A Topological Approach to Constrained Points

P. Cassidy, R.C. Churchill and J.J. Kovacic

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Department of Mathematics
Graduate Center, CUNY
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Generic points and specializations are concepts from algebraic geometry which were quite popular when the Weil approach was fashionable, but which receive scant attention in the standard contemporary texts (e.g., [Hart] and [S]). Moreover, when these modern texts do address the concepts the definitions tend to be topological rather than algebraic. (There are exceptions, e.g., [L, vdW].)

In differential algebraic geometry the analogous concepts are still generally treated in the classical spirit, despite the fact that it is quite easy to bring the language up to date. Moreover, in this subject one encounters, along with generic points and specializations, the unintuitive concept of a "constrained point," accompanied by a definition in the classical spirit. The purpose of these notes is to give a simple topological definition.

Although our definition does apply, constrained points in the context of differential algebraic geometry are not treated in these notes. For an introduction to that topic see, e.g., [Cas].

This exposition can be viewed as a summary of a conversation between the authors on 2 November, following the weekly meeting of the Kolchin Seminar.

Throughout the notes X *denotes a non-empty topological space, and the closure of a subset* $Y \subset X$ *is denoted* $\mathrm{cl}(Y)$.

1. Basic Definitions

We begin with the contemporary topological definitions of locus, specialization, generic specialization and generic point[1]. Let $x \in X$.

- The *locus* of x is the name (occasionally) given to the closure $\mathrm{cl}(\{x\})$ of x.
- Any point $y \in \mathrm{cl}(\{x\})$ is a *specialization* of x, and we indicate this by writing $x \to y$ (read: x specializes to y.).
- A point $y \in X$ is a *generic specialization* of x if $x \to y$ and $y \to x$, which we indicate by writing $x \leftrightarrow y$. In other words: y is a generic specialization of x if and only if $\mathrm{cl}(\{x\}) = \mathrm{cl}(\{y\})$.
- The point x is a *generic point* of X if $\mathrm{cl}(\{x\}) = X$.

For examples of generic points and specializations in classical affine algebraic geometry see, e.g., [Chu, §14]. In particular, Proposition 14.5 of that reference relates the topological definitions given above to the classical algebraic definitions. For an example of generic specialization consider the classical $(\mathbb{Z}, \mathbb{R})$-affine algebraic set[2] $X := \{-\sqrt{2}, \sqrt{2}\} \subset \mathbb{R}$ defined by the polynomial $x^2 - 2 \in \mathbb{Z}[x]$. When the $(\mathbb{Z}, \mathbb{R})$-Zariski topology is assumed on X one has $\mathrm{cl}(\{-\sqrt{2}\}) = \mathrm{cl}(\{\sqrt{2}\}) = X$; hence each point is a generic specialization of the other.

We isolate one particular example for later reference.

Example 1.1 : Let R be a commutative ring with unity and let $X := \mathrm{Spec}(R)$ with the Zariski topology[3]. Then for any prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ one has

$$\text{(i)} \qquad \mathrm{cl}(\{\mathfrak{p}\}) = V(\mathfrak{p}) := \{\, \mathfrak{q} \in \mathrm{Spec}(R) \,:\, \mathfrak{p} \subset \mathfrak{q} \,\}$$

(e.g., see [Chu, §9, Proposition 9.25(i)]). Using the arrow notation this gives $\mathfrak{p} \to \mathfrak{q} \Leftrightarrow \mathfrak{p} \subset \mathfrak{q}$, and therefore $\mathfrak{p} \leftrightarrow \mathfrak{q}$ if and only if $\mathfrak{p} = \mathfrak{q}$.

Proposition 1.2 : *Suppose $x, y \in X$ satisfy $x \leftrightarrow y$. Then y is contained in every open neighborhood of x.*

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[1] The definitions of locus, specialization are generic point are from [Mac, p. 15], but can be found in many other places. The definition of a generic specialization is adapted from [Kol, Chapter 0, §14, p. 33].

