Differential schemes

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Abstract

We start with the definition of differential scheme using the treatment of Hartshorne. Immediately we find that there are problems: the global section functor of an affine differential scheme does not recover the original ring. We give some examples to show what goes wrong. We shall discuss what is known and not known.

Notation

All rings are commutative and unitary. We use the prefix Δ instead of the word "differential". The theory makes no distinction between ordinary and partial Δ-rings, but all the examples will be ordinary.

Throughout, R is a Δ-ring. For simplicity we assume that R is a Ritt algebra, i.e. it is an algebra over $\mathbb{Q}$. $X = \operatorname{Diffspec} R$ (which is defined below).

Diffspec

Definition

$X = \operatorname{Diffspec} R$ is the set of prime Δ-ideals of R.

The topology

Definition

If $\mathfrak{a}$ is a Δ-ideal then

$$V(\mathfrak{a}) = \{\mathfrak{p} \in X \mid \mathfrak{a} \subset \mathfrak{p}\}.$$

These sets are the closed sets in the *Kolchin* topology.

Theorem

$V(\mathfrak{a}) \approx \text{Diffspec}\, R/\mathfrak{a}$.

Definition

If $a \in R$ then

$$D(a) = \{\mathfrak{p} \in X \mid a \notin \mathfrak{p}\}.$$

These sets are the basic opens.

Theorem

$D(a) \approx \text{Diffspec}\, R_a$.

If the derivation on R is trivial $\operatorname{Diffspec} R = \operatorname{Spec} R$.
In general, $\operatorname{Diffspec} R \subsetneqq \operatorname{Spec} R$ and the definitions ensure that $\operatorname{Diffspec} R$ is given the induced topology.
However $\operatorname{Diffspec} R$ is, in general, neither open nor closed in $\operatorname{Spec} R$.

Consider $\mathbb{Q}[x]$, where $x' = 1$.

Lemma

$\mathbb{Q}[x]$ is Δ-simple, i.e. it has no proper non-zero Δ-ideal.

Proof.

If $\mathfrak{a} \subset \mathbb{Q}[x]$ is a Δ-ideal then

$$\mathfrak{a} = (P)$$

for some polynomial P. However $P' \in (P)$ so

$$P' = AP$$

for some polynomial A. If $A \neq 0$. the degree on the left is smaller than the degree on the right. Therefore $A = 0$, and $P \in \mathbb{Q}$, so $\mathfrak{a}$ contains 1. □

(This is false in characteristic p since $P = X^p$ has derivative 0 but is not a constant polynomial.)

Diffspec $\mathbb{Q}[x]$

Because (0) is the unique prime Δ-ideal of $\mathbb{Q}[x]$,

$$\operatorname{Diffspec} \mathbb{Q}[x] = \{(0)\}.$$

On the other hand, $\operatorname{Spec} \mathbb{Q}[x]$ is the set of prime ideals (not Δ-ideals) of $\mathbb{Q}[x]$, of which there are many.
Of course, (0) is one of them. The closure of $\{(0)\}$ in $\operatorname{Spec} \mathbb{Q}[x]$ is the set of all prime ideals of $\mathbb{Q}[x]$ that contain (0), i.e. all of $\operatorname{Spec} \mathbb{Q}[x]$. It is the generic point of $\operatorname{Spec} \mathbb{Q}[x]$.
In particular $\operatorname{Diffspec} \mathbb{Q}[x]$ is dense in $\operatorname{Spec} \mathbb{Q}[x]$.

In fact this is always the case, since minimal prime ideals of a ring (containing $\mathbb{Q}$) are always Δ-ideals. I.e. $\operatorname{Diffspec} R$ is dense in $\operatorname{Spec} R$.

X is neither open nor closed

Theorem

$X = \operatorname{Diffspec} \mathbb{Q}[x]$ *is neither open nor closed in* $\operatorname{Spec} \mathbb{Q}[x]$.

Proof.

The closure of X is $\operatorname{Spec} \mathbb{Q}[x]$.
If X were open, then

$$\operatorname{Spec} \mathbb{Q}[x] \setminus X = V(I)$$

for some ideal $I = (f) \subset \mathbb{Q}[x]$.
Choose any irreducible polynomial g that is not a factor of f.
Then g does not divide f and hence

$$I = (f) \not\subset (g)$$

i.e.

$$(g) \notin V(I), \qquad \text{hence} \qquad (g) \in X = \{(0)\},$$

which is a contradiction. □

If we do not assume that R is a Ritt algebra, then $\operatorname{Diffspec} R$ might be empty even though R is not the 0 ring.

Example

Consider $\mathbb{Z}[x]$, where $x' = 1$, and the Δ-ideal (p, x^p) (p being a prime number). Then

$$R = \mathbb{Z}[x]/(p, x^p) = \mathbb{F}_p[\overline{x}]$$

is a non-zero Δ-ring where

$$\overline{x}' = 1 \quad \text{and} \quad \overline{x}^p = 0.$$

So any prime Δ-ideal $\mathfrak{p}$ of R must contain $\overline{x}$ and therefore must contain $1 = \overline{x}'$, so cannot be prime. Thus R has *no* prime Δ-ideal, i.e. $\operatorname{Diffspec} R = \emptyset$.

