Difference Algebraic Groups

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Outline

Motivation: σ-Galois theory of linear differential equations

Difference algebraic geometry

The limit degree and algebraic σ-groups (Kowalski, Pillay)

A decomposition theorem for σ-étale σ-algebraic groups

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Galois theory of (parameterized) linear differential and difference equations

- **Classical (no parameters):** Galois groups are algebraic groups, measure the algebraic relations among the solutions
- **Continuous parameters:** Galois groups are differential algebraic groups, measure the differential algebraic relations among the solutions with respect to some auxiliary derivations ($\geq$ 2006, Singer, Cassidy, Landesman, Hardouin, Ovchinnikov, Di Vizio, Mitschi, Minchenko, Arreche, Dreyfus, Maier, W.,...)
- **Discrete parameters:** Galois groups are difference algebraic groups, measure the difference algebraic relations among the solutions with respect to some auxiliary difference operator ($\geq$ 2013, Di Vizio, Hardouin, Ovchinnikov, W.)

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Example

The Bessel function $J_\alpha(x)$ solves

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

and satisfies

$$xJ_{\alpha+2}(x) - 2(\alpha+1)J_{\alpha+1}(x) + xJ_\alpha(x) = 0.$$

The σ-Galois group of Bessel's equation is

$$G = \{g \in \mathrm{SL}_2 \mid \sigma(g) = g\} \leq \mathrm{SL}_2 .$$

Algebraization: $K = \mathbb{C}(\alpha, x)$ is a $\delta\sigma$-field: $\delta = \frac{\partial}{\partial x}$, $\sigma(f(\alpha, x)) = f(\alpha+1, x)$, $\delta\sigma = \sigma\delta$.

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The Galois theory of linear differential equations depending on a discrete parameter

K a $\delta\sigma$-field of char 0, $\delta\colon K \to K$ derivation, $\sigma\colon K \to K$ endomorphism
$\delta\sigma = \sigma\delta \Rightarrow k = K^\delta = \{a \in K |\ \delta(a) = 0\}$ is a σ-field
Ex.: $K = \mathbb{C}(\alpha, x) \Rightarrow k = \mathbb{C}(\alpha)$

A σ-Picard-Vessiot extension for

$$\delta(y) = Ay,\ A \in K^{n\times n}$$

is a $\delta\sigma$-field extension L of K such that

- $\exists\ Y \in \mathrm{GL}_n(L)\colon \delta(Y) = AY$ and $L = K(Y_{ij}, \sigma(Y_{ij}), \sigma^2(Y_{ij}), \ldots)$
- $L^\delta = K^\delta$.

$S := K[Y_{ij}, \frac{1}{\det(Y)}, \sigma(Y_{ij}), \frac{1}{\det(\sigma(Y))}, \ldots]$ σ-Picard-Vessiot ring

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The σ-Galois group

The σ-Galois group of $L|K$ is the functor

$$G\colon \{k\text{-}\sigma\text{-algebras}\} \longrightarrow \{\text{groups}\}$$

given by

$$G(R) = \mathrm{Aut}^{\delta\sigma}(S \otimes_k R | K \otimes_k R).$$

It is a difference algebraic group (i.e. representable).

Example

$K = \mathbb{C}(x)$, $\delta = \frac{\partial}{\partial x}$, $\sigma(f(x)) = f(x+1)$,

$$\delta(y) = \frac{1}{2x} y$$

$L = S = K(\sqrt{x}, \sqrt{x+1}, \sqrt{x+2}, \ldots)$, $G(R) = \{g \in R^\times | \; g^2 = 1\}$

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$$\{\text{intermediate } \delta\sigma\text{-fields of } L|K\} \overset{1:1}{\longleftrightarrow} \{\sigma\text{-algebraic subgroups of } G\}$$

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An application, σ-independence of special functions

Theorem

Let $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ be two $\mathbb{C}$-linearly independent solutions of $y'' = xy$. Then

$$\mathrm{Ai}(x), \mathrm{Bi}(x), \mathrm{Ai}'(x), \mathrm{Ai}(x+1), \mathrm{Bi}(x+1), \mathrm{Ai}'(x+1), \mathrm{Ai}(x+2), \dots$$

are algebraically independent over $\mathbb{C}(x)$.

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Difference algebra

Definition

A *difference ring (σ-ring)* is a ring R together with a ring endomorphism $\sigma: R \to R$.

Example

$R = \mathbb{C}^{\mathbb{N}}$, $\sigma((a_n)_{n\in\mathbb{N}} = (a_{n+1})_{n\in\mathbb{N}}$

k a σ-field, e.g., $k = \mathbb{C}(\alpha)$ with $\sigma(f(\alpha)) = f(\alpha + 1)$. The σ-polynomial ring over k is

$$k\{y\} = k\{y_1, \dots, y_n\} = k[y_1, \dots, y_n, \sigma(y_1), \dots, \sigma(y_n), \sigma^2(y_1), \dots].$$

$F \subset k\{y\}$, R a k-σ-algebra

$$\mathbb{V}_R(F) = \{a \in R^n | \ f(a) = 0 \ \forall \ f \in F\}$$

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Difference varieties

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$a_{n+2} = a_{n+1} + a_n \rightsquigarrow \sigma^2(y_1) - \sigma(y_1) - y_1$
$k = \mathbb{C},\ R = \mathbb{C}^{\mathbb{N}} \rightsquigarrow$ Fibonacci-sequence$\in \mathbb{V}_R(\sigma^2(y_1) - \sigma(y_1) - y_1)$

Definition

A functor X of the form $R \rightsquigarrow X(R) = \mathbb{V}_R(F)$ is called a *σ-variety*.

$$\mathbb{I}(X) := \{f \in k\{y\} \mid f(a) = 0\ \forall\ a \in X(R),\ \forall\ R\} \subset k\{y\}$$

$$k\{X\} := k\{y\}/\mathbb{I}(X) \quad \text{coordinate ring of } X$$

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Difference algebraic groups

Definition

A *σ-algebraic group* G is a group object in the category of σ-varieties.