[2] As defined in [Chu, §2].

[3] See, e.g., [Chu, §5].

Proof : If there is an open neighborhood U of x not containing y then $X \backslash U$ would be a closed set containing y but not containing x, thereby contradicting $x \in \text{cl}(\{y\})$.
q.e.d.

The next definition[4] is preliminary to that of a "constrained point." A subset $Y \subset X$ is *locally closed* if there is a closed set $C \subset X$ and an open set $U \subset X$ such that $Y = C \cap U$.

Examples 1.3 :

(a) Any open set $V \subset X$ is locally closed: take $C := X$ and $U := V$.

(b) Any closed set $K \subset X$ is locally closed: take $C = K$ and $U := X$.

(c) When $x \in X$ is a closed point[5] the singleton set $\{x\}$ is locally closed: this is an important special case of (b) (take $K := \{x\}$).

(d) Assuming the usual topology any half-open interval $[a, b) \subset \mathbb{R}$ (with $a, b \in \mathbb{R}$ and $a < b$) is locally closed: take $C := [a, b]$, and $U := (a - 1, b)$. Similarly, any half-open interval $(a, b]$ is locally closed.

(e) The collection of locally closed subsets of a topological space is closed under finite intersection. For if $\{C_j \cap U_j\}_{j=1}^n$ is a collection of locally closed sets then one sees from $\cap_j (C_j \cap U_j) = (\cap_j C_j) \cap (\cap_j U_j)$ that the intersection is locally closed.

From the preceding examples one might begin to suspect that <u>all</u> subsets of a topological space are locally closed. Not so.

(f) Let $X = \{a, b, c\}$ (three distinct points), and let τ be the topology $\{X, \emptyset, \{a\}\}$ on X. Then the singleton set $\{b\}$ is not locally closed. (This is easy to prove directly, or one can use Proposition 1.4(b).)

Proposition 1.4 : *Suppose* $Y \subset X$ *and let* $\text{cl}(Y)$ *be endowed with the relative topology. Then the following statements are equivalent:*

(a) Y *is locally closed;*

(b) Y *is open in* (the topological space) $\text{cl}(Y)$, *and*

(c) $Y = \text{cl}(Y) \cap U$ *for some open set* $U \subset X$.

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[4]Which we take from [A-M, Chapter 5, p. 71].

[5]A *closed point* is a point $x \in X$ satisfying $\text{cl}(\{x\}) = \{x\}$.

Proof :

(a) $\Rightarrow$ (b) : Suppose $Y = C \cap U$, where $C \subset X$ is closed and $U \subset X$ is open. Then $Y \subset C \;\Rightarrow\; \mathrm{cl}(Y) \subset C \;\Rightarrow\; Y \subset \mathrm{cl}(Y) \cap U \subset C \cap U = Y$, and $Y = \mathrm{cl}(Y) \cap U$ is therefore open in $\mathrm{cl}(Y)$.

(b) $\Rightarrow$ (a) : Y is open in $\mathrm{cl}(Y)$ if and only if there is an open set $V \subset X$ such that $Y = \mathrm{cl}(Y) \cap V$. Taking $C := \mathrm{cl}(Y)$ and $U := V$ we conclude that $Y = C \cap U$ is locally closed.

(b) $\Leftrightarrow$ (c) : Immediate from the definition of the relative topology.

q.e.d.

2. Constrained Points

For any $x \in X$ define the *generic specialization set* GS_x *of* x by

$$GS_x := \{\, y \in X \mid x \leftrightarrow y \,\} = \{\, y \in X \, : \, \mathrm{cl}(\{y\}) = \mathrm{cl}(\{x\}) \,\}. \tag{2.1}$$

A point $x \in X$ is (*a*) *constrained* (*point*) if GS_x is locally closed.

Examples 2.2 :

(a) When $x \in X$ is a closed point the singleton set $\{x\}$ is a constrained point (by Example 1.3(c)).

