Sparse Differential Resultant for Laurent Differential Polynomials

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Motivations

- Sparse Resultant is a basic concept in algebraic geometry and a powerful tool in algebraic elimination theory with important applications.
- Differential polynomials from practice are usually sparse, and their differential resultant may vanish identically.
- **Sparse differential resultant** is not studied before.

Work on Sparse Resultant

- Gelfand, Kapranov, and Zelevinsky (1991, 1994) introduced the sparse resultant.
- Sturmfels (1993, 1994) proved basic properties for the sparse resultant.
- Canny and Emiris (1993, 1995, 2000) gave matrix formulas for sparse resultants and efficient algorithms.
- D'Andrea (2002) proved a sparse resultant is the quotient of two determinants.

Work on Differential Resultant

- Ritt (1932): Differential resultant for two differential polynomials in one diff indeterminate.
- Ferro (1997): Differential resultant as algebraic Macaulay resultant.

  Ferrro's resultant of two generic diff polynomials of degree larger than one is always zero.
- Chardin (1991): Resultant for differential operators.
- Rueda-Sendra (2010): Differential resultant of a linear system.
- Gao, Li, Yuan (2010): Rigorous definition of differential resultant of $n+1$ differential polynomials in n indeterminates.

Outline of the Talk

- Sparse diff resultant for Laurent diff polynomials
- Properties of sparse diff resultant
- Criterion for Laurent diff essential system in terms of supports
- A single exponential algorithm to compute sparse diff resultant
- Sparse diff resultant for diff polynomial with non-vanishing degree zero terms
- Summary

Sparse Differential Resultant for Laurent Differential Polynomials

Notations

Ordinary differential field: $(\mathcal{F}, \delta)$. e.g. $(\mathbf{Q}(x), \frac{d}{dx})$

Universal differential field of $\mathcal{F}$: $(\mathcal{E}, \delta)$.

Diff Indeterminates: $\mathbb{Y} = \{y_1, \ldots, y_n\}$.

Notation: $y_i^{(k)} = \delta^k y_i$, $\mathbb{Y}^{[t]} = \{y_i^{(k)} : k \leq t\}$.
Differential Monomial: $M = \prod_{k=1}^{n} \prod_{l=0}^{o} (y_k^{(l)})^{d_{kl}}$ with $d_{kl} \in \mathbb{Z}_{\geq 0}$;
$\mathfrak{m}$: set of diff monomials in $\mathbb{Y}$.

Differential polynomial ring: $\mathcal{F}\{\mathbb{Y}\} = \mathcal{F}[y_i^{(k)} : k \geq 0]$.

Notations: For $S \subset \mathcal{F}\{\mathbb{Y}\}$,

$[S]$ = the diff ideal generated by S

$[S] : \mathfrak{m} = \{f \in \mathcal{F}\{\mathbb{Y}\} \,|\, \exists M \in \mathfrak{m}, \text{ s.t. } M \cdot f \in [S]\}$.

Laurent Differential Polynomial Ring

- **Laurent Diff Monomial:** $M = \prod_{k=1}^{n} \prod_{l=0}^{o} (y_k^{(l)})^{d_{kl}}$ with $d_{kl} \in \mathbb{Z}$;
- **Laurent Diff Polynomial:** $f = \sum_{k=1}^{m} a_k M_k$, M_k Laurent diff monomials.

  Support of f: $\mathcal{A} = \{M_1, \ldots, M_m\}$.

  Norm Form of f: $f^N = Mf$, where M is the denominator of f.

  Laurent Diff Polynomials Ring: $\mathcal{F}\{\mathbb{Y}, \mathbb{Y}^{-1}\}$.

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- $\mathcal{E}$: universal diff field of $\mathcal{F}$.

  $\mathcal{E}^{\wedge} \triangleq \mathcal{E} \backslash \{a \in \mathcal{E} : \exists k \geq 0 \text{ s.t. } a^{(k)} = 0\}$.

  Non-polynomial Diff Solution: $\xi \in (\mathcal{E}^{\wedge})^n$ s.t. $f(\xi) = 0$.

