Rota-Baxter Type Operators, Rewriting Systems, and Gröbner-Shirshov Bases, Part II

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Part II Outline

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TRS Defined by Certain Operated Polynomials

- Let $\varphi(x,y) \in \mathbf{k}\|x,y\|$ be an operated polynomial of the form $\lfloor x\rfloor\lfloor y\rfloor - \lfloor B(x,y)\rfloor$, where $B(x,y) \in \mathbf{k}\|x,y\|$.
- **The rewriting system $\Pi_\varphi(Z)$ defined by φ on $\mathbf{k}\|Z\|$ with basis $\mathfrak{M}(Z)$** is the relation

$$\Pi_\varphi(Z) := \{\, (q|_{\lfloor u\rfloor\lfloor v\rfloor}, q|_{\lfloor B(u,v)\rfloor}) \;\big|\; q \in \mathfrak{M}^\star(Z), u, v \in \mathfrak{M}(Z) \,\}$$

- We write $\rightarrow_\varphi$ instead of $\rightarrow_{\Pi_\varphi(Z)}$ when Z is fixed.
- **Proposition**: $\Pi_\varphi(Z)$ is simple.
- $f \rightarrow_\varphi g$ if for some $u, v \in \mathfrak{M}(Z)$, g is obtained from f by replacing *exactly once* a subword $\lfloor u\rfloor\lfloor v\rfloor$ in *one* monomial $w \in \mathrm{Supp}(f)$ by $\lfloor B(u,v)\rfloor$. A bracketed polynomial $g \in \mathbf{k}\|Z\|$ is said to be **a normal φ-form** for f if g is in RBNF and $f \xrightarrow{*}_\varphi g$.

Totally Linear Expressions

- For $\varphi = \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x,y) \rfloor$ to be of a linear operator, we need to put some conditions on $B(x,y)$.

- An expression $B \in \mathbf{k}\|X\|$ is **totally linear in X** if **every** variables $x \in X$, when counted with multiplicity in repeated multiplications, appears exactly once in **every** monomial $w \in \mathrm{Supp}(B)$.

- **Examples**: Let $X = \{x, y\}$. The expression $x\lfloor y \rfloor + \lfloor x \rfloor \lfloor y \rfloor + xy$ is totally linear in X, but the monomials $\lfloor x \rfloor$ (missing y), $x^2\lfloor y \rfloor$ and $x\lfloor y \rfloor^2$ are not.q

Rota-Baxter Type OPI, Operators, and Algebras

An OPI $\varphi \in \mathbf{k}\|x, y\|$ is **of Rota-Baxter type** if φ has the form $\lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x, y) \rfloor$ for some $B(x, y) \in \mathbf{k}\|x, y\|$ and if:

(a) : $B(x, y)$ is totally linear in x, y;

(b) : $B(x, y)$ is in RBNF;

(c) : for every well-ordered set Z, the rewriting system $\Pi_{\varphi}(Z)$ defined by φ is terminating;

(d) : for every well-ordered set Z, the expression $B(B(u, v), w) - B(u, B(v, w))$ is φ-reducible to zero for all $u, v, w \in \mathfrak{M}(Z)$.

If $\varphi := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x, y) \rfloor$ is of Rota-Baxter type, then we say the defining **operator** $P = \lfloor\ \rfloor$ **of a** φ**-algebra** R, and (by abuse) the **expression** $B(x, y)$, are **of Rota-Baxter type**.
By a **Rota-Baxter type algebra**, we mean **some** φ-algebra R where φ is **some** OPI in $\mathbf{k}\|x, y\|$ of Rota-Baxter type.

Examples

- Let $B(x,y) := x\lfloor y\rfloor$. Then $\varphi = 0$ is the OPI defining the **average operator** and it is of Rota-Baxter type. The identity defining a **Rota-Baxter operator** and that defining a **Nijenhuis operator** are OPIs of Rota-Baxter type.
- The expression $B(x,y) := y\lfloor x\rfloor$ is not of Rota-Baxter type. This is because in $\mathbf{k}\|u,v,w\|$, the operated **monomial** $B(B(u,v),w) = w\lfloor B(u,v)\rfloor = \mathbf{w}\lfloor\mathbf{v}\lfloor\mathbf{u}\rfloor\rfloor$ is in RBNF, while $B(u,B(v,w)) = B(v,w)\lfloor u\rfloor = w\lfloor v\rfloor\lfloor u\rfloor \to_\varphi \mathbf{w}\lfloor\mathbf{u}\lfloor\mathbf{v}\rfloor\rfloor$ is also in RBNF and the two operated **monomials** are not φ-joinable. By the Hierarchy Lemma,

$$B(B(u,v),w) - B(u,B(v,w)) \not\xrightarrow{*}_\varphi 0.$$

This also shows the base fork $(\lfloor u\rfloor\lfloor v\rfloor\lfloor w\rfloor \to_\varphi \lfloor B(u,v)\rfloor\lfloor w\rfloor, \lfloor u\rfloor\lfloor v\rfloor\lfloor w\rfloor \to_\varphi \lfloor u\rfloor\lfloor B(v,w)\rfloor)$ is not φ-joinable, and the TRS $\Pi_\varphi(u,v,w)$ is not confluent.