(b) When R is a commutative ring with unity we see from Example 1.1 that for any prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ we have

(i) $$GS_{\mathfrak{p}} = \{\mathfrak{p}\}.$$

A point $\mathfrak{p} \in \mathrm{Spec}(R)$ is therefore constrained if and only if $\{\mathfrak{p}\}$ is locally closed. Since maximal ideals are closed points[6], it follows from Example 1.3(b) that maximal ideals are constrained.

(c) When $x \in X$ is constrained the same is true for all elements of GS_x (because $y \in GS_x$ if and only if $GS_x = GS_y$).

For any $y \in X$ we (obviously) have $y \in \mathrm{cl}(\{y\})$, and if $x \in X$ and $y \in GS_x$ then from (2.1) we see that $y \in \mathrm{cl}(\{x\})$. An immediate consequence is

$$GS_x \subset \mathrm{cl}(\{x\}), \tag{2.3}$$

whereupon from $\{x\} \subset GS_x$ we conclude that

$$\mathrm{cl}(GS_x) = \mathrm{cl}(\{x\}). \tag{2.4}$$

Proposition 2.5 : *The following assertions regarding a point* $x \in X$ *are equivalent:*

(a) x *is a constrained point; and*

(b) *there is an open neighborhood* $U \subset X$ *of* x *such that*

(i) $$GS_x = \mathrm{cl}(\{x\}) \cap U.$$

Proof : Use Proposition 1.4(c) and (2.4). **q.e.d.**

[6]See, e.g., [Chu, Proposition 12.5].

Corollary 2.6 : *Suppose* $\{D_\alpha\}_{\alpha\in A}$ *is a basis for the topology on* X. *Then for any* $x \in X$ *the following assertions are equivalent:*

(a) x *is a constrained point; and*

(b) *there is an index* $\beta \in A$ *such that*

(i) $$GS_x = \mathrm{cl}(\{x\}) \cap D_\beta.$$

When either (and therefore both) of these conditions hold one says that x is *constrained* by β, and that β is a *constraint* of x. Since one can index in many ways one cannot expect constraints to be unique.

Proof :

(a) ⇒ (b): By Proposition 2.5 there is an open set $U \subset X$ such that (i) of that statement holds, and since $\{D_\alpha\}_{\alpha\in A}$ is a basis there is an index $\beta \in A$ such that

(ii) $$x \in D_\beta \subset U.$$

We claim that

$$\mathrm{cl}(\{x\}) \cap D_\beta = \mathrm{cl}(\{x\}) \cap U.$$

Indeed, from (ii) we have $\mathrm{cl}(\{x\}) \cap D_\beta \subset \mathrm{cl}(\{x\}) \cap U$, and if the reverse inclusion fails there must be an element $y \in \mathrm{cl}(\{x\}) \cap U$ such that $y \notin D_\beta$. But this would place y is the closed set $X \setminus D_\beta$, hence $x \notin \mathrm{cl}(\{y\})$, and this contradicts $y \in GS_x$.

(b) ⇒ (a) : Take $U := D_\beta$ in Proposition 2.5.

q.e.d.

3. Constrained Points in Spec($\boldsymbol{R}$)

In this section R *is a commutative ring with unity* 1.

For any element $r \in R$ define

$$D(r) := \{\, \mathfrak{p} \in \operatorname{Spec}(R) \;:\; r \notin \mathfrak{p} \,\}. \tag{3.1}$$

The collection $\{D(r)\}_{r \in R}$ is a basis for the Zariski topology[7] on $\operatorname{Spec}(R)$. Note that, since prime ideals are (by definition) proper ideals,

$$D(1) = \operatorname{Spec}(R). \tag{3.2}$$

For any ideal $\mathfrak{i} \subset R$ define

$$V(\mathfrak{i}) := \{\, \mathfrak{q} \in \operatorname{Spec}(R) \;:\; \mathfrak{i} \subset \mathfrak{q} \,\}. \tag{3.3}$$

