\delta_m x}{x})$ is a differential rational homomorphism defined over $\mathbb{Q}$ with kernel $\mathbf{G}_{\mathrm{m}\mathcal{K}} = \mathcal{K}^*$ and image $\mathbf{I}$, which is a differential algebraic subgroup of $\mathbf{G}_{\mathrm{a}}^m$ with corresponding linear differential ideal generated by $\left\{ \delta_i y_j - \delta_j y_i \,\middle|\, 1 \leqslant i < j \leqslant m \right\}$.

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- (Cassidy) Every infinite differential algebraic subgroup A of $\mathbf{G}_{\mathrm{m}}$ contains $\mathbf{G}_{\mathrm{m}\mathcal{K}}$, is connected, corresponds to a differential algebraic subgroup of $\mathbf{I}$, and is defined over $\mathfrak{F}$ if and only if $\ell\Delta(A)$ is. These are all.

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- (Cassidy) Every infinite differential algebraic subgroup A of $\mathbf{G}_{\mathrm{m}}$ contains $\mathbf{G}_{\mathrm{m}\mathcal{K}}$, is connected, corresponds to a differential algebraic subgroup of $\mathbf{I}$, and is defined over $\mathcal{F}$ if and only if $\ell\Delta(A)$ is. These are all.
- The finite (differential) algebraic subgroups of $\mathbf{G}_{\mathrm{m}}$ are the cyclic groups $\mathbf{P}_n$ consisting of n-th roots of unity.

Differential Algebraic Subgroups of $ST(2)$

- B is a differential algebraic subgroup of $\mathbf{G}_{\mathrm{a}}$ defined over $\mathfrak{F}$;
  A is a (finite or infinite) differential algebraic subgroup of $\mathbf{G}_{\mathrm{m}\mathfrak{K}(B)} = \mathfrak{K}(B)^*$, also defined over $\mathfrak{F}$;
  $\beta \in \mathfrak{U}$ is rational over $\mathfrak{F}$ mod B.

Differential Algebraic Subgroups of $ST(2)$

- B is a differential algebraic subgroup of $\mathbf{G}_{\mathrm{a}}$ defined over $\mathfrak{F}$; A is a (finite or infinite) differential algebraic subgroup of $\mathbf{G}_{\mathrm{m}\mathfrak{K}(B)} = \mathfrak{K}(B)^*$, also defined over $\mathfrak{F}$; $\beta \in \mathfrak{U}$ is rational over $\mathfrak{F}$ mod B.
- To such a triple (B, A, β), we define

$$ST(B, A, \beta) = \left\{ \begin{pmatrix} a & (a - a^{-1})\beta + b \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a \in A, b \in B \right\}.$$

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- $ST(B, A, \beta)$ is conjugate over $\mathfrak{F}$ to $ST(B', A', \beta')$ if and only if $A = A'$, and there exists $f \in \mathfrak{F}^*$ such that $fB = B'$, and $(f\beta - \beta' + B') \cap \mathfrak{F} \neq \emptyset$.

Dense Cases for $ST(2)$, $U^n(2)$, $SD(2)$

- For all $H = ST(B, A, \beta)$, one and only one of the following holds:
  1. $B \neq 0$ and A is infinite; dense in $ST(2)$.
  2. $B \neq 0$ and $A = \mathbf{P}_n$; dense in $U^n(2)$.
  3. $B = 0$ and A is infinite; then $\beta \in \mathfrak{F}$, H is conjugate over $\mathfrak{F}$ to $ST(0, A, 0)$, which is dense in $SD(2)$.
  4. $B = 0$ and $A = \mathbf{P}_n$; then $\beta \in \mathfrak{F}$, H is finite and conjugate over $\mathfrak{F}$ to $C(n)$.

A Modified Logarithmic Derivative on $SO^e(2)$

- Recall for $e \in \mathfrak{F}^*$, $SO^e(2) = \left\{ \begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix} \,\middle|\, \alpha^2 - e\gamma^2 = 1 \right\}$.