Remarks on Definition of Rota-Baxter Type OPIs

- The **well-order on Z** need not be given if Z is denumerable or if Z is uncountable and we accept the Axiom of Choice.
- **Total linearity on $B(x, y)$** is imposed since we are considering linear operators.
- $B(x, y)$ **in RBNF** and $\Pi_\varphi(Z)$ **terminating** are necessary to avoid obvious infinite rewriting under $\Pi_\varphi(Z)$. This rules out the Reynolds identity:

$$\lfloor x\rfloor\lfloor y\rfloor = \lfloor x\lfloor y\rfloor\rfloor + \lfloor\lfloor x\rfloor y\rfloor + \lfloor\lfloor \mathbf{x}\rfloor\lfloor \mathbf{y}\rfloor\rfloor.$$

- **φ-reduction to zero of $B(B(u, v), w) - B(u, B(v, w))$ for all $u, v, w \in \mathfrak{M}(Z)$** is to ensure $\Pi_\varphi(Z)$ is confluent.
- Note that in the definition for φ to be of Rota-Baxter type, there is no mention of monomial orders on $\mathfrak{M}(Z)$ and hence no requirement of compatibility.

List of Rota-Baxter Type Operators

Conjecture: For any $c, \lambda \in \mathbf{k}$, the operated polynomial $\varphi := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x,y) \rfloor$, where $B(x,y)$ is taken from the list below, is of Rota-Baxter type. Moreover, any OPI φ of Rota-Baxter type is necessarily defined as above by a $B(x,y)$ from among this list.

- (a) : $x\lfloor y \rfloor$ (average)
- (b) : $\lfloor x \rfloor y$ (inverse average)
- (c) : $\underline{x\lfloor y \rfloor + y\lfloor x \rfloor}$, (symmetric average)
- (d) : $\underline{\lfloor x \rfloor y + \lfloor y \rfloor x}$, (symmetric inverse average)
- (e) : $x\lfloor y \rfloor + \lfloor x \rfloor y - \lfloor xy \rfloor$ (Nijenhuis)
- (f) : $x\lfloor y \rfloor + \lfloor x \rfloor y + \lambda xy$ (Rota-Baxter)
- (g) : $\underline{x\lfloor y \rfloor - x\lfloor 1 \rfloor y + \lambda xy}$, (average TD)
- (h) : $\underline{\lfloor x \rfloor y - x\lfloor 1 \rfloor y + \lambda xy}$, (inverse average TD)

List continued

$$
\begin{aligned}
(i):&\ \underline{x\lfloor y\rfloor + \lfloor x\rfloor y - x\lfloor 1\rfloor y + \lambda xy}\ \text{(TD)}\\
(j):&\ \underline{x\lfloor y\rfloor + \lfloor x\rfloor y - x\lfloor 1\rfloor y - xy\lfloor 1\rfloor + \lambda xy},\quad \text{(right TD)}\\
(k):&\ \underline{x\lfloor y\rfloor + \lfloor x\rfloor y - x\lfloor 1\rfloor y - \lfloor xy\rfloor + \lambda xy},\ \text{(Nijenhuis TD)}\\
(l):&\ \underline{x\lfloor y\rfloor + \lfloor x\rfloor y - x\lfloor 1\rfloor y - \lfloor 1\rfloor xy + \lambda xy},\quad \text{(left TD)}\\
(m):&\ \underline{cx\lfloor 1\rfloor y + \lambda xy}\ \text{(generalized endomorphism)}\\
(n):&\ \underline{cy\lfloor 1\rfloor x + \lambda yx}\ \text{(generalized antimorphism)}
\end{aligned}
$$

- A new type is underlined followed by a proposed name. When λ is present, "of weight λ" should be added.

Main Result on RBT OPI, Sufficiency

Theorem. Let $\mathbf{k}$ be a field. Let $\varphi := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x,y) \rfloor \in \mathbf{k}\|x, y\|$, where $B(x,y)$ is in the list in the Conjecture. Then for every well-ordered set $(Z, \leqslant_Z)$, the following statements hold :

1. The operated polynomial φ is of **Rota-Baxter type**.
2. $\leqslant_Z$ can be extended to the **monomial order** $\leqslant_{\mathrm{db},Z}$ on $\mathfrak{M}(Z)$ with which φ **compatible**.
3. The rewriting system $\Pi_\varphi(Z)$ on $\mathbf{k}\|Z\|$ is **convergent**.
4. The set $S_\varphi(Z)$ is a **Gröbner-Shirshov basis** in $\mathbf{k}\|Z\|$ with respect to the monomial order $\leqslant_{\mathrm{db},Z}$.
5. There is a (uniform) **construction of the free φ-algebra over Z**, which is $(\mathbf{k}\cdot\mathfrak{R}(Z), \diamond, P_r)$.

Moreover, the constructions of $\Pi_\varphi(Z), \leqslant_{\mathrm{db},Z}, S_\varphi(Z)$ and $(\mathbf{k}\cdot\mathfrak{R}(Z), \diamond, P_r)$ are uniformly defined.

Main Result on RBT OPI: Necessity

Theorem. Let $\mathbf{k}$ be a field. Let $\varphi := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x,y) \rfloor$. Then φ is of Rota-Baxter type and is compatible with $\leqslant_{\mathrm{db}.Z}$ on $\mathfrak{M}(Z)$ for some well-ordered set Z (possibly the empty set) if and only if $B(x,y)$ is among the list given in Conjecture.