The sets $V(\mathfrak{i})$ are the closed sets of the Zariski topology[8] on $\operatorname{Spec}(R)$, and for any prime ideal $\mathfrak{p} \subset R$ one has[9]

$$\operatorname{cl}(\{\mathfrak{p}\}) = V(\mathfrak{p}). \tag{3.4}$$

Proposition 3.5 : *For any prime ideal* $\mathfrak{p} \in \operatorname{Spec}(R)$ *the following assertions are equivalent:*

(a) $\mathfrak{p}$ *is constrained;*

(b) *there is an element* $r \in R$ *such that*

$$\{\mathfrak{p}\} = V(\mathfrak{p}) \cap D(r); \tag{i}$$

(c) *there is an element* $r \in R$ *such that*

$$\{\mathfrak{p}\} = \{\, \mathfrak{q} \in \operatorname{Spec}(R) \;:\; \mathfrak{p} \subset \mathfrak{q} \;\; \textit{and} \;\; r \notin \mathfrak{q} \,\}; \tag{ii}$$

and

(d) *there is an element* $r \in R$ *such that* $\mathfrak{p}$ *is maximal* (w.r.t. inclusion) *among the prime ideals of* R *not containing* r.

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[7] See, e.g., the paragraph immediately following (5.4) of [Chu].

[8] See, e.g., see Corollary 9.22 of [Chu].

[9] See, e.g., Proposition 9.25 of [Chu].

Any element r as in (b) or (c) is called a *constraint* of $\mathfrak{p}$, and $\mathfrak{p}$ is said to be *constrained* by r. Constraints need not be unique. The result motivates the "constraint" terminology, i.e., both the location (within R) and size of $\mathfrak{p}$ are "limited" by the ring element r.

Proof : (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) : Recall from Example 2.2(b) that $GS_{\mathfrak{p}} = \{\mathfrak{p}\}$, whereupon it follows from (3.4) that $\mathrm{cl}(GS_{\mathfrak{p}}) = V(\mathfrak{p})$. Since the collection $\{D(r)\}_{r\in R}$ is a basis for the Zariski topology condition (i) of Corollary 2.6 is equivalent in the present context to

(i) $$\{\mathfrak{p}\} = V(\mathfrak{p}) \cap D(r) \qquad \text{for some} \qquad r \in R,$$

which upon recalling the definitions is seen to be equivalent to

$$\{\mathfrak{p}\} = \{\, \mathfrak{q} \in \mathrm{Spec}(R) \,:\, \mathfrak{p} \subset \mathfrak{q} \text{ and } r \notin \mathfrak{q}\,\}.$$

The equivalences of (a), (b) and (c) follow.

(c) $\Leftrightarrow$ (d) : Each assertion is a restatement of the other.

q.e.d.

Corollary 3.6 : *For any prime ideal* $\mathfrak{p} \in \mathrm{Spec}(R)$ *the following assertions are equivalent:*

(a) $\mathfrak{p}$ *is maximal; and*

(b) $\mathfrak{p}$ *is constrained by* 1.

Proof : For any ideal $\mathfrak{m} \in \mathrm{Spec}(R)$ see from the definition of "maximal" together with (3.2) that

$$\begin{aligned} \mathfrak{m} \text{ is maximal } &\Leftrightarrow \{\mathfrak{m}\} = V(\mathfrak{m}) \\ &\Leftrightarrow \{\mathfrak{m}\} = V(\mathfrak{m}) \cap \mathrm{Spec}(R) \\ &\Leftrightarrow \{\mathfrak{m}\} = V(\mathfrak{m}) \cap D(1). \end{aligned}$$

q.e.d.

Corollary 3.7 : *Suppose* R *is an integrally closed Noetherian integral domain. Then* R *is a Dedekind domain if and only if every non-zero prime ideal is constrained by* 1.