Sparse Laurent Diff Polynomials

$\mathcal{A} = \{M_1, \ldots, M_m\}$: finite set of Laurent Diff monomials.

- **Notation:** $\mathcal{L}(\mathcal{A}) = \{\sum_{k=1}^{m} b_k M_k : b_k \in \mathcal{E}\}$.
- **Sparse Laurent Diff Polynomial w.r.t. $\mathcal{A}$:** $f \in \mathcal{L}(\mathcal{A})$.
- **Generic Sparse Laurent Diff Polynomial w.r.t. $\mathcal{A}$:**

  $\mathbb{P} \in \mathcal{L}(\mathcal{A})$ with coefficients diff independent over $\mathbb{Q}$.

Given $\mathcal{A}_i$ $(i = 0, \ldots, n)$ and $f_i \in \mathcal{L}(\mathcal{A}_i)$, we ask:

> *f_i have a common non-polynomial solution*
>
> $\Longleftrightarrow$ **?**

Sparse Differential Resultant

- **Generic Sparse Laurent Diff polynomials w.r.t. $\mathcal{A}_i$:**
  $\mathbb{P}_i = \sum_{j=0}^{l_i} u_{ij} M_{ij} \in \mathcal{L}(\mathcal{A}_i)$ with coefficients $\mathbf{u}_i$.

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  $\mathbb{P}_i = \sum_{j=0}^{l_i} u_{ij} M_{ij} \in \mathcal{L}(\mathcal{A}_i)$ with coefficients $\mathbf{u}_i$.
- **Sparse Differential Resultant Exists**:
  $[\mathbb{P}_0, \ldots, \mathbb{P}_n] \cap \mathbf{Q}\{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n\} = \mathbf{sat}(R(\mathbf{u}_0, \ldots, \mathbf{u}_n))$ is of codimension 1

Definition

R is defined to be the **Sparse Differential Resultant** of $\mathbb{P}_i$, denoted by $\mathrm{Res}_{\mathbb{P}_0,\ldots,\mathbb{P}_n}$, or $\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}$.

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$\Leftrightarrow$ $\mathbb{P}_i$ **are Laurent differentially essential:**

There exist k_i $(i = 0, \ldots, n)$ with $1 \le k_i \le l_i$ such that

$$\text{d.tr.deg}\, \mathbf{Q}\langle \frac{M_{0k_0}}{M_{00}}, \frac{M_{1k_1}}{M_{10}}, \ldots, \frac{M_{nk_n}}{M_{n0}} \rangle / \mathbf{Q} = n.$$

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Facts:

Laurent diff essential does not depend on M_{i0}:

Lemma

There exist (k_i, j_i) *with* $k_i \neq j_i$ *s.t.* $\mathrm{d.tr.deg}\, \mathbf{Q}\langle \frac{M_{0k_0}}{M_{0j_0}}, \ldots, \frac{M_{nk_n}}{M_{nj_n}} \rangle / \mathbf{Q} = n$.

$\Longleftrightarrow$ *There exist* k_i *with* $1 \leq k_i \leq l_i$ *s.t.* $\mathrm{d.tr.deg}\, \mathbf{Q}\langle \frac{M_{0k_0}}{M_{00}}, \ldots, \frac{M_{nk_n}}{M_{n0}} \rangle / \mathbf{Q} = n$.

Definition for Sparse Diff Resultant relies on the fact:

Theorem

- $[\mathbb{P}_0^N, \ldots, \mathbb{P}_n^N] : \mathrm{m}$ *is a prime differential ideal in* $\mathbf{Q}\{\mathbb{Y}, \mathbf{u}_0, \ldots, \mathbf{u}_n\}$.
- $([\mathbb{P}_0^N, \ldots, \mathbb{P}_n^N] : \mathrm{m}) \cap \mathbf{Q}\{\mathbf{u}_0, \ldots, \mathbf{u}_n\}$ *is of codimension 1 if and only if* $\mathbb{P}_0, \ldots, \mathbb{P}_n$ *are Laurent diff essential.*
- $[\mathbb{P}_0, \ldots, \mathbb{P}_n] \cap \mathbf{Q}\{\mathbf{u}_0, \ldots, \mathbf{u}_n\} = ([\mathbb{P}_0^N, \ldots, \mathbb{P}_n^N] : \mathrm{m}) \cap \mathbf{Q}\{\mathbf{u}_0, \ldots, \mathbf{u}_n\}$.

