Rational and Algebraic General Solutions of First-Order Algebraic ODEs

N. Thieu Vo

RISC, Johannes Kepler University, Linz, Austria
KSDA Seminar, The Graduate Center, CUNY, New York

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Contents

Contents

Objects

- $\mathbb{K} := \overline{\mathbb{Q}}$, or an (computable) algebraically closed field.
- First-order AODE: $F(x, y, y') = 0$,
- $F \in \mathbb{K}[x, y, z] \setminus \mathbb{K}[x, y]$ irreducible.
- In $\mathbb{K}(x)\{y\}$, we have

$$\{F\} = \left(\{F\} : \frac{\partial F}{\partial y'}\right) \cap \left\{F, \frac{\partial F}{\partial y'}\right\}$$

  where the first component is a prime differential ideal.
- A general solution := a generic zero of $\left(\{F\} : \frac{\partial F}{\partial y'}\right)$.

Objects

- Let C be the field of constants of the universal extension of $\mathbb{K}$.
- An algebraic solution is a solution in $\overline{C(x)}$.
- Algebraic general solution = algebraic solution + general solution.
- A rational solution is a solution in $C(x)$.
- Rational general solution = rational solution + general solution.
- Example: $y(x) = \frac{1}{x-c}$ is a rational general solution of the Riccati equation $y' + y^2 = 0$.

The problems and results

Problem

Check the existence of a rational/algebraic general solution of the AODE $F(x, y, y') = 0$, and compute it in the affirmative case.

Results

- An algorithm for computing a rational general solution of the form $y(x, c) \in \mathbb{K}(x, c)$.
- A procedure for finding an algebraic general solution of a parametrizable first-order AODE.

Recent works

- 2004 R. Feng and X.S. Gao: Rat. gen. sol.s of $F(y, y') = 0$
- 2005 J.M. Aroca et. al.: Alg. gen. sol.s of $F(y, y') = 0$
- 2005 G. Chen and Y. Ma: Rat. gen. sol.s of $F(x, y, y') = 0$ without movable critical points
- 2010 L.X.C. Ngô and F. Winkler: Rat. gen. sol.s of $F(x, y, y') = 0$ with parametrizable condition.
- 2013 L.X.C. Ngô and F. Winkler: extended to higher order $F(x, y, \ldots, y^{(m)}) = 0$
- 2015 G. Grasseger and F. Winkler: Symbolic solutions of $F(x, y, y') = 0$ and $F(u, u'_x, u'_y) = 0$.

Contents

The AODE as an algebraic curve

- Consider the AODE $F(x, y, y') = 0$
- The algebraic curve $\mathcal{C}$ in $\mathbb{A}^2(\overline{\mathbb{K}(x)})$ defined by $F(x, y, z) = 0$ is called the corresponding curve.
- A parametrization of $\mathcal{C}$ is a rational map

$$\mathcal{P} = (p_1, p_2) : \mathbb{A}^1(\overline{\mathbb{K}(x)}) \to \mathcal{C} \subset \mathbb{A}^2(\overline{\mathbb{K}(x)})$$

  for some $p_1, p_2 \in \overline{\mathbb{K}(x)}(t)$, such that $\mathrm{Im}\,\mathcal{P}$ is dense in $\mathcal{C}$.
- If furthermore $\mathcal{P}$ is a birational equivalence, then it is called a proper parametrization.
- $\mathcal{C}$ has a parametrization iff $\mathcal{C}$ is of genus 0[1]

[1] Sendra, Winkler, Pére-Díaz. Rational Algebraic Curves: A Computer Algebra Approach

Recall $F \in \mathbb{K}[x,y,z] \setminus \mathbb{K}[x,y]$ irreducible.

Theorem

If the DE $F(x,y,y')=0$ has a rational solution $y = y(x,c)$ in $\mathbb{K}(x,c) \setminus \mathbb{K}(x)$, then

- F is still irreducible over $\overline{\mathbb{K}(x)}$, and
- $\mathcal{C}$ is of genus 0, and moreover, $\mathcal{C}$ can be parametrized by a pair of $(p_1, p_2) \in \mathbb{K}(x,t)^2$.