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- Let $\epsilon^2 = e$, $s = \begin{pmatrix} 1 & \epsilon \\ -1 & \epsilon \end{pmatrix}$ and let $\tau_s(u) = sus^{-1}$, $\ell_\epsilon(x) = \epsilon x$.

  Then the map $\pi_e : SO^e(2) \xrightarrow{\tau_s} SD(2) \xrightarrow{p} \mathbf{G}_{\mathrm{m}} \xrightarrow{\ell\Delta} \mathbf{G}_{\mathrm{a}}^m \xrightarrow{\ell_\epsilon} \mathbf{G}_{\mathrm{a}}^m$ is a differential rational homomorphism defined over $\mathbb{Q}$, given by

$$\pi_e\left(\begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix}\right) = \begin{cases} \left(\frac{\delta_1 \alpha}{\gamma}, \ldots, \frac{\delta_m \alpha}{\gamma}\right) & \text{if } \gamma \neq 0; \\ (0, \ldots, 0) & \text{if } \gamma = 0. \end{cases}$$

  with

$$\operatorname{Ker} \pi_e = \left\{ \begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix} \,\middle|\, \alpha \in \mathfrak{K}, \gamma \in \mathfrak{U}, \alpha^2 - e\gamma^2 = 1 \right\}.$$

  Image = differential algebraic $\mathfrak{F}$-subgroup $\mathbf{I}_e$ of $\mathbf{G}_{\mathrm{a}}^m$ defined by

$$\delta_i y_j - \delta_j y_i + \frac{1}{2}(\ell\delta_j e) y_i - \frac{1}{2}(\ell\delta_i e) y_j \qquad (1 \leqslant i < j \leqslant m).$$

Dense Differential Algebraic Subgroups of $SO^e(2)$

- $p \circ \tau_s : SO^e(2) \rightarrow \mathbf{G}_{\mathrm{m}}$ is a rational isomorphism;
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- Exact sequence: $1 \to \mathrm{Ker}\,\pi_e \to SO^e(2) \xrightarrow{\pi_e} \mathbf{I}_e \to 1$.
- There is a bijection between the set of differential algebraic subgroups B of $\mathbf{I}_e$ and the set of differential algebraic subgroup of $SO^e(2)$ containing $\mathrm{Ker}\,\pi_e$. This bijection is given by $B \mapsto SO^e(B) = \pi_e^{-1}(B)$, B is defined over $\mathfrak{F}$ if and only if $SO^e(B)$ is.

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- A differential algebraic subgroup H is dense in $SO^e(2)$ if and only if $H \supset \operatorname{Ker} \pi_e$, in which case $H = SO^e(B)$ for a unique differential algebraic subgroup B of $\mathbf{I}_e$, H is connected, and H is defined over $\mathfrak{F}$ if and only if B is. These are all.

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- $SO^e(B)$ and $SO^{e'}(B')$ are conjugate over $\mathfrak{F}$ if and only if there exists $f \in \mathfrak{F}^*$ such that $e'^{-1}e = f^2$ and $fB' = B$.

A Modified Logarithmic Derivative on $SO^e(2)^*$

- Recall for $e \in \mathfrak{F}^*$,

$$SO^e(2)^* = \left\{ \begin{pmatrix} \alpha & -e\gamma \\ \gamma & -\alpha \end{pmatrix} \;\middle|\; \alpha^2 - e\gamma^2 = -1 \right\}.$$

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- The map $\pi_e^* : SO^e(2)^* \xrightarrow{\tau_s^*} SD(2)^* \xrightarrow{p^*} \mathbf{G}_{\mathrm{m}} \xrightarrow{\ell\Delta} \mathbf{G}_{\mathrm{a}}^m \xrightarrow{\ell_\epsilon} \mathbf{G}_{\mathrm{a}}^m$ is a differential rational map defined over $\mathbb{Q}$ given by

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with

$$\mathrm{Ker}\,\pi_e^* = \left\{ \begin{pmatrix} \alpha & -e\gamma \\ \gamma & -\alpha \end{pmatrix} \,\middle|\, \alpha \in \mathfrak{K}, \gamma \in \mathfrak{U}, \alpha^2 - e\gamma^2 = -1 \right\}.$$

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Image is the differential algebraic subgroup $\mathbf{I}_e$ of $\mathbf{G}_{\mathrm{a}}^m$.