A Property on Differential Specialization

Key fact used in the proof:

Theorem

$\mathbb{U} = (u_1, \ldots, u_r)$ and $\mathbb{Y} = (y_1, \ldots, y_n)$: sets of diff indeterminates.

$P_i(\mathbb{U}, \mathbb{Y}) \in \mathcal{F}\langle\mathbb{Y}\rangle\{\mathbb{U}\}$ $(i = 1, \ldots, m)$.

If $P_i(\mathbb{U}, \mathbb{Y})$ are diff dependent over $\mathcal{F}\langle\mathbb{U}\rangle$, then for any specialization $\mathbb{U}$ to $\overline{\mathbb{U}} \subset \mathcal{F}$ over $\mathcal{F}$, $P_i(\overline{\mathbb{U}}, \mathbb{Y})$ are diff dependent over $\mathcal{F}$.

Examples

Example (1)

$\mathbb{P}_0 = u_{00} + u_{01} y_1 y_2$

$\mathbb{P}_1 = u_{10} + u_{01} y_1' y_2'$

$\mathbb{P}_2 = u_{20} + u_{21} y_1' y_2.$

$\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2$ form a Laurent diff essential system.

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Using diff characteristic set method, we can compute

$\mathbf{Res}_{\mathbb{P}_0,\mathbb{P}_1,\mathbb{P}_2} =$

$-u_{11} u_{20}^2 u_{01}^2 - u_{01} u_{00} u_{21}^2 u_{10} + u_{01} u_{11} u_{20} u_{21} u_{00}' - u_{11} u_{20} u_{00} u_{21} u_{01}'.$

Examples

Example (2)

$n = 1$ and $\mathcal{A}_0 = \mathcal{A}_1 = \{y, y', y^2\}$.

$\mathcal{A}_0, \mathcal{A}_1$ are Laurent diff essential.

$$\begin{aligned}\mathrm{Res}_{\mathcal{A}} = & -u_{12}u_{01}u_{00}u_{10} - u_{12}u_{01}^2u_{10}' + u_{12}u_{01}u_{11}'u_{00} + u_{12}u_{01}u_{11}u_{00}' \\ & -u_{11}u_{02}u_{00}u_{10} + u_{11}u_{02}u_{10}'u_{01} + u_{02}u_{01}u_{10}^2 - u_{11}^2u_{02}u_{00}' \\ & +u_{11}u_{02}u_{01}'u_{10} + u_{11}u_{00}^2u_{12} + u_{11}^2u_{02}'u_{00} - u_{11}u_{02}'u_{01}u_{10} \\ & -u_{11}u_{01}u_{12}'u_{00} + u_{01}^2u_{12}'u_{10} - u_{11}u_{01}'u_{12}u_{00} - u_{11}'u_{02}u_{01}u_{10}.\end{aligned}$$

Basic Properties of Sparse Differential Resultant

Properties of Sparse Differential Resultant

Theorem (Order and Diff homogeneity)

- *Diff resultant is diff homogeneous in each* $\mathbf{u}_i$ *and is of order* $h_i = s - s_i$ *in* $\mathbf{u}_i$ $(i = 0, \ldots, n)$ *where* $s = \sum_{l=0}^{n} s_l$.
- *Sparse diff resultant is diff homogeneous in each* $\mathbf{u}_i$ *and is of order* $h_i \leq s - s_i$ *in* $\mathbf{u}_i$.

Degree Bound will be given later!