- **Remark:** A solution of the form $y(x,c) \in \mathbb{K}(x,c) \setminus \mathbb{K}(x)$ is a rational general solution. But the converse is not always true.
- **Example:** The AODE

$$x^3 y'^3 - (3x^2 y - 1) y'^2 + 3xy^2 y' - y^3 - 1 = 0$$

has a rational general solution $y(x) = cx + (c^2+1)^{\frac{1}{3}}$, and the corresponding curve has genus 1.

Definition

- A rational solution of the form $y(x,c) \in \mathbb{K}(x,c) \setminus \mathbb{K}(x)$ is called a strong rational general solution.
- A parametrization of the corresponding curve with coefficients in $\mathbb{K}(x)$ is called a strong parametrization.
- A first-order AODE is called strongly parametrizable if its corresponding curve admits a strong parametrization.

- Almost all first-order AODEs from Kamke collection are strongly parametrizable.

Contents

Construction of the associated differential equation

- Consider $F(x, y, y') = 0$, strongly parametrizable,
- Assume that $\mathcal{P} = (p_1, p_2) \in \mathbb{K}(x, t)^2$ is given,
- And that $y(x) \in \mathbb{K}(x)$ is a rational solution.
- In generic, $(y(x), y'(x)) = \mathcal{P}(\omega(x))$ for some $\omega(x) \in \overline{\mathbb{K}(x)}$.
- Then $p_2(x, \omega(x)) = \frac{d}{dx} p_1(x, \omega(x))$.
- Therefore $\omega(x)$ is a solution of the first-order first-degree AODE

$$\omega' = \frac{p_2 - \frac{\partial p_1}{\partial x}}{\frac{\partial p_1}{\partial t}} \tag{1}$$

- (1) is called the associated DE of the AODE $F(x, y, y') = 0$ w.r.t $\mathcal{P}$.

Theorem

There is a one-to-one correspondence between rational general solutions of the AODE $F(x, y, y') = 0$ and its associated DE. In particular,

- If $y(x) \in C(x)$ is a rational general solution of $F(x, y, y') = 0$, then $\omega(x) = \mathcal{P}^{-1}(y(x), y'(x))$ is a rational general solution of the associates DE.
- Conversely, if $\omega(x) \in C(x)$ is a rational general solution of the associated DE, then $y(x) = p_1(x, \omega(x))$ is a rational general solution of $F(x, y, y') = 0$.

Theorem[2]

If a first-order first-degree AODE has a rational general solution, then it must be a linear or a Riccati equation[3].

Remaining question: Given $F(x, y, y') = 0$, how to check the strongly parametrizability, and compute a strong parametrization in the affirmative case?

[2] Behloul, Cheng. 2011. Computation of rational solutions for a first-order nonlinear differential equation. EJDE.

[3] Kovacic. 1986. An algorithm for solving second order linear homogeneous differential equations. JSC.

Contents

Optimal Parametrization Problem

- Let $\mathbb{F}$ be a field of char 0, $\overline{\mathbb{F}}$ its algebraic closure,
- Consider an algebraic curve $\mathcal{C}$ of genus 0 in $\mathbb{A}^2(\overline{\mathbb{F}})$ defined by $G(y,z)=0$ for some $G \in \mathbb{F}[y,z]$ irreducible.
- A proper parametrization of $\mathcal{C}$ is a pair $\mathcal{P}=(p_1(t),p_2(t)) \in \overline{\mathbb{F}}(t)^2$.
- The field extended from $\mathbb{F}$ by coefficients of p_1, p_2 is called the field of the parametrization $\mathcal{P}$.

Problem

Determine a proper parametrization such that the field of the parametrization is as smallest as possible.

Optimal Parametrization Problem

- Done in case $\mathbb{F} = \mathbb{Q}$. [4]
- Done in case $\mathbb{F} = \mathbb{Q}(x)$. [5]
- Done in case $\mathbb{F} = \mathbb{K}(x)$: a straight-forward generalization from the case $\mathbb{F} = \mathbb{Q}(x)$.

[4] Sendra, Winkler, Pérez-Díaz. 2008. Rational Algebraic Curves. A Computer Algebra Approach.

[4] Hillgarter, Winkler. 1997. Points on algebraic curves and the parametrization problem. LNCS.

Contents

Require: $F(x, y, y') = 0$, irreducible.

Ensure: A strong rational general solution, or "No strong rational general solution exists".

1: Compute the genus g of the corresponding curve.

2: If $g = 0$, compute an optimal parametrization $(p_1, p_2) \in \overline{\mathbb{K}(x)}(t)^2$.