Examples

(Affine) algebraic groups

$$G(R) = \{g \in \mathrm{SL}_2(R) |\ \sigma(g) = g\} \leq \mathrm{SL}_2(R)$$

$$G(R) = \{g \in R^\times |\ g\sigma^2(g)^3 = 1\} \leq \mathbb{G}_m(R)$$

$$G(R) = \{g \in R |\ \sigma^n(g) + \lambda_{n-1}\sigma^{n-1}(g) + \ldots + \lambda_0 y = 0\} \leq \mathbb{G}_a(R)$$

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Difference algebraic groups

Facts:

- The category of σ-varieties is anti–equivalent to the category of finitely σ-generated k-σ-algebras.
- The category of σ-algebraic groups is anti–equivalent to the category of finitely σ-generated k-σ-Hopf algebras.

$$G \leftrightarrow k\{G\}$$

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The limit degree

Theorem

Any σ-algebraic group is isomorphic to a σ-algebraic subgroup of some GL_n.

Fix an embedding $G \hookrightarrow \mathrm{GL}_n$.

$$\mathbb{I}(G) \subset k\{\mathrm{GL}_n\} = k\{X, \tfrac{1}{\det(X)}\}$$

For $i \geq 0$ the ideal

$$\mathbb{I}(G) \cap k[X, 1/\det(X), \ldots, \sigma^i(X), 1/\det(\sigma^i(X))]$$

defines an algebraic subgroup $G[i]$ of GL_n^{i+1} and we have morphisms

$$\pi_i\colon G[i] \to G[i-1],\ (g_0, \ldots, g_i) \mapsto (g_0, \ldots, g_{i-1}).$$

Theorem (Existence of the limit degree)

$\mathrm{ld}(G) = \lim_{i\to\infty} \deg(\pi_i)$ exists and does not depend on the embedding $G \hookrightarrow \mathrm{GL}_n$.

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$$\mathrm{ld}(G) = \beta_n$$

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Algebraic σ-groups in the sense of Kowalski and Pillay

Definition (Kowalski, Pillay)

An algebraic σ-group is an algebraic group G together with a morphism of algebraic groups $\sigma\colon G \to {}^{\sigma}G$.

Theorem

The category of algebraic σ-groups is equivalent to the category of σ-algebraic groups of limit degree one (and σ-dimension zero).

Idea of proof: $\mathrm{ld}(G) = 1 \Leftrightarrow k\{G\}$ is finitely generated as a k-algebra.

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Examples

$G(R) = \{g \in R^\times | \ g^n = 1\} \leq \mathbb{G}_m(R)$ iff $\mathrm{char}(k) \nmid n$

G étale algebraic group $\Rightarrow$ G σ-étale σ-algebraic group

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$$1 \to G^o \to G \to G/G^o \to 1$$

G^o connected (Spec($k\{G^o\}$) connected), G/G^o σ-étale

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$\mathcal{G}$ finite group, $\sigma\colon \mathcal{G} \to \mathcal{G}$ endomorphism, $k\{G\} = k^{\mathcal{G}}$,
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$$1 \to G^o \to G \to G/G^o \to 1$$

G^o connected ($\mathrm{Spec}(k\{G^o\})$ connected), G/G^o σ-étale

σ-étale σ-algebraic groups

Definition

A σ-algebraic group G is *σ-étale* if $k\{G\}$ is a union of étale k-algebras $\Leftrightarrow$ every element of $k\{G\}$ satisfies a separable polynomial over k.

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The σ-topology

For a σ-ring R we have an induced map $\sigma\colon \operatorname{Spec}(R) \to \operatorname{Spec}(R)$

Definition

A subset of $\operatorname{Spec}(R)$ is *σ-closed* if it is closed and stable under σ.

$$\mathfrak{a} \mapsto \mathcal{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(R) |\ \mathfrak{a} \subset \mathfrak{p}\}$$

is a bijection between the radical σ-ideals of R and the σ-closed subsets of $\operatorname{Spec}(R)$.

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In algebraic geometry:

Spec(R) is connected if and only if R has no non-trivial idempotent elements.

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connected σ-algebraic groups, algebraic groups

The σ-connected σ-algebraic subgroup

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of G corresponding to the σ-connected component of $\mathrm{Spec}(k\{G\})$ that contains the identity, is called the *σ-identity component* of G.

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Idea of proof: Induction on $\operatorname{ld}(G)$

Main step: G σ-connected, $\nexists\ N \trianglelefteq G$ with $1 < \operatorname{ld}(N) < \operatorname{ld}(G)$

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Example

$G(R) = \{g \in R^\times \mid g^4 = 1,\ \sigma(g)^2 = 1\} \leq \mathbb{G}_m(R)$

$2 \nmid \mathrm{char}(k) \Rightarrow G$ σ-étale
$G = G^{\sigma o}$
$N(R) = \{g \in R^\times \mid g^4 = 1,\ \sigma(g) = 1\} \Rightarrow N \trianglelefteq G$ σ-infinitesimal

$$G = G^{\sigma o} \supset N \supset 1$$

$H(R) = \{g \in R^\times \mid g^2 = 1\} \leq \mathbb{G}_m(R)$

$$G \to H,\ g \mapsto \sigma(g)$$

is surjective with kernel $N \Rightarrow G/N \simeq H$ is benign.

Thank you!

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