Proof : An integral domain is a Dedekind domain if and only if it is Noetherian, integrally closed, and every non-zero prime ideal is maximal[10]. **q.e.d.**

[10]See, e.g., [Hun, Chapter VIII, §6, Theorem 6.10(viii), pp. 405-6].

4. Constrained Points in Affine n-Space

In this section $B \supset A$ is an extension of integral domains, n is a positive integer, and $B[x] \supset A[x]$ are the usual polynomial algebras in indeterminates $x = (x_1, x_2, \ldots, x_n)$. All rings are assumed commutative with unities.

We recall a few definitions and facts from basic algebraic geometry and commutative algebra.

- The *zero set* of any subset $S \subset A[x]$ is defined by

  $$\mathcal{V}(S) = \{ c = (b_1, b_2, \ldots, b_n) \in B^n : p(c) = 0 \quad \text{for all} \quad p \in S \}.$$

  Subsets of B^n of this form are also called (A, B)-*affine algebraic sets*[11].

- The collection of (A, B)-affine algebraic subsets of B^n constitute the closed sets of a topology[12] on B^n, called the (A, B)-*Zariski topology*[13]. When the prefix "Zariski" is used in connection with subsets of B^n this topology is being assumed, e.g., a subset $X \subset B^n$ is: *Zariski closed* if X is closed in this topology; *Zariski dense* if the closure $\mathrm{cl}(X)$ of X is equal to B^n.

For the remainder of the section B^n is assumed endowed with the (A, B)-Zariski topology.

For each polynomial $p \in A[x]$ define[14]

(4.1) $$D_p := \{ c = (b_1, b_2, \ldots, b_n) \in B^n : p(c) \neq 0 \}.$$

For example,

(4.2) $$D_1 = B^n.$$

Observe that D_p is the complement of the closed set

$$\mathcal{V}(\{p\}) = \{c \in B^n : p(c) = 0\},$$

and is therefore open.

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[11] In [Chu] they are called *classical* (A, B)-*affine algebraic sets.*

[12] See, e.g., [Chu, §4, Corollary 4.6]. The integral domain hypothesis on B is needed here: the assertion regarding a topology is false if B is merely assumed a ring.

[13] The same name is given to the topology we have already encountered on $\mathrm{Spec}(R)$. Fortunately, it is generally easy to tell from context which topology is under consideration.

[14] The notation D_p is not standard. It is introduced to conform with that of Corollary 2.6, and so as not to clash with the (completely standard) notation $D(r)$ introduced in (3.1).

Proposition 4.3 *The collection* $\{D_p\}_{p\in A[x]}$ *is a basis for the* (A,B)*-Zariski topology on* B^n.

Proof : Let $c \in B^n$ and let U be an open neighborhood of c. We must show there exists a polynomial $p \in A[X]$ such that

$$b \in D_p \subset U.$$

Since U is open, the complement $B^n \setminus U$ is closed, hence of the form

$$B^n \setminus U = \mathcal{V}(\mathfrak{i})$$

for some ideal $\mathfrak{i} \subset A[x]$. Because U is a neighborhood of c we have $c \notin \mathcal{V}(\mathfrak{i})$, and as a consequence there exists $p \in \mathfrak{i}$ with $p(c) \neq 0$. In particular

$$x \in D_p.$$

If $y \in D_p$ then $p(y) \neq 0$ also holds, hence y cannot be in $\mathcal{V}(\mathfrak{i})$. Thus $y \in U$. We therefore have

$$x \in D_p \subset U$$

as required. **q.e.d.**

Corollary 4.4 : *A point* $c \in B^n$ *is constrained if and only if there exists a polynomial* $p \in A[x]$ *such that*

$$GS_c = \mathrm{cl}(\{c\}) \cap D_p.$$

In this context one would say that c is *constrained by* p, and that p is a *constraint* of c.