3: If $p_1, p_2 \in \mathbb{K}(x, t)$, then compute $f = \frac{p_2 - \frac{\partial p_1}{\partial x}}{\frac{\partial p_1}{\partial x}}$.

4: If $f \in \mathbb{K}(x)[t]$ and $\deg_t f \leq 2$, solve the DE $\omega' = f(x, \omega)$.

5: If $\omega(x)$ is a rational general solution, then return $y(x) = p_1(x, \omega(x))$.

6: If one of the above steps is fail, return "No strong rational general solution exists".

Algorithm 1: Strong rational general solution of first-order AODEs

Example

Example 1.537 in Kamke collection[6]:

- $F(x, y, y') = x^3 y'^3 - 3x^2 y y'^2 + (x^6 + 3xy^2)y' - y^3 - 2x^5 y = 0.$
- A strong parametrization of the corresponding curve:

$$\mathcal{P}(t) = \left(-\frac{x^5 t^3 - x^6 t^2 + (t-x)^3}{x^5 t^3}, -\frac{2x^5 t^3 - 2x^6 t^2 + (t-x)^3}{x^6 t^3} \right)$$

- The associated DE: $\omega' = \frac{1}{x^2}\omega(2\omega - x)$.
- Solve the associated DE: $\omega(x) = \frac{x}{1+cx^2}$.
- A rational general solution: $y(x) = cx(x + c^2)$.

[1] E. Kamke. 1997. Differentialgleichungen: Lösungsmethoden und Lösungen.

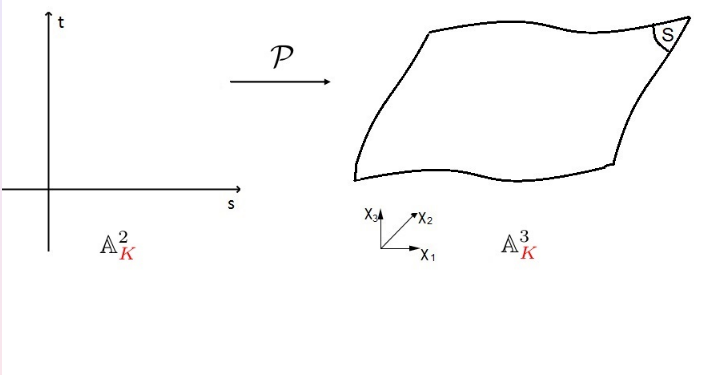
Contents

The AODE as an algebraic surface

- Consider $F(x, y, y') = 0$, where F is irreducible.
- $F(x, y, z) = 0$ defines an algebraic surface, say $\mathcal{S}$, in $\mathbb{A}^3(\mathbb{K})$.
- We assume that $\mathcal{S}$ is birational equivalent to $\mathbb{A}^2(\mathbb{K})$, and a birational map $\mathcal{P} = (\chi_1, \chi_2, \chi_3) \in \mathbb{K}(s, t)^3$ is given.[7]
- $\mathcal{S}$ is called the corresponding surface, and $F(x, y, y') = 0$ is called a parametrizable first-order AODE.

[6] J. Schicho. 1998. Rational Parametrization of Surfaces. JSC.