- When Z is the empty set, $\mathfrak{M}(Z)$ is the free operated monoid $\mathfrak{M}(1)$ on 1 where 1 is the monoid identity element and $\mathfrak{M}(1)$ **contains** (not consists of) monomials of the form $M = \prod_{i=1}^{n} P^{e_i}(1)$ and $P(P^i(1)) = P^{i+1}(1)$. It also contains $P(M)$.
- **Remark**. Recall that in the definition for φ to be of Rota-Baxter type, there is no mention of monomial orders on $\mathfrak{M}(Z)$ and hence no requirement of compatibility. So, there **might** be some φ of Rota-Baxter type (as we define it) that is not on the list.

A Local Characterization Theorem

Theorem: Let $\mathbf{k}$ be a field. Let Z be a set, let $\leqslant_Z$ be a monomial order on $\mathfrak{M}(Z)$, let $B(x, y)$ be in RBNF and totally linear in x, y, and let $\varphi(x, y) := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x, y) \rfloor \in \mathbf{k}\|x, y\|$ be compatible with $\leqslant_Z$ on $\mathfrak{M}(Z)$. TFAE:

(*a*): For all $u, v, w \in \mathfrak{M}(Z)$,
$B(B(u, v), w) - B(u, B(v, w)) \xrightarrow{*}_{\varphi} 0$.

(*b*): For all $u, v, w \in \mathfrak{M}(Z)$,
$B(B(u, v), w) \downarrow_{\varphi} B(u, B(v, w))$.

(*c*): $\Pi_{\varphi}(Z)$ is confluent.

(*d*): $\Pi_{\varphi}(Z)$ is convergent.

Moreover, these hold if the following holds, and the converse is true if Z has at least 3 elements.

(*e*): $B(B(\mu, \nu), \omega) \downarrow_{\Pi_{\varphi}(\mu,\nu,\omega)} B(\mu, B(\nu, \omega))$ under the rewriting system $\Pi_{\varphi}(\mu, \nu, \omega)$ for $\mathbf{k}\|\mu, \nu, \omega\|$.

A Global Characterization for Rota-Baxter Type

- Let **k** be a field. Let $B(x, y) \in \mathbf{k}\|x, y\|$ be in RBNF and totally linear in x, y. Let $\varphi(x, y) := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x, y) \rfloor$. Let $Z' = \{ \mu, \nu, \omega \}$.
- Suppose for **every** well-ordered set $(Z, \leqslant_Z)$, there is a monomial order with which φ is compatible (also denoted by $\leqslant_Z$) extending $\leqslant_Z$ from Z to $\mathfrak{M}(Z)$. Then the following four statements are equivalent.
  - (a) φ is a Rota-Baxter type OPI.
  - (b) The two expressions $B(B(\mu, \nu), \omega)$ and $B(\mu, B(\nu, \omega)) \in \mathbf{k}\|Z'\|$ are joinable under $\Pi_\varphi(Z')$.
  - (c) $\Pi_\varphi(Z')$ is convergent.
  - (d) $\Pi_\varphi(Z)$ is convergent for every well-ordered set Z.

Monomial Order on $\mathfrak{M}(Z)$

- A **monomial order on $\mathfrak{M}(Z)$** is a **well order** $\leqslant$ on $\mathfrak{M}(Z)$ such that:

$$u < v \Longrightarrow q|_u < q|_v, \quad \text{for all } u, v \in \mathfrak{M}(Z) \text{ and all } q \in \mathfrak{M}^\star(Z).$$

- Given a monomial order $\leqslant$, and $f \in \mathbf{k}\|Z\|$, $f \notin \mathbf{k}$, the notions of **leading bracketed monomial** $\overline{f}$ and **leading coefficient** $c(f)$ are clear. The **remainder** $R(f)$ is defined as $f - c(f)\overline{f}$.
- If $f \in \mathbf{k}$ (including $f = 0$), we define the leading monomial of f to be $\overline{f} = 1 \in \mathfrak{M}(Z)$, the leading coefficient to be $c(f) = f$, and the remainder is $R(f) = 0$.
- We say f is **monic** with respect to $\leqslant$ if $c(f) = 1$. A subset $S \subset \mathbf{k}\|Z\|$ is **monic** if every $s \in S$ is.

TRS on Free Operated Algebras For Subsets

- Let $\leqslant$ be a monomial order on $\mathfrak{M}(Z)$. Let $S \subset \mathbf{k}\|Z\|$ be monic with respect to $\leqslant$.
- We associate a term-rewriting system $\Pi_{S,\leqslant}(Z)$ on $\mathbf{k}\|Z\|$ with basis $\mathfrak{M}(Z)$:

$$\Pi_{S,\leqslant}(Z) := \{(q|_{\bar{s}}, q|_{-R(s)}) \mid s \in S,\ q \in \mathfrak{M}^\star(Z)\} \subseteq \|Z\| \times \mathbf{k}\|Z\|.$$

- We will often abbreviate $\Pi_{S,\leqslant}(Z)$ to Π_S, and denote the relation by $\to_S$ and its reflexive transitive closure by $\xrightarrow{*}_S$.
- If a rule $(q|_{\bar{s}}, q|_{-R(s)}) \in \Pi_S$ is used to reduce $f \in \mathbf{k}\|Z\|$ to $g \in \mathbf{k}\|Z\|$ in one step, we shall write $f \xrightarrow{q,s}_S g$, in which case, $f - g = q|_s$, and $f \in \mathrm{Id}(S)$ if and only if $g \in \mathrm{Id}(S)$, where $\mathrm{Id}(S)$ is the operated ideal of $\mathbf{k}\|Z\|$ generated by S.