- Note that $p^* \circ \tau_s^*$ is a bijective birational map.

Dense Differential Algebraic Subgroups of $SO^e(2)^\dagger$

- Recall that $SO^e(2)^\dagger = SO^e(2) \cup SO^e(2)^*$.

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- Let $e \in \mathfrak{F}^*$, B a differential algebraic subgroup of $\mathbf{I}_e$ defined over $\mathfrak{F}$, and $\beta \in \mathbf{I}_e$ be such that $\beta + B \in \mathbf{G}_{\mathrm{a}}^m / B$ is rational over $\mathfrak{F}$.

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- The set $SO^e(B,\beta)^\dagger = \pi_e^{-1}(B) \cup \pi_e^{*\,-1}(\beta + B)$ is a Zariski dense differential algebraic subgroup of $SO^e(2)^\dagger$ defined over $\mathfrak{F}$ with $\pi_e^{-1}(B) = SO^e(B)$ as its component of the identity, and these are all.

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- Recall that $SO^e(2)^\dagger = SO^e(2) \cup SO^e(2)^*$.
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- $SO^e(B, \beta)^\dagger$ is conjugate over $\mathfrak{F}$ to $SO^{e'}(B', \beta')^\dagger$ if and only if there exists $f \in \mathfrak{F}^*$ such that $e'^{-1} e = f^2$, $fB' = B$, and

$$SO^e_{\mathfrak{F}}(2) \cap \pi_e^{-1}\big((f\beta' - \beta + B) \cup (f\beta' + \beta + B)\big) \neq \emptyset.$$

Example: Dense DA_G of $SO^e(2)^\dagger$

- Example. Let $\mathfrak{F} = \mathbb{Q}(i \sin x, i \cos x)$, $\delta = d/dx$. Then $\mathfrak{C} = \mathbb{Q}$ and the field of constants of $\mathfrak{F}(i)$ is $\mathbb{Q}(i)$ by linear disjointness.

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- Example. Let $\mathfrak{F} = \mathbb{Q}(i \sin x, i \cos x)$, $\delta = d/dx$. Then $\mathfrak{C} = \mathbb{Q}$ and the field of constants of $\mathfrak{F}(i)$ is $\mathbb{Q}(i)$ by linear disjointness.
- Let $e = -1$, $B = 0$, $h = i \sin x$, $g = i \cos x$, and $\beta = \frac{1}{2}$. Then $G_5 = SO^e(0, \beta)^\dagger$ and $G_5' = SO^e(0, 2\beta)^\dagger$ are both dense differential algebraic subgroups of $SO^e(2)^\dagger$ contained in $\mathbf{P}(\mathfrak{F}\mathfrak{K})$, but neither is conjugate over $\mathfrak{F}$ to $SO^e_{\mathfrak{K}}(2)^\dagger = SO^e(0,0)^\dagger$, which is the union of

$$\operatorname{Ker} \pi_e = \left\{ \begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix} \middle| \, \alpha \in \mathfrak{K}, \gamma \in \mathfrak{U}, \alpha^2 - e\gamma^2 = 1 \right\}$$

and

$$\operatorname{Ker} \pi_e^* = \left\{ \begin{pmatrix} \alpha & -e\gamma \\ \gamma & -\alpha \end{pmatrix} \middle| \, \alpha \in \mathfrak{K}, \gamma \in \mathfrak{U}, \alpha^2 - e\gamma^2 = -1 \right\}.$$