Proof : Corollary 2.5 and the above proposition. **q.e.d.**

Corollary 4.5 : *A point* $c \in B^n$ *is constrained with constraint* 1 *if and only if* GS_c *is closed.*

Proof : Using Corollary 4.4 and (4.2) one sees that a point $c \in B^n$ has constraint 1 if and only if

$$GS_c = \mathrm{cl}(\{c\}) \cap D_1 = \mathrm{cl}(\{c\}).$$

q.e.d.

At this point we need a few more recollections from commutative algebra and basic algebraic geometry.

- Note that for any two subsets $S, T \subset K[x]$ one has

$$S \subset T \qquad \Rightarrow \qquad \mathcal{V}(T) \subset \mathcal{V}(S).$$

- The *radical* of an ideal $\mathfrak{i}$ within a ring R is defined by

$$\sqrt{\mathfrak{i}} := \{\, r \in R \,:\, r^n \in \mathfrak{i} \ \text{ for some integer } \ n \geq 1\,\}.$$

One easily verifies that $\sqrt{\mathfrak{i}}$ is an ideal satisfying $\mathfrak{i} \subset \sqrt{\mathfrak{i}}$. The ideal $\mathfrak{i}$ is *radical* if $\mathfrak{i} = \sqrt{\mathfrak{i}}$. Any prime ideal is radical, but the converse is false, e.g., the principal ideal $(6) \subset \mathbb{Z}$ is radical but not prime.

- For any subset $S \subset A[x]$ let $(S) \subset A[x]$ denote the ideal generated by S. Then[15]

$$\mathcal{V}(S) = \mathcal{V}((S)) = \mathcal{V}(\sqrt{(S)}\,).$$

In other words, when considering zero sets noting is lost by restricting to zero sets of radical ideals. In particular, the mapping $\mathfrak{r} \mapsto \mathcal{V}(\mathfrak{r})$ from radical ideals of $A[x]$ to zero sets within B^n is a surjection. However, without further restrictions on A and B it need not be an injection.

- The *defining* (A, B)*-ideal* of any subset $C \subset B^n$ is given by

$$\mathfrak{i}(C) := \{\, p \in A[x] \,:\, p(b) = 0 \ \text{ for all } \ b \in C\,\}.$$

Note that

$$C \subset D \qquad \Rightarrow \qquad \mathfrak{i}(D) \subset \mathfrak{i}(C).$$

That $\mathfrak{i}(C)$ is an ideal is easily established, and from the integral domain hypothesis on $B \supset A$ one easily verifies that $\mathfrak{i}(C)$ is radical. Thus

$$\sqrt{\mathfrak{i}(C)} = \mathfrak{i}(C).$$

When $C = \{c\}$ is a singleton the ideal $\mathfrak{i}(\{c\})$ is prime.

For our purposes the importance of the defining ideal is that for any subset $C \subset B^n$ one has[16]

$$\mathcal{V}(\mathfrak{i}(C)) = \mathrm{cl}(C),$$

i.e., this ideal determines the Zariski closure of C.

For any subset $S \subset A[x]$ one should keep in mind the inclusions

(i) $$S \subset (S) \subset \mathfrak{i}(\mathcal{V}(S)).$$

[15]This is easily established. See, e.g., [Chu, §4, Proposition 4.2].
[16]See, e.g., [Chu, §9, Corollary 9.7(a)].

- When $K := A$ and $L := B$ are fields, with L algebraically closed, there is a much sharper formulation of the final inclusion of (i) of the previous item: the Hilbert Nullstellensatz asserts that for any subset $S \subset K[x]$ one has[17]

  $$\mathfrak{i}(\mathcal{V}(S)) = \sqrt{(S)}.$$

  In particular, for radical ideals $\mathfrak{r} \subset K[x]$ one has

  $$\mathfrak{i}(\mathcal{V}(\mathfrak{r})) = \mathfrak{r},$$

  and the previously encountered mapping $\mathfrak{r} \mapsto \mathcal{V}(\mathfrak{r})$ is thereby seen to be a bijection between radical ideals of $K[x]$ and Zariski closed subsets of L^n.