Compatible Rewriting Systems

- Let Π be a rewriting system on $V = \mathbf{k} \cdot \mathfrak{M}(Z)$ with basis $W = \mathfrak{M}(Z)$. We say Π is **compatible with the monomial order $\leqslant$ on $\mathfrak{M}(Z)$** if $\overline{v} < t$ for all $(t, v) \in \Pi$.
- Every rewriting system $\Pi_{\leqslant,S}(Z)$ is by definition compatible with respect to $\leqslant$.
- Let $\varphi = \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x,y) \rfloor \in \mathbf{k}\|x,y\|$. We say φ is **compatible with $\leqslant$ on $\mathfrak{M}(Z)$** if the rewriting system $\Pi_{\varphi}(Z)$ is compatible with $\leqslant$ on $\mathfrak{M}(Z)$, or equivalently, if the rewriting subsystem $\Pi_{\varphi}^{\dagger}(Z)$ is, or equivalently, if $\overline{\lfloor B(u,v) \rfloor} < \lfloor u \rfloor \lfloor v \rfloor$ for every $u, v \in \mathfrak{M}(Z)$.
- **If φ is compatible with $\leqslant$, then $\Pi_{\varphi}(Z) = \Pi_{\leqslant,S}(Z)$, with $S = \{\, \varphi(u,v) \mid u, v \in \mathfrak{M}(Z) \,\}$.** If $s = \varphi(u,v)$, then $f \xrightarrow{q,u,v}_{\varphi} g$ if and only if $f \xrightarrow{q,s}_{S} g$. Hence $f \in \mathbf{k}\|Z\|$ is Π_{φ}-reducible if and only if f is Π_S-reducible. The set $\mathrm{Irr}(S)$ of Π_S-irreducible monomials is precisely the set $\mathfrak{R}(Z)$ of monomials in RBNF.

The Rewriting System Π_S is Terminating

Let $\leqslant$ be a monomial order on $\mathfrak{M}(Z)$ for a set Z. Let $S \subset \mathbf{k}\|Z\|$ be monic and let $f \in \mathbf{k}\|Z\|$.

- For the rewriting system Π_S, we define the **leading S-reducible monomial of f** to be the monomial $L(f)$ maximal with respect to $\leqslant$ among monomials m appearing in f that are not in $\mathrm{Irr}(S)$, that is,

$$L(f) := \max\{m \mid m \in \mathrm{Supp}(f),\ m \notin \mathrm{Irr}(S)\}.$$

- **Lemma:** Suppose $g, g' \in \mathbf{k}\|Z\|$ are both S-reducible and for some $q \in \mathfrak{M}^{\star}(Z)$ and $s \in S$, $g \xrightarrow{(q,s)}_S g'$. Then $L(g') \leqslant L(g)$, where equality holds if and only if $L(g) \neq q|_{\bar{s}}$.
- **Theorem:** The rewriting system Π_S is terminating. In particular, the rewriting system $\Pi_{\varphi}(Z)$ is terminating if $\varphi := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x,y) \rfloor \in \mathbf{k}\|x, y\|$ is compatible with $\leqslant$.

Descent of Leading Term After Rewriting

The following are more refined versions of the previous Lemma.

- **Lemma:** Let $g \in \mathbf{k}\|Z\|$ be Π_S-reducible and for some $s \in S$, $q \in \mathfrak{M}^\star(Z)$, suppose $g \xrightarrow{(q,s)}_S g' \in \mathbf{k}\|Z\|$. Then $\overline{g'} \leqslant \overline{g}$, where equality holds if and only if $q|_{\overline{s}} < \overline{g}$.

- **Corollary:** Let $\varphi = \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x,y) \rfloor \in \mathbf{k}\|Z\|$ be compatible with $\leqslant$ on $\mathfrak{M}(Z)$. Let $g \in \mathbf{k}\|Z\|$ be φ-reducible and let $u, v \in \mathfrak{M}(Z)$, $q \in \mathfrak{M}^\star(Z)$ and $g' \in \mathbf{k}\|Z\|$ be such that $g \xrightarrow{q,u,v}_\varphi g'$. Then $\overline{g'} \leqslant \overline{g}$, where equality holds if and only if $q|_{\lfloor u \rfloor \lfloor v \rfloor} < \overline{g}$.

Restriction to a Rewritng Subsystem

- Let W be a subset of $\mathfrak{M}(Z)$ and let V be the free $\mathbf{k}$-submodule of $\mathbf{k}\|Z\|$ with basis W. We say the rewriting system $\Pi_\varphi(Z)$ on $\mathbf{k}\|Z\|$ **restricts to** a rewriting system Π on V with basis W if for all $f \in V$ and all triples (q, u, v) for f, we have $f \xrightarrow{q,u,v}_\varphi g$ implies $g \in V$.
- A necessary and sufficient condition that $\Pi_\varphi(Z)$ restricts to Π is that for all $f \in V$ and all triples (q, u, v) of f, $q|_{\lfloor u\rfloor\lfloor v\rfloor} \in W$ implies $q|_{\lfloor B(u,v)\rfloor} \in V$.
- A similar definition can be given for the more general TRS $\Pi_S(Z)$ for a subset $S \subset \mathbf{k}\|Z\|$.