Contents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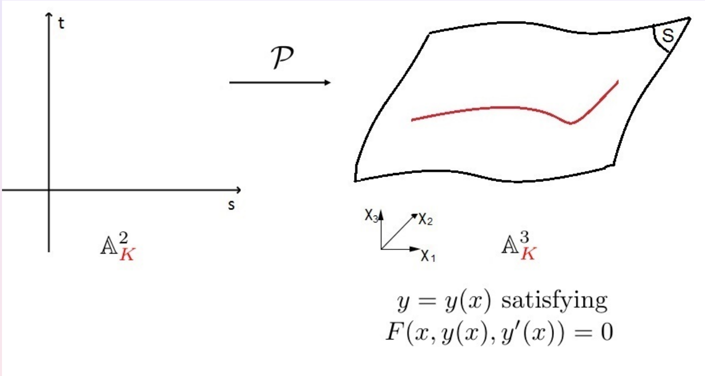
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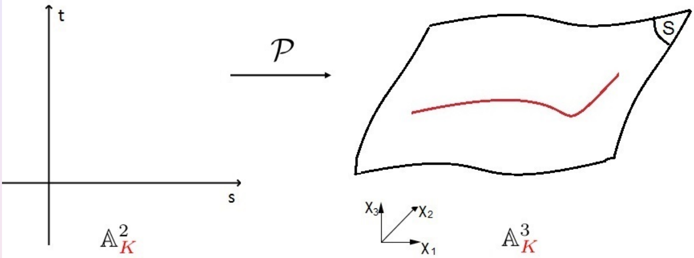
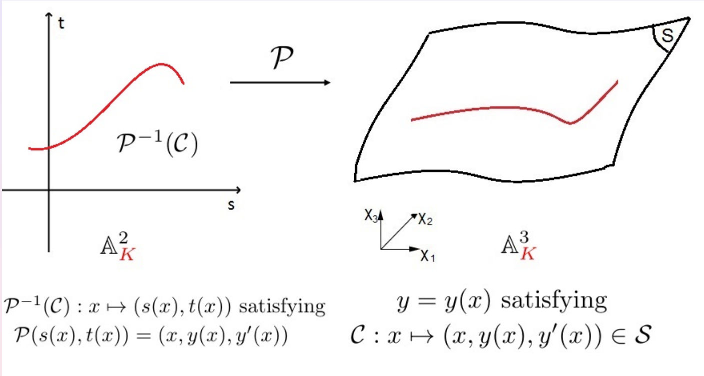


$y = y(x)$ satisfying

$$\mathcal{C} : x \mapsto (x, y(x), y'(x)) \in \mathcal{S}$$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$\mathcal{P}(s(x), t(x)) = (x, y(x), y'(x))$
$y = y(x)$ satisfying
$\mathcal{C} : x \mapsto (x, y(x), y'(x)) \in \mathcal{S}$

$$\mathcal{P}(s(x), t(x)) = (x, y(x), y'(x))$$

$$\Rightarrow \begin{cases} \chi_1(s(x), t(x)) = x \\ \chi_2'(s(x), t(x)) = \chi_3(s(x), t(x)) \end{cases}$$

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$$\mathcal{P}(s(x), t(x)) = (x, y(x), y'(x))$$

$$\Rightarrow \begin{cases} s'(x)\chi_{1s}(s(x), t(x)) + t'(x)\chi_{1t}(s(x), t(x)) = 1 \\ s'(x)\chi_{2s}(s(x), t(x)) + t'(x)\chi_{2t}(s(x), t(x)) = \chi_3(s(x), t(x)) \end{cases}$$

$\Rightarrow (s(x), t(x))$ is an algebraic solution of the system

$$\begin{cases} s' = \frac{f_1(s,t)}{g(s,t)} \\ t' = \frac{f_2(s,t)}{g(s,t)} \end{cases}$$

where

$g := \chi_{1s}\chi_{2t} - \chi_{2s}\chi_{1t}$, $f_1 := -\chi_{2t} + \chi_3\chi_{1t}$, $f_2 := -\chi_{1s}\chi_3 - \chi_{2s}$.

$\Rightarrow (s(x), t(x))$ is an algebraic solution of the system

$$\begin{cases} s' = \frac{f_1(s,t)}{g(s,t)} \\ t' = \frac{f_2(s,t)}{g(s,t)} \end{cases} \tag{2}$$

where

$g := \chi_{1s}\chi_{2t} - \chi_{2s}\chi_{1t}$, $f_1 := -\chi_{2t} + \chi_3\chi_{1t}$, $f_2 := -\chi_{1s}\chi_3 - \chi_{2s}$.

Definition

The system (2) is called the associated system of the differential equation $F(x, y, y') = 0$.

Theorem

There is a one-to-one correspondence between algebraic general solutions of $F(x, y, y') = 0$ and its associated system. In particular,

- If $y(x)$ is an algebraic general solution of $F(x, y, y') = 0$, then
$$(s(x), t(x)) := \mathcal{P}^{-1}(x, y(x), y'(x))$$
is an algebraic general solution of the associated system.
- If $(s(x), t(x))$ is an algebraic general solution of the associated system, then
$$y(x) := \chi_2(s(2x - \chi_1(s(x), t(x))), t(2x - \chi_1(s(x), t(x))))$$
is an algebraic general solution of $F(x, y, y') = 0$.

Procedure
Input: $F(x, y, y') = 0$.
Output: An algebraic general solution $y(x)$ if any.