Conditions for Existence of Non-polynomial Solutions

- $\mathcal{A}_0, \ldots, \mathcal{A}_n$: Laurent diff essential;

  $(F_0, \ldots, F_n) \in \mathcal{L}(\mathcal{A}_0) \times \cdots \times \mathcal{L}(\mathcal{A}_n)$: $F_i = \sum_{j=0}^{l_i} v_{ij} M_{ij}$.
- $\mathcal{Z}_0(\mathcal{A}_0, \ldots, \mathcal{A}_n)$: set of F_i having a common **non-polynomial solution**.
- $\mathcal{Z}(\mathcal{A}_0, \ldots, \mathcal{A}_n)$: Kolchin diff closure of $\mathcal{Z}_0(\mathcal{A}_0, \ldots, \mathcal{A}_n)$.

Theorem

$$\mathcal{Z}(\mathcal{A}_0, \ldots, \mathcal{A}_n) = \mathbb{V}\big(\mathbf{sat}(\mathrm{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n})\big).$$

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Example

The need to consider non-polynomial solutions:

Example

$n = 2$, and $(\mathcal{F}, \delta) = (\mathbf{Q}(x), \frac{d}{dx})$.

$\mathbb{P}_i = u_{i0} y_1'' + u_{i1} y_1''' + u_{i2} y_2''' \, (i = 0, 1, 2)$.

$\mathbb{P}_i$ form a Laurent diff essential system for $\mathrm{d.tr.deg}\, \mathbf{Q}\langle \frac{y_1'''}{y_1''}, \frac{y_2'''}{y_1''} \rangle / \mathbf{Q} = 2$.

The sparse differential resultant is

$$\mathrm{Res}_{\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2} = \begin{vmatrix} u_{00} & u_{01} & u_{02} \\ u_{10} & u_{11} & u_{12} \\ u_{20} & u_{21} & u_{22} \end{vmatrix} \neq 0.$$

But $\xi = (c_{11}x + c_{10}, c_{22}x^2 + c_{21}x + c_{20}) \notin (\mathcal{E}^\wedge)^2$ is a non-zero solution of $\mathbb{P}_i = 0 \, (i = 0, 1, 2)$ where c_{ij} are distinct arbitrary constants.

Condition for Existence of Unique Non-polynomial Solutions

Assume 1) Any n of the $\mathcal{A}_i$ diff independent: $\mathbf{ord}(R, \mathbf{u}_i) \geq 0$ for each i.

2) $\mathbf{e}_j \in \mathrm{Span}_{\mathbb{Z}}\{\alpha_{ij} - \alpha_{i0}\}$.

Theorem

$(\mathbb{P}_i, \mathbf{u}_i)$ specialize to $(\overline{\mathbb{P}}_i, \mathbf{v}_i)$.

- *If $R(\mathbf{v}_0, \ldots, \mathbf{v}_n) = 0$ and $\frac{\partial R}{\partial u_{ik}^{(h_i)}}(\mathbf{v}_0, \ldots, \mathbf{v}_n) \neq 0$, then*

  $\overline{\mathbb{P}}_i = 0$ have at most a unique common non-polynomial solution.

- *$\mathcal{Z}_1(\mathcal{A}_0, \ldots, \mathcal{A}_n)$: set of $(\mathbf{v}_0, \ldots, \mathbf{v}_n)$ s.t. $\overline{\mathbb{P}}_i$ have a unique non-polynomial solution. Then*

  $\overline{\mathcal{Z}_1(\mathcal{A}_0, \ldots, \mathcal{A}_n)} = \mathbb{V}\big(\mathbf{sat}(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n})\big)$

Differential Toric Variety

- $\mathcal{A} = \{M_0, \ldots, M_l\}$: $\text{d.tr.deg}\,\mathbf{Q}\langle \frac{M_1}{M_0}, \ldots, \frac{M_l}{M_0} \rangle / \mathbf{Q} = n$.
- $\mathbf{P}(l)$: l-dimensional diff projective space.
- Consider

$$\begin{array}{rccc} \phi_{\mathcal{A}}: & (\mathcal{E}^{\wedge})^n & \longrightarrow & \mathbf{P}(l) \\ & \xi & & (M_0(\xi), M_1(\xi), \ldots, M_l(\xi)) \end{array}$$

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Definition

The Kolchin closure of $\phi_{\mathcal{A}}((\mathcal{E}^{\wedge})^n)$ is defined to be the *diff toric variety w.r.t.* $\mathcal{A}$, denoted by $X_{\mathcal{A}}$. That is, $X_{\mathcal{A}} = \overline{\phi_{\mathcal{A}}((\mathcal{E}^{\wedge})^n)}$.