A Crucial Lemma on Confluence

- Let Z be a set, let $\leqslant$ be a monomial order on $\mathfrak{M}(Z)$, let $Y \subset \mathfrak{M}(Z)$ be monic and let $\varphi(x, y) := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x, y) \rfloor \in \mathbf{k}\|x, y\|$ be compatible with $\leqslant$ on $\mathfrak{M}(Z)$.

- Suppose the rewriting system $\Pi_\varphi(Z)$ restricts to a rewriting system $\Pi_{\varphi,Y}$ on $\mathbf{k}\cdot Y$ with basis Y and suppose $\Pi_{\varphi,Y}$ is confluent.

- For $1 \leqslant i \leqslant n$, let $c_i \in \mathbf{k}$, $q_i \in \mathfrak{M}^\star(Z)$ and $s_i \in S_\varphi(Z)$ be such that $q_i|_{\overline{s_i}} \in Y$ and $\sum_{i=1}^n c_i q_i|_{\overline{s_i} - s_i} = 0$.

- **Lemma:** Then $\sum_{i=1}^n c_i q_i|_{\overline{s_i}} \xrightarrow{*}_\varphi 0$.

Basics on Gröbner-Shirshov Basis

- Let $f, g \in \mathbf{k}\|Z\|$ be two distinct bracketed polynomials, each monic with respect to $\leqslant$. Let $\overline{f}$ be the leading monomial of f, and let $|\overline{f}|$ be the breadth of $\overline{f}$.
- If there exist $\mu, \nu, b \in \mathfrak{M}(Z)$ such that $b = \overline{f}\mu = \nu\overline{g}$ with $\max\{|\overline{f}|, |\overline{g}|\} < |b| < |\overline{f}| + |\overline{g}|$, we call the operated polynomial

$$(f, g)_b^{\mu,\nu} := f\mu - \nu g$$

the **intersection composition of f and g with respect to (μ, ν)**.
- If there exist $q \in \mathfrak{M}^\star(Z)$ and $b \in \mathfrak{M}(Z)$ such that $b = \overline{f} = q|_{\overline{g}}$, we call the operated polynomial

$$(f, g)_b^q := f - q|_g$$

the **including composition of f and g with respect to q**.

Gröbner-Shirshov Basis

- Let $b \in \mathfrak{M}(Z)$, let S be a set of monic bracketed polynomials in $\mathbf{k}\|Z\|$, and let $\mathrm{Id}(S)$ be the bracketed ideal generated by S.
- An operated polynomial $f \in \mathbf{k}\|Z\|$ is called **trivial modulo S with bound b** or, in short, **trivial modulo (S, b)** if $f \in \mathrm{Id}(S)$ and can be expressed as a sum $\sum_i c_i q_i|_{s_i}$ where $0 \neq c_i \in \mathbf{k}$, $q_i \in \mathfrak{M}^\star(Z)$, $s_i \in S$ and $q_i|_{\overline{s_i}} < b$.
- The polynomial 0, expressed as the empty sum, is trivial modulo (S, b) for any S and b.
- A set $S \subseteq \mathbf{k}\|Z\|$ of monic bracketed polynomials is called a **Gröbner-Shirshov basis with respect to $\leqslant$** if, for all pairs $f, g \in S$ with $f \neq g$, every intersection composition of the form $(f, g)_b^{\mu,\nu}$ is trivial modulo S with bound b, and every including composition of the form $(f, g)_b^q$ is trivial modulo S with bound b.

The Composition-Diamond Lemma

Let $\eta\colon \mathbf{k}\|Z\| \to \mathbf{k}\|Z\|/\mathrm{Id}(S)$ be the canonical homomorphism of $\mathbf{k}$-modules. Then the following statements are equivalent.

(a) : S is a Gröbner-Shirshov basis in $\mathbf{k}\|Z\|$.

(b) : For every non-zero $f \in \mathrm{Id}(S)$, $\overline{f} = q|_{\overline{s}}$ for some $q \in \mathfrak{M}^\star(Z)$ and some $s \in S$.

(c) : For every non-zero $f \in \mathrm{Id}(S)$, f can be expressed in **triangular form**:

$$f = c_1 q_1|_{s_1} + c_2 q_2|_{s_2} + \cdots + c_n q_n|_{s_n}, \qquad (1)$$

where $c_i \in \mathbf{k}$ $(c_i \neq 0)$, $s_i \in S$, $q_i \in \mathfrak{M}^\star(Z)$ for $1 \leqslant i \leqslant n$, and

$$q_1|_{\overline{s_1}} > q_2|_{\overline{s_2}} > \cdots > q_n|_{\overline{s_n}}.$$

(d) : As $\mathbf{k}$-modules, $\mathbf{k}\|Z\| = \mathbf{k}\cdot\mathrm{Irr}(S) \oplus \mathrm{Id}(S)$ and $\eta(\mathrm{Irr}(S))$ is a $\mathbf{k}$-basis of $\mathbf{k}\|Z\|/\mathrm{Id}(S)$.