1. Compute a rational parametrization
2. Compute the associated system
3. Find an algebraic general solution $(s(x), t(x))$ of the associated system
4. Return $y(x)$ as in the previous theorem.

Example:

- $F = 4x(x-y)y'^2 + 2xyy' - 5x^2 + 4xy - y^2 = 0$
- A parametrization of the corresponding surface:

$$\mathcal{P}(s,t) = \left(s, -\frac{t^2 - 5ts + 5s^s}{s}, -\frac{t^2 - 4st + 5s^2}{2s(t-2s)}\right)$$

- The associated system:

$$\begin{cases} s' = 1 \\ t' = \frac{t^2 - 3s^2}{2s(t-2s)} \end{cases}$$

- Solving the system yields $(s(x), t(x)) = (x, t(x,c))$, where $t(x,c)$ is a root of the equation $T^2 - 4xT + 3x^2 - cx = 0$.
- Solution: $y(x) = \frac{1}{c}(\sqrt{cx(cx+1)} - 1)$.

Example:

- $F = 4x(x-y)y'^2 + 2xyy' - 5x^2 + 4xy - y^2 = 0$
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Contents

Planar rational systems

- Planar rational system:

$$\begin{cases} s' = M(s,t) \\ t' = N(s,t) \end{cases} \quad (3)$$

where $M = \frac{M_1}{M_2}, N = \frac{N_1}{N_2} \in \mathbb{K}(s,t)$.

- A rational first integral is a function $R(s,t) \in \mathbb{K}(s,t) \setminus \mathbb{K}$ such that $M.R'_s + N.R'_t = 0$.
- An algebraic curve defined by $G(s,t) = 0$ is an invariant algebraic curve if $M_1 N_2 G'_s + M_2 N_1 G'_t = GH$ for some $H \in \mathbb{K}[s,t]$.

Some remarks

- If $(s(x), t(x)) \in \overline{\mathbb{K}(x)}^2$ is an algebraic solution of (3), and $G \in \mathbb{K}[s, t]$ is an irreducible polynomial such that $G(s(x), t(x)) = 0$, then $G(s, t) = 0$ defines an irreducible invariant algebraic curve.
- If (3) has "enough" number of irreducible invariant algebraic curves, then it admits a rational first integral.
- If $F(x, y, y') = 0$ has an algebraic general solution, then the associated system has a rational first integral.

The Process

Find an algebraic general solution of $F(x, y, y') = 0$

$\Uparrow$

Find an explicit alg. gen. sol. $(s(x), t(x))$ of the associated system

$\Uparrow$

Find an implicit alg. gen. solution $G(s, t) = 0$

$\equiv$ Find all irr. invariant algebraic curves

$\Updownarrow$

Find a rational first integral

Remaining Problems

Given a planar rational system.

- **Problem 1 (Poincaré Problem)**: Find a bound N for the degree of a rational first integral (or irr. invariant algebraic curves).
- **Problem 2**: Given N, check the existence of a rational first integral of degree at most N, and compute it in the affirmative case.

Remaining Problems

Given a planar rational system.

- **Problem 1 (Poincaré Problem)**: Find a bound N for the degree of a rational first integral.
  Current status: open
- **Problem 2**: Given N, check the existence of a rational first integral of degree at most N, and compute it in the affirmative case.
  Current status: Effective algorithms established[8][9]

[8] Bostan et. al.. 2013. Efficient algorithm for computing rational first integral and Darboux polynomials of planar vector fields. Arxiv.

[9] Prelle, Singer. 1983. Elementary first integrals of differential equations. Trans. AMS.

Poincaré Problem

- 1870. Darboux: proposed an algebraic method to solve first order first degree AODEs over $\mathbb{P}^2(\mathbb{C})$; proved the existence of a degree bound for irreducible invariant algebraic curves.
- 1979. Jouanolou: In generic case, a planar rational system has no invariant algebraic curve. But it is "hard" to find such an example.
- 1983. Prelle, Singer: procedure to find an elementary/Liouvilian first integral.
- 1994. Carnicer: A degree bound for invariant algebraic curves without dicritical singularities.

Conclusion

We proposed:

- An algorithm for computing a strong rational general solution of a first-order AODE.
- An algorithm for computing a rational general solution of a strongly parametrizable first-order AODE.
- A procedure for computing an algebraic general solution of a parametrizable first-order AODE.

Still open:

- An algorithm for finding a rational/algebraic general solution of first-order AODEs.