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Theorem

$X_{\mathcal{A}}$ is an irreducible projective diff variety of dim n over $\mathbf{Q}$.

The Relation of Diff Toric variety and Diff Chow Form

- **Laurent Diff Polynomial:**

$$\mathbb{P}_i = u_{i0}M_0 + u_{i1}M_1 + \cdots + u_{il}M_l$$

- **Generic Projective Diff Hyperplane in P(l):**

$$\mathbb{L}_i = u_{i0}z_0 + u_{i1}z_1 + \cdots + u_{il}M_l$$

Theorem

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Example (Continue from Example 2)

In this example, $X_{\mathcal{A}} = \mathbb{V}(\mathbf{sat}(z_1z_2 - (z_0z_2' - z_0'z_2)))$. And $\mathrm{Res}_{\mathcal{A}}$ is the diff Chow form of $X_{\mathcal{A}}$.

Poisson-Type Product Formula

- Algebraic Resultant: $\text{R}es(A(x), B(x)) = c \prod_{\eta, B(\eta)=0} A(\eta)$.

Poisson-Type Product Formula

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- Differential Resultant:
  $\delta\mathbf{Res}(\mathbf{u}_0, \ldots, \mathbf{u}_n) = A(\mathbf{u}_0, \ldots, \mathbf{u}_n)\prod_{\tau=1}^{t_0} \mathbb{P}_{\mathbf{0}}(\eta_{\tau\mathbf{1}}, \ldots, \eta_{\tau\mathbf{n}})^{(\mathbf{h_0})}$.
  And $(\eta_{\tau 1}, \ldots, \eta_{\tau n})$ are generic points of $[\mathbb{P}_1, \ldots, \mathbb{P}_n]$.

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  And $(\eta_{\tau 1}, \ldots, \eta_{\tau n})$ are generic points of $[\mathbb{P}_1, \ldots, \mathbb{P}_n]$.
- Sparse Differential Resultant:
  $\mathbf{Res}(\mathbf{u}_0, \ldots, \mathbf{u}_n) = A\prod_{\tau=1}^{t_0}(u_{00} + \sum_{k=1}^{l_0} u_{0k}\xi_{\tau k})^{(h_0)}$.

Possion Type Product Formula (Extended)

Theorem

When 1) Any n of the $\mathcal{A}_i$ diff independent and

2) $\mathbf{e}_j \in \mathrm{Span}_{\mathbb{Z}}\{\alpha_{ij} - \alpha_{i0}\}$,

the result can be strengthened:

$$\mathrm{Res}(\mathbf{u}_0, \ldots, \mathbf{u}_n) = A \prod_{\tau=1}^{t_0} \left(\frac{\mathbb{P}_{\mathbf{0}}(\eta_{\tau\mathbf{1}}, \ldots, \eta_{\tau\mathbf{n}})}{\mathbf{M}_{\mathbf{00}(\eta_{\tau\mathbf{1}}, \ldots, \eta_{\tau\mathbf{n}})}}\right)^{(\mathbf{h_0})}.$$

And $\eta_\tau = (\eta_{\tau 1}, \ldots, \eta_{\tau n})$ *lie on* $\mathbb{P}_1, \ldots, \mathbb{P}_n$.

Furthermore, they are generic points of $[\mathbb{P}_0^N, \ldots, \mathbb{P}_n^N] : \mathrm{m}$.

Representation of the Sparse Resultant

- Algebraic Resultant:

$$\mathrm{Res}(\mathrm{A}(\mathrm{x}), \mathrm{B}(\mathrm{x})) = \mathrm{A}(\mathrm{x})\mathrm{T}(\mathrm{x}) + \mathrm{B}(\mathrm{x})\mathrm{W}(\mathrm{x}),$$

  where $\mathbf{deg}(T) < \mathbf{deg}(B), \mathbf{deg}(W) < \mathbf{deg}(A)$.

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  where $h_i \leq s - s_i$.