Second Local Characterization Theorem

Let Z be a set, and let $\leqslant$ be a monomial order on $\mathfrak{M}(Z)$. Let $\varphi(x,y) := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x,y) \rfloor \in \mathbf{k}\|x,y\|$ with $B(x,y)$ in RBNF and totally linear in x, y be compatible with $\leqslant$. Then the following conditions are equivalent.

- (a) : The rewriting system $\Pi_\varphi(Z)$ is convergent.
- (b) : With respect to $\leqslant$, the set $S := S_\varphi(Z)$ is a Gröbner-Shirshov basis in $\mathbf{k}\|Z\|$.

Construction of Free φ-Algebras

Let Z be a set, let $\leqslant$ be a monomial order on $\mathfrak{M}(Z)$, and let $\varphi(x,y) := \lfloor x\rfloor\lfloor y\rfloor - \lfloor B(x,y)\rfloor \in \mathbf{k}\|x,y\|$ be of Rota-Baxter type and compatible with $\leqslant$. Let $S = S_\varphi(Z)$. Then the following holds:

- Recall that $\mathfrak{R}(Z) \subset \mathfrak{M}(Z)$ is the set of monomials in RBNF and $\eta\colon \mathbf{k}\|Z\| \to \mathbf{k}\|Z\|/\mathrm{Id}(S) =: \mathbf{k}_\varphi\|Z\|$ is the canonical homomorphism of $\mathbf{k}$-modules.
- The composition $\eta\circ\iota\colon \mathbf{k}\cdot\mathfrak{R}(Z) \xrightarrow{\iota} \mathbf{k}\cdot\mathfrak{M}(Z) \equiv \mathbf{k}\|Z\| \xrightarrow{\eta} \mathbf{k}_\varphi\|Z\|$ is an isomorphism of operated $\mathbf{k}$-modules, taking the $\mathbf{k}$-basis $\mathfrak{R}(Z)$ of $\mathbf{k}\cdot\mathfrak{R}(Z)$ to a $\mathbf{k}$-basis of $\mathbf{k}_\varphi\|Z\|$ and P_r the operator induced by P, that is, $P_r(f + \mathrm{Id}(S)) = P(f) + \mathrm{Id}(S)$.
- Let $\alpha : \mathbf{k}_\varphi\|Z\| \to \mathbf{k}\cdot\mathfrak{R}(Z)$ be the inverse of $\eta\circ\iota$ and let $\rho := \rho_\varphi$ be the composition $\mathbf{k}\|Z\| \xrightarrow{\eta} \mathbf{k}_\varphi\|Z\| \xrightarrow{\alpha} \mathbf{k}\cdot\mathfrak{R}(Z)$. For $u, v \in \mathfrak{R}(Z)$, let $u = u_1\cdots u_s$ and $v = v_1\cdots v_t$ be standard decompositions.

Construction of Diamond Product

- Let $\diamond$ be the multiplication on $\mathbf{k}\cdot\mathfrak{R}(Z)$ that is uniquely determined by bilinearity and the products $u \diamond v$, where
  1. $1 \diamond v = v$ and $u \diamond 1 := u$, if either $u = 1$ or $v = 1$ (or both) where 1 is the empty word in $\mathfrak{M}(Z)$;
  2. $u \diamond v := uv$ if either $u \in S(Z)$ or $v \in S(Z)$ (or both);
  3. $u \diamond v := \lfloor \rho(B(u^*, v^*)) \rfloor$ if $u = \lfloor u^* \rfloor$ and $v = \lfloor v^* \rfloor$ are both in $\lfloor \mathfrak{R}(Z) \rfloor$;
  4. $u \diamond v := u_1 \cdots u_{s-1}(u_s \diamond v_1)v_2 \cdots v_t$ if either $s > 1$, or $t > 1$ (or both). Here the multiplications are concatenations, except for $u_s \diamond v_1$, where $\diamond$ is defined recursively by Step 2 or 3.

  With the above construction, $(\mathbf{k}\cdot\mathfrak{R}(Z), \diamond, P_r)$ is a free φ-algebra on the set Z.

Monomial order on $\mathfrak{M}(Z)$

We construct a monomial order on $\mathfrak{M}(Z)$ inductively in several steps. Recall that $\mathfrak{M}(Z)$ is the direct limit of $\iota_n : \mathfrak{M}_n(Z) \to \mathfrak{M}_{n+1}(Z)$.

- The **degree lexicographical order $\leqslant_{\rm dlex}$ on $M(Z)$**.
- A well-order $\leqslant_n$ on $\mathfrak{M}_n(Z)$ for every non-negative integer n which is induced by $\leqslant_Z$ on $\mathfrak{M}_n(Z)$ and the preorders induced by degree and breadth.
- The **degree-breadth lexicographic order $\leqslant_{{\rm db},Z}$ is the direct limit of $\leqslant_n$**.
- **Lemma**. $\leqslant_{db,Z}$ is a well order and the restriction of $\leqslant_Z$ to $\mathfrak{M}_n(Z)$ is $\leqslant_n$.
- **Theorem**. $\leqslant_{{\rm db},Z}$ is a monomial order on $\mathfrak{M}(Z)$.

Degree Lexicographic Order $\leqslant_{\rm dlex}$ on $M(Z)$

Let Z be a set with a well order $\leqslant_Z$ and $M(Z)$ the free monoid on Z.