Infinite DA_G H of $SL(2)$ contained in $\mathbf{P}(\mathcal{FK})$

- If H is infinite, then H is conjugate over $\mathfrak{F}$ to one of the following:
  1. $G_1 = sSL_{\mathfrak{K}}(2)s^{-1}$, where $s \in \mathbf{P}(\mathfrak{F}\mathfrak{C}_{\mathrm{a}})$ and $\ell\Delta s \in sl_{\mathfrak{F}}(2)^m$.
  2. $G_2 = ST(B, \mathfrak{K}^*, 0)$, where B is a nonzero differential algebraic subgroup of $\mathbf{G}_{\mathrm{a}}$ defined over $\mathfrak{F}$, and B, as a vector space over $\mathfrak{K}$, has a finite basis in $\mathfrak{F}$.
  3. $G_3 = ST(B, \mathbf{P}_{2n}, 0)$, where $n \in \mathbb{N}$, and B is as above.
  4. $G_4 = SO^e_{\mathfrak{K}}(2)$, where $e \in \mathfrak{C}^*$.
  5. $G_5 = SO^e(0, \beta)^\dagger$ where $e \in \mathfrak{C}^*$, $\beta = (\frac{1}{2}\frac{\delta_1 h}{g}, \ldots, \frac{1}{2}\frac{\delta_m h}{g})$ for some $h, g \in \mathfrak{F}$, $g \neq 0$, and $h^2 - eg^2 \in \mathfrak{C}^*$.

Finite DA_G H of $SL(2)$ contained in $\mathbf{P}(\mathfrak{FK})$

- If H is finite and if $\mathfrak{C}$ is algebraically closed, then $H \subset \mathbf{P}(\mathfrak{F})$, and H is conjugate over $\mathfrak{F}$ to one of the following:
  1. $G_6 = C(2n)$, where $n \in \mathbb{N}$.
  2. $G_7 = D(4n, e)$, where $n \in \mathbb{N}$, $e \in \mathfrak{F}^*$.
  3. $G_8 = D(8, e, f, g)$, where $e, f, g \in \mathfrak{F}$, $eg \neq 0$, and $f^2 + eg^2 = 1$.
  4. $G_9 = T$
  5. $G_{10} = OC$
  6. $G_{11} = DI$.

$$D(4n, e) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a^n = 1 \right\} \cup \left\{ \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix} \,\middle|\, b^{2n} = e^n \right\}$$

$$D(8, e, f, g) = \left\{ \pm 1, \pm \begin{pmatrix} 0 & \epsilon \\ -\epsilon^{-1} & 0 \end{pmatrix}, \pm \begin{pmatrix} \alpha & e\gamma \\ \gamma & -\alpha \end{pmatrix}, \pm \begin{pmatrix} -\epsilon\gamma & \epsilon\alpha \\ \epsilon^{-1}\alpha & \epsilon\gamma \end{pmatrix} \right\}$$

$$\epsilon^2 = e, \quad \alpha^2 + e\gamma^2 = -1, \quad \alpha^2 - e\gamma^2 = f, \quad 2\alpha\gamma = g$$

Example of $D(8, e, f, g)$

- Let $\mathfrak{F} = \mathbb{C}(\sin^2 x, \sin x \cos x)$, δ/dx.

Example of $D(8, e, f, g)$

- Let $\mathfrak{F} = \mathbb{C}(\sin^2 x, \sin x \cos x)$, δ/dx.
- Let $e = \sec^2 x$, $f = i \tan x$, and $g = 1$.

Example of $D(8, e, f, g)$

- Let $\mathfrak{F} = \mathbb{C}(\sin^2 x, \sin x \cos x)$, δ/dx.
- Let $e = \sec^2 x$, $f = i \tan x$, and $g = 1$.
- Let $\epsilon = \sec x$, $\alpha = (\frac{-1+i \tan x}{2})^{1/2}$, $\gamma = \frac{1}{2}\alpha^{-1}$.