Degree bound for this linear combination will be given later.

Criterion for Laurent Diff Essential in terms of Supports

Basic Notions

$B_i = \prod_{j=1}^{n} \prod_{k=0}^{s} (y_j^{(k)})^{d_{ijk}}$ $(i = 1, \ldots, m)$: Laurent Diff Monomials.

- New algebraic indeterminates: $x_1, \ldots, x_n$.
- $d_{ij} = \sum_{k=0}^{s} d_{ijk} x_j^k$ $(i = 1, \ldots, m; j = 1, \ldots, n)$.
- **Symbolic Support Vector (SSV) of B_i**: $\beta_i = (d_{i1}, \ldots, d_{in})$.
- **Symbolic Support Matrix (SSM) of $B_1, \ldots, B_m$**:

$$\mathbf{M} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & \ldots & d_{1n} \\ d_{21} & d_{22} & \ldots & d_{0n} \\ & & \ddots & \\ d_{m1} & d_{m2} & \ldots & d_{mn} \end{pmatrix}$$

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$$\mathbf{M} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & \ldots & d_{1n} \\ d_{21} & d_{22} & \ldots & d_{0n} \\ & & \ddots & \\ d_{m1} & d_{m2} & \ldots & d_{mn} \end{pmatrix}$$

Def: Call $B_1, \ldots, B_m$ **Reduced** if $-\infty \neq \mathbf{deg}(d_{ii}) > \mathbf{deg}(d_{ji})(j > i)$ for each $i \leq \min(m, n)$.

Basic Notions

$B_i = \prod_{j=1}^{n} \prod_{k=0}^{s} (y_j^{(k)})^{d_{ijk}}$ $(i = 1, \ldots, m)$: Laurent Diff Monomials.

- New algebraic indeterminates: $x_1, \ldots, x_n$.
- $d_{ij} = \sum_{k=0}^{s} d_{ijk} x_j^k$ $(i = 1, \ldots, m; j = 1, \ldots, n)$.
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Lemma

$B_1, \ldots, B_m$ Reduced. Then $\text{d.tr.deg}\, \mathbf{Q}\langle B_1, B_2, \ldots, B_m\rangle / \mathbf{Q} = \min(m, n)$.

Laurent Diff monomials in T-shape

Def: Call $B_1, \ldots, B_m$ **in T-shape with index** (i, j) if there exist (i, j) s.t.

1) $(M_1)_{i\times i}$ and $(M_2)_{(n-i)\times j}$ are reduced sub-matrices,
2) Z_1 and Z_2 constitute an $(m-i)\times(n-j)$ zero matrix.

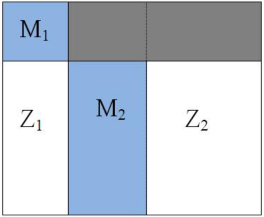

Figure: T-shape Matrix

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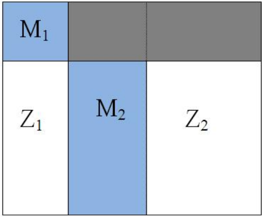

Figure: T-shape Matrix

Theorem

$B_1, \ldots, B_m$ *in T-shape.* $\text{d.tr.deg}\,\mathbf{Q}\langle B_1, B_2, \ldots, B_m\rangle/\mathbf{Q} = \text{rk}(M) = i + j.$

General Laurent Diff Monmials

- M: **SSM** of arbitrary $B_1, \ldots, B_m$.
- **Q-elementary Transformations:** row(column) interchanging, and adding a **Q**-multiple of one row to another.