- For $u = u_1 \cdots u_r \in M(Z)$ with $u_1, \cdots, u_r \in Z$, define $\deg_Z(u) = r$ if $u \neq 1$ and $\deg_Z(1) = 0$.
- Define the **degree lexicographical order** $\leqslant_{\rm dlex}$ on $M(Z)$ by taking, for any $u = u_1 \cdots u_r, v = v_1 \cdots v_s \in M(Z) \backslash \{1\}$, where $u_1, \cdots, u_r, v_1, \cdots, v_s \in Z$,

$$u \leqslant_{\rm dlex} v \Leftrightarrow \begin{cases} \deg_Z(u) < \deg_Z(v), \\ \text{or } \deg_Z(u) = \deg_Z(v)(= r) \text{ and } u_1 \cdots u_r \leqslant_{\rm lex} v_1 \cdots \end{cases}$$

where $\leqslant_{\rm lex}$ is the lexicographical order on $M(Z)$, with the convention that the empty word $1 \leqslant_{\rm dlex} u$ for all $u \in M(Z)$.

Lemma. If $\leqslant_Z$ is a well order on Z, then $\leqslant_{\rm dlex}$ is a well order on $M(Z)$.

Preorders and Pre-linear Orders

Let Y be a nonempty set.

- A **preorder** or **quasiorder** $\leqslant_Y$ on Y is a binary relation that is reflexive and transitive, that is, for all $x, y, z \in Y$, we have
  1. $x \leqslant_Y x$; and
  2. if $x \leqslant_Y y, y \leqslant_Y z$, then $x \leqslant_Y z$.

- We denote $x =_Y y$ if $x \leqslant_Y y$ and $y \leqslant_Y x$. If $x \leqslant_Y y$ but $x \neq_Y y$, we write $x <_Y y$ or $y >_Y x$.

- A **pre-linear order** $\leqslant_Y$ on Y is a preorder $\leqslant_Y$ such that either $x \leqslant_Y y$ or $y \leqslant_Y x$ (or both) for all $x, y \in Y$.

P-degree and P-breadth: Examples of Preorders

- Let $u = u_0\lfloor u_1^*\rfloor u_1\lfloor u_2^*\rfloor \cdots \lfloor u_r^*\rfloor u_r \in \mathfrak{M}(Z)$, where $u_0, u_1, \cdots, u_r \in M(Z)$ and $u_1^*, u_2^*, \cdots, u_r^*, \in \mathfrak{M}(Z)$.
- Same for $v = v_0\lfloor v_1^*\rfloor v_1\lfloor v_2^*\rfloor \cdots \lfloor v_s^*\rfloor v_s \in \mathfrak{M}(Z)$.
- Define

$$u \leqslant_{\mathrm{dgp}} v \Leftrightarrow \deg_P(u) \leqslant \deg_P(v), \tag{2}$$

where the ***P*-degree** $\deg_P(u)$ of u is the number of occurrences of $P = \lfloor\ \rfloor$ in u.
- Define

$$u \leqslant_{\mathrm{brp}} v \Leftrightarrow r \leqslant s \quad (\text{that is } |u|_P \leqslant |v|_P), \tag{3}$$

where $|u|_P := r$ is the ***P*-breadth** of u.
- **Lemma**. $\leqslant_{\mathrm{dgp}}$ and $\leqslant_{\mathrm{brp}}$ are pre-linear orders satisfying the descending chain condition on $\mathfrak{M}(Z)$.

Isomorphisms between Ordered Free Monoids

Proposition. For every $n \geqslant 0$, we can construct a well order $\leqslant_n$ on $\mathfrak{M}_n(Z)$ depending only on $\leqslant_Z$ such that the natural embeddings ι_n induces an isomorphism of ordered free monoids between $(\mathfrak{M}_n(Z), \leqslant_n)$ and $(\mathrm{Im}(\iota_n), \leqslant'_{n+1})$, where $\leqslant'_{n+1}$ is the restriction of $\leqslant_{n+1}$ to $\mathrm{Im}(\iota_n)$.

- For natural numbers $m \geqslant 0$, let

$$\mathfrak{M}^m(Z) := \{u \in \mathfrak{M}(Z) \;\big|\; |u|_P = m\}$$

and let

$$\mathfrak{M}_n^m(Z) := \mathfrak{M}_n(Z) \cap \mathfrak{M}^m(Z).$$

- Then $\mathfrak{M}_n(Z)$ is the infinite disjoint union $\bigcup_{m=0}^{\infty} \mathfrak{M}_n^m(Z)$.

Linear Order $\leqslant_{\mathrm{lex}_n}$ on $\mathfrak{M}_n(Z)$

- Define a relation $\leqslant_{\mathrm{lex}_n^m}$ on $\mathfrak{M}_n^m(Z)$ by $u \leqslant_{\mathrm{lex}_n^m} v \Leftrightarrow$

$$(u_1^*, u_2^* \cdots , u_m^*, u_0, \cdots , u_m) \leqslant_{\mathrm{clex}_{n,m}} (v_1^*, v_2^*, \cdots , v_m^*, v_0, \cdots , v_m).$$

- Then $\leqslant_{\mathrm{lex}_n^m}$ is a well order on $\mathfrak{M}_n^m(Z)$ for all $m \geqslant 0$.
- Now $\mathfrak{M}_n(Z) = \bigcup_{m=0}^{\infty} \mathfrak{M}_{m=0}^{\infty}$, hence we may define a relation $\leqslant_{\mathrm{lex}_n}$ on $\mathfrak{M}_n(Z)$ by $u \leqslant_{\mathrm{lex}_n} v \Leftrightarrow$

$$\begin{cases} u <_{\mathrm{brp}} v, \\ \text{or } u =_{\mathrm{brp}} v \text{ and } u \leqslant_{\mathrm{lex}_n^m} v, \ \text{where } m = |u|_P = |v|_P. \end{cases}$$

- $\leqslant_{\mathrm{lex}_n}$ a linear order on $\mathfrak{M}_n(Z)$, and in fact, it is a well order.