Example of $D(8, e, f, g)$

- Let $\mathfrak{F} = \mathbb{C}(\sin^2 x, \sin x \cos x)$, δ/dx.
- Let $e = \sec^2 x$, $f = i \tan x$, and $g = 1$.
- Let $\epsilon = \sec x$, $\alpha = (\frac{-1+i\tan x}{2})^{1/2}$, $\gamma = \frac{1}{2}\alpha^{-1}$.
- Then if $H = D(8, e, f, g)$, H is not conjugate over $\mathfrak{F}$ to $D(8, e)$.

$$D(4n, e) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| \, a^n = 1 \right\} \cup \left\{ \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix} \middle| \, b^{2n} = e^n \right\}$$

$$D(8, e, f, g) = \left\{ \pm 1, \pm \begin{pmatrix} 0 & \epsilon \\ -\epsilon^{-1} & 0 \end{pmatrix}, \pm \begin{pmatrix} \alpha & e\gamma \\ \gamma & -\alpha \end{pmatrix}, \pm \begin{pmatrix} -\epsilon\gamma & \epsilon\alpha \\ \epsilon^{-1}\alpha & \epsilon\gamma \end{pmatrix} \right\}$$

$$\epsilon^2 = e, \quad \alpha^2 + e\gamma^2 = -1, \quad \alpha^2 - e\gamma^2 = f, \quad 2\alpha\gamma = g$$

Strongly Normal Subfields: Infinite Galois Group

- If $\mathcal{E}$ is a differential subfield of $\mathcal{G} = \mathcal{F}\langle t \rangle$ such that $\mathcal{G}$ is strongly normal over $\mathcal{E}$ and $G(\mathcal{G}/\mathcal{E})$ is infinite, then $\mathcal{E}$ is conjugate over $\mathcal{F}$ to one of the following:
  1. $\mathcal{E}_1 = \mathcal{F}\langle S\delta_1(t^s), \ldots, S\delta_m(t^s) \rangle$, where $s \in \mathbf{P}(\mathcal{F}\mathcal{C}_{\mathrm{a}})$ and $\ell\Delta(s) \in sl_{\mathcal{F}}(2)^m$.
  2. $\mathcal{E}_2 = \mathcal{F}\langle \big(\ell\delta_i L(t)\big)_{1 \leqslant i \leqslant m, L \in \Gamma(B)} \rangle$, where B is a nonzero differential algebraic subgroup of $\mathbf{G}_{\mathrm{a}}$ defined over $\mathcal{F}$ and B (as a vector space over $\mathcal{K}$) has a finite basis in $\mathcal{F}$, and $\Gamma(B)$ is the set of all non-zero linear homogeneous differential polynomials vanishing on B.
  3. $\mathcal{E}_3 = \mathcal{F}\langle \big(L(t)^n\big)_{L \in \Gamma(B)} \rangle$, where $n > 0 \in \mathbb{N}$, and $B, \Gamma(B)$ as above.
  4. $\mathcal{E}_4 = \mathcal{F}\langle \big(\frac{\delta_i t}{e - t^2}\big)_{1 \leqslant i \leqslant m} \rangle$, where $e \in \mathcal{C}^*$.
  5. $\mathcal{E}_5 = \mathcal{F}\langle \big((\frac{2\delta_i t}{e - t^2} - \frac{\delta_i h}{2eg})^2\big)_{1 \leqslant i \leqslant m} \rangle$, where $e \in \mathcal{C}^*$, and $h, g \in \mathcal{F}$, $g \neq 0$, and $h^2 - eg^2 \in \mathcal{C}^*$.

Strongly Normal Subfields: Finite Galois Group

- If $G(\mathcal{G}/\mathcal{E})$ is finite, and if C is algebraically closed, then $\mathcal{E}$ is conjugate over $\mathfrak{F}$ to one of the following, and $\mathcal{G}$ is Galois over $\mathcal{E}$.
  1. $\mathcal{E}_6 = \mathfrak{F}\langle t^n \rangle$, where $n > 0 \in \mathbb{N}$.
  2. $\mathcal{E}_7 = \mathfrak{F}\langle t^n + \frac{e^n}{t^n} \rangle$, where $n > 0 \in \mathbb{N}$, $e \in \mathfrak{F}^*$.
  3. $\mathcal{E}_8 = \mathfrak{F}\langle \frac{g(t^2+e)^2}{t(gt^2 - 2ft - eg)} \rangle$, where $e, f, g \in \mathfrak{F}$, $eg \neq 0$, and $f^2 + eg^2 = 1$.
  4. $\mathcal{E}_9 = \mathfrak{F}\langle \frac{t^{12} - 33t^8 - 33t^4 + 1}{(t^4-1)^2 t^2} \rangle$.
  5. $\mathcal{E}_{10} = \mathfrak{F}\langle \frac{(t^{12} - 33t^8 - 33t^4 + 1)^2}{(t^4-1)^4 t^4} \rangle$.
  6. $\mathcal{E}_{11} = \mathfrak{F}\langle \frac{(t^{30} - 228t^{15} + 494t^{10} + 228t^5 + 1)^3}{t^5(t^{10} + 11t^5 - 1)^5} \rangle$.