Proposition 4.6 : *Suppose $L \supset K$ is an extension of fields with L algebraically closed. Then a point $c \in L^n$ is constrained if and only if there exists a polynomial $p \in K[x]$ such that the ideal $\mathfrak{i}(\{c\}) \subset K[x]$ is maximal with respect to the property that it contain no power of p.*

Again the terminology would be: c is *constrained* by p; p is a *constraint* for c.

Proof :

$\Rightarrow$: By Corollary 4.4, we have

$$GS_c = \mathrm{cl}(\{c\}) \cap D_p$$

for some polynomial $p \in K[x]$. Because $p(c) \neq 0$, no power of p can be in $\mathfrak{i} := \mathfrak{i}(\{c\})$. We must show that $\mathfrak{i}$ is maximal with respect to this property.

Suppose $\mathfrak{j} \subset K[x]$ is an ideal properly containing $\mathfrak{i}$. Set $Y = \mathcal{V}(\mathfrak{j}) \subset L^n$. Then

$$Y = \mathcal{V}(\mathfrak{j}) \subset \mathcal{V}(\mathfrak{i}) = \mathrm{cl}(\{c\}).$$

We claim that $Y \neq \mathrm{cl}(\{c\})$. Otherwise the Nullstellensatz gives

$$\mathfrak{i} = \mathfrak{i}(\{c\}) \supset \mathfrak{i}(\mathrm{cl}(\{c\})) = \mathfrak{i}(Y) = \mathfrak{i}(\mathcal{V}(\mathfrak{j})) = \sqrt{\mathfrak{j}} \supset \mathfrak{j},$$

implying $\mathfrak{j} \subset \mathfrak{i}$, and this contradicts the assumption that $\mathfrak{j}$ properly includes $\mathfrak{i}$. We therefore have

$$Y \subsetneq \mathrm{cl}(\{c\}),$$

[17] For a proof see, e.g., [Chu, §19, Corollary 19.2].

and as a result we see that there must be a point $y \in Y$ such that $\mathrm{cl}(\{y\}) \neq \mathrm{cl}(\{c\})$, i.e., which is not a generic specialization of c. This in turn gives

$$y \notin GS_c = \mathrm{cl}(\{c\}) \cap D_p.$$

We conclude that $y \notin D_p$, i.e., that $p(y) = 0$, whereupon from the Nullstellensatz we see that some power of p must lie in $\mathfrak{j}$.

$\Leftarrow$: Suppose $\mathfrak{i}$ is an ideal maximal with respect to the condition that it contain no power of p. We claim that

$$GS_c = \mathrm{cl}(\{c\}) \cap D_p.$$

By Proposition 1.2 we have

$$GS_c \subset \mathrm{cl}(\{c\}) \cap D_p,$$

hence we only need establish the opposite inclusion.

Let $y \in \mathrm{cl}(\{c\}) \cap D_p$ and set $\mathfrak{j} := \mathfrak{i}(\{y\})$. Then

$$y \in \mathrm{cl}(\{c\}) \quad \Rightarrow \quad \mathfrak{i} \subset \mathfrak{j},$$

and $y \in D_p$ implies that no power of p is in $\mathfrak{j}$. By maximality of $\mathfrak{i}$ we must have $\mathfrak{j} = \mathfrak{i}$, and

$$\mathrm{cl}(\{y\}) = \mathrm{cl}(\{c\})$$

follows.

q.e.d.

Corollary 4.7 : *A point $c \in L^n$ is constrained with constraint 1 if and only if $\mathfrak{i}(\{c\})$ is a maximal ideal.*

Proposition 4.8 *Suppose $L \supset K$ is an extension of fields and L is algebraically closed. Then for any point $c = (b_1, b_2, \ldots, b_n) \in L^n$ the following assertions are equivalent.*

(a) *c is constrained.*

(b) *GS_c is locally closed.*

(c) *c is constrained with constraint 1.*

(d) *GS_c is closed.*

(e) *Each* b_i *is algebraic over* K, $i = 1, 2, \ldots, n$.