Well Order $\leqslant_n$ on $\mathfrak{M}_n(Z)$ and $\leqslant_{\mathrm{db},Z}$ on $\mathfrak{M}(Z)$

- Finally, define the relation $\leqslant_n$ on $\mathfrak{M}_n(Z)$ by:

$$u \leqslant_n v \Leftrightarrow \begin{cases} u <_{\mathrm{dgp}} v, \\ \text{or } u =_{\mathrm{dgp}} v \text{ and } u \leqslant_{\mathrm{lex}_n} v, \end{cases} \tag{4}$$

where we have restricted $\leqslant_{\mathrm{dgp}}$ to $\mathfrak{M}_n(Z)$.

- We prove by induction on n that

$$u \leqslant_n v \iff \iota_n(u) \leqslant'_{n+1} \iota_n(v)$$

for all $n \geqslant 0$, and so the induced map $\mathfrak{M}_n(Z) \to \mathrm{Im}(\iota_n)$ is an isomorphism between ordered free monoids.

- $\{(\mathfrak{M}_n(Z), \leqslant_n)\}_{n=0}^{\infty}$ as a filtration of ordered free monoids for $(\mathfrak{M}(Z), \leqslant_{\mathrm{db}})$, where $\mathfrak{M}(Z) = \bigcup_{n=0}^{\infty} \mathfrak{M}_n(Z)$ and $\leqslant_{\mathrm{db}} = \varinjlim \leqslant_n$. More specifically, for $u, v \in \mathfrak{M}(Z)$,

$$u \leqslant_{\mathrm{db},Z} v := u \leqslant_n v \text{ for any } n \text{ such that } u, v \in \mathfrak{M}_n(Z). \tag{5}$$

$\leqslant_{\mathrm{db},Z}$ is a Monomial Order

- A well order $\leqslant$ on $\mathfrak{M}(Z)$ is called **bracket compatible** if $u \leqslant v \Rightarrow \lfloor u \rfloor \leqslant \lfloor v \rfloor$ for all $u, v, w \in \mathfrak{M}(Z)$.
- A well order $\leqslant$ on $\mathfrak{M}(Z)$ is called **left (multiplicatively) compatible** if $u \leqslant v \Rightarrow wu \leqslant wv$ for all $u, v, w \in \mathfrak{M}(Z)$.
- A well order $\leqslant$ on $\mathfrak{M}(Z)$ is called **right (multiplicatively) compatible** if $u \leqslant v \Rightarrow uw \leqslant vw$ for all $u, v, w \in \mathfrak{M}(Z)$.
- **Lemma**. A well order $\leqslant$ is a monomial order on $\mathfrak{M}(Z)$ if and only if $\leqslant$ is bracket, left, and right compatible.
- **Theorem**. Let $(Z, \leqslant_Z)$ be a well-ordered set. The order $\leqslant_{\mathrm{db},Z}$ is a monomial order on $\mathfrak{M}(Z)$, extending $\leqslant_Z$.

Equivalences for Compatibility with $\leqslant_{\mathrm{db},Z}$

Proposition. Let $\varphi(x,y) = \lfloor x\rfloor\lfloor y\rfloor - \lfloor B(x,y)\rfloor \in \mathbf{k}\|x,y\|$ and suppose $B(x,y)$ is totally linear and in RBNF. TFAE:

(a) For **every** well-ordered set $(Z, \leqslant_Z)$, the rewriting system $\Pi_\varphi(Z)$ is compatible with $\leqslant_{\mathrm{db},Z}$ on $\mathfrak{M}(Z)$.

(b) For **some** well-ordered set $(Z', \leqslant_{Z'})$ (possibly the empty set), the rewriting system $\Pi_\varphi(Z')$ is compatible with $\leqslant_{\mathrm{db},Z'}$.

(c) The total P-degree of $B(x,y)$ is $\leqslant 1$.

(d) $B(x,y)$ has the form:

$$a_1 y\lfloor x\rfloor + a_2 x\lfloor y\rfloor + a_3\lfloor y\rfloor x + a_4\lfloor x\rfloor y + a_5\lfloor yx\rfloor + a_6\lfloor xy\rfloor$$

$$+a_7 y\lfloor 1\rfloor x + a_8 x\lfloor 1\rfloor y + a_9 yx\lfloor 1\rfloor + a_{10} xy\lfloor 1\rfloor + a_{11}\lfloor 1\rfloor yx + a_{12}\lfloor 1\rfloor xy$$

$$+a_{13} yx + a_{14} xy$$

where $a_j \in \mathbf{k}$ $(1 \leqslant j \leqslant 14)$.