The Differential Galois Group for $ST(B, \mathcal{K}^*, 0)$

- $G_2 = ST(B, \mathcal{K}^*, 0) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathcal{K}^*, b \in B \right\}$ where B is a nonzero differential algebraic subgroup of $\mathbf{G}_\mathrm{a}$ defined over $\mathcal{F}$, and B, as a vector space over $\mathcal{K}$, has a finite basis in $\mathcal{F}$.

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- $\mathcal{E}_2 = \mathfrak{F}\langle \big(\ell\delta_i L(t)\big)_{1 \leqslant i \leqslant m, L \in \Gamma(B)} \rangle$, where $\Gamma(B)$ is the set of all non-zero linear homogeneous differential polynomials vanishing on B. ($\Gamma(B)$ can be replaced by a finite subset.)

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- $\mathcal{E}_2 = \mathcal{F}\langle \big(\ell\delta_i L(t)\big)_{1 \leqslant i \leqslant m, L \in \Gamma(B)} \rangle$, where $\Gamma(B)$ is the set of all non-zero linear homogeneous differential polynomials vanishing on B. ($\Gamma(B)$ can be replaced by a finite subset.)
- G_2 is $\mathcal{C}$-isomorphic to the group $G(\ell, 0)$ of $(\ell+1) \times (\ell+1)$ matrices of the form

$$g(b_1, \ldots, b_\ell; a) = \begin{pmatrix} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_\ell \\ 0 & 0 & \cdots & 0 & a \end{pmatrix}$$

where $\ell = \dim_{\mathcal{K}} B$, and $b_1, \ldots, b_\ell, a \in \mathcal{K}$.

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- $G_3 = ST(B, \mathbf{P}_{2n}, 0) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a^{2n} = 1, b \in B \right\}$ where B is a nonzero differential algebraic subgroup of $\mathbf{G}_\mathrm{a}$ defined over $\mathfrak{F}$, and B, as a vector space over $\mathfrak{K}$, has a finite basis in $\mathfrak{F}$.

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- $\mathcal{E}_3 = \mathfrak{F}\langle \left(L(t)^n\right)_{L \in \Gamma(B)} \rangle$, where $n > 0 \in \mathbb{N}$, and $B, \Gamma(B)$ as before. ($\Gamma(B)$ can be replaced by a finite subset.)

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- G_3 is $\mathfrak{C}$-isomorphic to the group $G(\ell, 0)$ of $(\ell+1) \times (\ell+1)$ matrices of the form

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where $\ell = \dim_{\mathfrak{K}} B$, and $b_1, \ldots, b_\ell, a^{2n} = 1$, $a \neq 0$.

Open Questions

- $G_1 = sSL_{\mathfrak{K}}(2)s^{-1}$, where $s \in \mathbf{P}(\mathfrak{F}\mathfrak{C}_{\mathrm{a}})$ and $\ell\Delta s \in sl_{\mathfrak{F}}(2)^m$.

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- $\mathcal{E}_5 = \mathfrak{F}\langle\left(\left(\frac{2\delta_i t}{e-t^2} - \frac{\delta_i h}{2eg}\right)^2\right)_{1\leqslant i\leqslant m}\rangle$, where $e \in \mathfrak{C}^*$, and $h, g \in \mathfrak{F}$, $g \neq 0$, and $h^2 - eg^2 \in \mathfrak{C}^*$.