Theorem

M can be reduced to a T-shape matrix by finite **Q***-elementary transformations.*

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Q-elementary transformations keep diff transcendence degree:

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Generic Laurent Diff Polynomials

$$\mathbb{P}_i = \sum_{j=0}^{l_i} u_{ij} M_{ij} \; (i = 1, \ldots, m).$$

- **SSV of M_{ij}/M_{i0}:** β_{ij}

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- **SSV of $\mathbb{P}_i$:** $\beta_i = \sum_{j=0}^{l_i} u_{ij}\beta_{ij}$.
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Theorem

1) $\mathrm{d.tr.deg}\, \mathbf{Q}\langle\hat{\mathbf{u}}\rangle\langle\frac{\mathbb{P}_1}{M_{10}}, \ldots, \frac{\mathbb{P}_m}{M_{m0}}\rangle/\mathbf{Q}\langle\hat{\mathbf{u}}\rangle = \mathrm{rk}(\mathbf{M}_{\mathbb{P}})$ $(\hat{\mathbf{u}} = \cup_{i=1}^{m}\mathbf{u}_i)$.

2) $\mathrm{Codim}([\mathbb{P}_1^N, \ldots, \mathbb{P}_m^N] : \mathrm{m}) \cap \mathbf{Q}\{\mathbf{u}_0, \ldots, \mathbf{u}_n\} = n + 1 - \mathrm{rk}(M_{\mathbb{P}})$.

Criterion for Laurent Diff Essential

$$\mathbb{P}_i = \sum_{j=0}^{l_i} u_{ij} M_{ij} \, (i = 0, \ldots, n).$$

Theorem

The $\mathbb{P}_i$ are Laurent differentially essential

$\Longleftrightarrow \mathrm{rk}(\mathbf{M}_{\mathbb{P}}) = n$

$\Longleftrightarrow$ *there exist k_i $(1 \leq k_i \leq l_i)$ s.t.* $\mathrm{rk}(\mathbf{M}_{k_0,\ldots,k_n}) = n$ *where* $\mathbf{M}_{k_0,\ldots,k_n}$ *is the* **SSM** for $M_{0k_0}/M_{00}, \ldots, M_{nk_n}/M_{n,0}$.

Remark: Here, a diff problem is translated to a linear algebraic one.

A Single Exponential Algorithm to Compute Sparse Diff Resultant

Degree of Sparse Differential Resultant

Laurent Diff Essential System: $\mathbb{P}_0, \ldots, \mathbb{P}_n$

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R : the sparse resultant of $\mathbb{P}_0, \ldots, \mathbb{P}_n$.

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3. $\prod_{i=0}^{n} N_{i0}^{(h_i+1)\mathbf{deg}(R)} \cdot R = \sum_{i=0}^{n} \sum_{j=0}^{h_i} G_{ij} \left(\mathbb{P}_i^N\right)^{(j)}$ *where* $\mathbf{deg}(G_{ij}(\mathbb{P}_i^N)^{(j)}) \le [m + 1 + \sum_{i=0}^{n} (h_i + 1)\mathbf{deg}(N_{i0})]\mathbf{deg}(R)$.

Key Ingredients to Prove the Theorem

$$R \in [\mathbb{P}_0^N, \ldots, \mathbb{P}_n^N] : \mathfrak{m} \cap \mathbf{Q}\{\mathbf{u}_0, \ldots, \mathbf{u}_n\}$$ Differential

$$\Downarrow 1)$$

$$(R) = (\mathbb{P}_i^{(k)} : k \le h_i) : \mathfrak{m}^{[h]} \cap \mathbf{Q}[U] \text{ where } h_i \le s - s_i$$ Algebraic

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- Key facts used in 2):
  - Bézout Inequality: $\mathbf{deg}(V) \le \prod_{i=1}^{r} \mathbf{deg}(f_i)$ for each component V of $\mathbb{V}(f_1, \ldots, f_r)$.
  - Degree of Elimination Ideal: $\mathbf{deg}(\mathcal{I}_k) \le \mathbf{deg}(\mathcal{I})$.