(f) *The field extension* $K[b_1, b_2, \ldots, b_n] \supset K$ *is algebraic.*

Proof : $(a) \Leftrightarrow (b)$ by the definition of constrained. $(c) \Leftrightarrow (d)$ is Corollary 4.5. $(d) \Rightarrow (b)$ and $(e) \Leftrightarrow (f)$ are trivial. It remains to prove $(e) \Rightarrow (c)$ and $(a) \Rightarrow (e)$.

(e) $\Rightarrow$ (c) : Let $p_i \in K[x_i]$ be the minimal polynomial for b_i over K. Then c is a zero of the ideal $\kappa = (p_1, ..., p_n) \in K[X]$ so $\kappa \subset \mathfrak{i}(\{c\})$. We claim that κ is a maximal ideal (and the same therefore holds for $\mathfrak{i}(\{c\})$).

If b_i is the image of x_i in quotient ring $K[x]/\kappa$ then $K[x]/\kappa = K[c] := K[b_1, b_2, \ldots, b_n]$. However each b_i is algebraic over K (being a zero of p_i), so $K[c]$ is an algebraic extension of K. It is therefore a field, and κ must be a maximal ideal. We now use Corollary 4.7 to obtain condition (c).

(a) $\Rightarrow$ (e): By Proposition 4.6 there exists a polynomial $p \in K[X]$ such that the ideal $\kappa = \mathfrak{i}(\{c\})$ is maximal with respect to the condition that it contain no power of p. Consider the ring of fractions

$$R = K[x]_p = K[x, \tfrac{1}{p}]$$

and the "extended ideal"

$$\kappa^e = \kappa R \subset R,$$

the elements of which consist of quotients of the form

$$\frac{q}{p^d}$$

with[18] $q \in \kappa$. We claim that κ^e is a maximal ideal of R.

To verify this we first observe that κ^e is proper. Indeed, if $1 \in \kappa^e$ then some power of p is in κ, which is a contradiction.

Now suppose that $\mathfrak{j} \subset R$ is an ideal that properly contains κ^e. Thus there exists $r/p^d \in \mathfrak{j}$ with $r \notin \kappa$. Because κ is maximal with respect to the condition that it contain no power of p we must have

$$p^e \in (\kappa, r)$$

for some natural number e (where (κ, r) denotes the ideal generated by κ and r). We therefore have

$$p^e = q + sr$$

[18] That $\kappa^e \subset R$ is an ideal is trivially verified.

for some $q \in \kappa$ and $s \in K[x]$, hence

$$p^{e-d} = \frac{q+sr}{p^d} = \frac{q}{p^d} + \frac{s}{1} \cdot \frac{r}{p^d} \in \mathfrak{j}\,.$$

However, since p^{e-d} is invertible in R it follows that $1 \in \mathfrak{j}$, and we conclude that κ^e is a maximal ideal of R.

Finally, consider the (surjective) substitution homomorphism

$$K[x] \to K[c], \qquad x_i \mapsto b_i.$$

The kernel is κ. Because $p(c) \neq 0$ this extends to a surjective homomorphism

$$R = K[x]_p \to K[c]_{p(c)}.$$

The kernel certainly contains κ^e, and since κ^e is maximal this last ideal must be the kernel. As an immediate consequence we see that $K[c]_{p(c)}$ must be a field. As it is finitely generated as a K algebra it follows from a well-known variation of the Nullstellensatz[19] that it must be an algebraic extension of K. This implies that each b_i is algebraic over K, which is exactly what we wanted to prove.

q.e.d.

[19]See, e.g., [Chu, §18, Corollary 18.7].

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P. Cassidy, R.C. Churchill and J.J. Kovacic
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