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An Algorithm for Sparse Differential Resultant

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1. Search for $R(\mathbf{u}_0,\ldots,\mathbf{u}_n)$
   with order $h_i = 0,\ldots,s-s_i$ and
   with degree from $D = 1,\ldots,\prod_{i=0}^{n}(m_i+1)^{h_i+1}$.
2. With fixed h_i and D, computing coefficients of R and G_{ik} by solving linear equations raising from

$$\prod_{i=0}^{n} N_{i0}^{(h_i+1)D} R(\mathbf{u}_0,\ldots,\mathbf{u}_n) = \sum_{i=0}^{n}\sum_{k=0}^{h_i} G_{ik}\mathbb{P}_i^{(k)}.$$

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Theorem (Computing Complexity)

$O(((n+1)(s+1))^{O(ls)}(m+1)^{O(nls^2)})$ $\mathbf{Q}$*-arithmetic operations.*

n: number of variables; s: order of system; l: **size of sparse system**

Sparse diff Resultant for Diff polynomials with Non-vanishing Degree Zero Terms

Diff Essential System and its Sparse Diff Resultant

- **Sparse Diff Polynomials:** $\mathbb{P}_i = u_{i0} + \sum_{j=1}^{l_i} u_{ij} M_{ij}\ (i = 0, \ldots, n)$.
- **Diff Essential System:** Laurent Diff Essential.

Resultant here has additional better properties.

Theorem

$(\mathbb{P}_i, \mathbf{u}_i)$ *specialize to* $(\overline{\mathbb{P}_i}, \mathbf{v}_i)$.

- $\mathcal{Z}_2$*: set of* $(\mathbf{v}_0, \ldots, \mathbf{v}_n)$ *s.t.* $\overline{\mathbb{P}_i}$ *have a common solution. Then* $\overline{\mathcal{Z}_2} = \mathbb{V}(\mathbf{sat}(\mathrm{Res}))$.
- *Assume 1) Any n of the* $\mathbb{P}_i$ *diff independent and 2)* $\mathbf{e}_j \in \mathrm{Span}_{\mathbb{Z}}\{\alpha_{ij}\}$. *If* $R(\mathbf{v}_0, \ldots, \mathbf{v}_n) = 0$ *and* $\frac{\partial R}{\partial u_{ik}^{(h_i)}}(\mathbf{v}_0, \ldots, \mathbf{v}_n) \neq 0$*, then*

  $\overline{\mathbb{P}}_i = 0$ *have a unique common solution.*

Theorem

$\mathbb{P}_0, \ldots, \mathbb{P}_n$: *Diff essential system.*

$\mathbf{ord}(\mathbb{P}_i) = s_i$ *and* $\mathbf{deg}(\mathbb{P}_i, \mathbb{Y}) = m_i$. *Then*

- $R = \sum_{i=0}^{n} \sum_{j=0}^{h_i} G_{ij}\mathbb{P}_i^{(j)}$ *with* $\mathbf{deg}(G_{ij}\mathbb{P}_i^{(j)}) \leq (m+1)\mathbf{deg}(R)$.
- *Better computing complexity* $O(n^{3.376}(s+1)^{O(n)}(m+1)^{O(nls^2)})$.

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Degree Bound for the Diff Resultant can be improved:

Theorem

Diff resultant has degree bounded by $\frac{s-s_k+1}{m_k}\prod_{i=0}^{n} m_i^{s-s_i+1}$ *in each* $\mathbf{u}_k$.

Key Tool: The degree of Generalized Chow form.

Summary

- We introduce the concepts of Laurent diff polynomials and Laurent diff essential systems, and give a criterion for Laurent differentially essential system in terms of the supports.
- Sparse differential resultant is defined and properties similar to that of the Macaulay resultant are given.
- A single exponential algorithm to compute the sparse differential resultant is given.

Problems for Future Research

- Find a determinant representation for the (sparse) differential resultant.
- Bound the degree of the sparse diff resultant in terms of the mixed volume of the polytopes generated by the supports of the diff polynomials.
- Find practically efficient algorithms to compute the sparse differential resultants.
- Study differential toric variety further to give a necessary and sufficient condition for the sparse differential resultant vanishing.

Reference:

- Wei Li, Xiao-Shan Gao, Chun-Ming Yuan. Sparse Differential Resultant. In *Proc. ISSAC 2011*, San Jose, CA, USA, 225-232, 2011. **ISSAC 2011 Distinguished Paper Award.**
- Wei Li, Chun-Ming Yuan, Xiao-Shan Gao. Sparse Differential Resultant for Laurent Differential Polynomials. ArXiv:1111.1084v1, page 1-57, 4 Nov 2011.