Affine line

Definition

Let y be a Δ-indeterminate over a Δ-field k. Then

$$X = \operatorname{Diffspec} k\{y\}$$

is called the *affine line*.

Since $k\{y\} = k[y, y', y'', \ldots]$, X is a subset of algebraic affine space of dimension $\aleph_0$.
Note that X has many "large" closed subsets (closed subsets of the algebraic affine line are all finite). Indeed for any differential polynomial P

$$V(P)$$

is closed and is thought of as the set of all solutions of the differential equation $P = 0$.
If k is differentially closed we can assert that every closed set has a rational point, but not much more.

Structure sheaf

The structure sheaf is defined exactly as in Hartshorne.

Theorem

If $s \in \mathcal{O}_X(U)$, then, for $\mathfrak{p} \in U$,

$$s(\mathfrak{p}) = \begin{cases} \frac{a_1}{b_1} & \text{for } \mathfrak{p} \in D(b_1) \\ \vdots & \vdots \\ \frac{a_n}{b_n} & \text{for } \mathfrak{p} \in D(b_n) \end{cases}$$

and $D(b_1) \cup \ldots \cup D(b_n) = U$.

Equivalently the structure sheaf is the sheafification of the presheaf defined on basic opens by

$$D(b) \mapsto R_b.$$

Usual results

Theorem

The stalk $\mathcal{O}_{X,\mathfrak{p}}$ is Δ-isomorphic to the local ring $R_{\mathfrak{p}}$.

Theorem

If $\phi\colon R \to S$ is a Δ-homomorphism of rings then there is a induced mapping

$$
\begin{aligned}
{}^{a}\phi\colon\ Y = \operatorname{Diffspec} S \to X = \operatorname{Diffspec} R,\\
\phi^{\#}\colon \mathcal{O}_X \to \phi_*\mathcal{O}_Y
\end{aligned}
$$

Buium defines a Δ-scheme as a *scheme* whose sheaf consists of Δ-rings. Umemura calls this “a scheme with derivations”.

Carra Ferro changes the definition of the sheaf. If $X = \operatorname{Diffspec} R$ and $Y = \operatorname{spec} R$ then $\mathcal{O}_X(U)$ is defined to be $\mathcal{O}_Y(V)$ where V is the largest open subset of Y with $V \cap X = U$.

Hrushovski, Chatzidakas, and others, have studied difference schemes.

Ovchinnikov uses Shafarevich as a model. He only gets a prescheme and asks when it is a scheme.

Keigher was the first to study Δ-schemes.

Global sections

Definition

$\widehat{R} = \mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X)$ is the ring of global sections of X.

Theorem

There is a canonical mapping

$$\iota \colon R \to \widehat{R}, \qquad \iota(a)(\mathfrak{p}) = \frac{a}{1} \in R_{\mathfrak{p}}.$$

However ι is neither injective nor surjective in general.

Example

Let $R = \mathbb{Q}[x]$ where $x' = 1$. Because R is Δ-simple,

$$\operatorname{Diffspec} R = \{(0)\}$$

consists of a single point.

The ring of global sections is then the ring of all fractions $a/b \in R$ whose denominator is not in (0), i.e.

$$\widehat{R} = R_{(0)} = \mathbb{Q}(x).$$

The canonical mapping

$$\iota\colon R \to \widehat{R}$$

is the inclusion

$$\mathbb{Q}[x] \subset \mathbb{Q}(x).$$

It is injective, but not surjective.

An isomorphism

In fact
$$\operatorname{Diffspec}\mathbb{Q}[x] \approx \operatorname{Diffspec}\mathbb{Q}(x).$$

If we take $R = \mathbb{Q}(x)$ then the canonical mapping
$$\iota\colon R \to \widehat{R}$$

is a bijection.

$xy = 0$

Example

Let y be a Δ-indeterminate over $\mathbb{Q}[x]$ and

$$R = \mathbb{Q}[x]\{y\}/[xy] = \mathbb{Q}[x]\{\overline{y}\}.$$

We have $x\overline{y} = 0$ so any prime Δ-ideal of R must contain $\overline{y}$.

Since $R/[\overline{y}] \approx \mathbb{Q}[x]$ is Δ-simple, $[\overline{y}]$ is a maximal Δ-ideal of R. Therefore

$$\operatorname{Diffspec} R = \{[\overline{y}]\}$$

is a single point.

It follows that

$$\widehat{R} = R_{[\overline{y}]}.$$

However $x\overline{y} = 0$ and from a previous lecture $x^{n+1}\overline{y}^{(n)} = 0$.
Because $x \notin [\overline{y}]$, we see that $\overline{y}^{(n)} = 0$ in $R_{[\overline{y}]}$. I.e. $[\overline{y}]$ is contained in the kernel of ι.
Also, no polynomial in x alone is in $[\overline{y}]$, hence every polynomial in x alone is invertible in $\widehat{R}$. So the kernel of ι is $[\overline{y}]$ and

$$\widehat{R} = \mathbb{Q}(x).$$

The canonical mapping is neither injective nor surjective.
In fact

$$\operatorname{Diffspec} \mathbb{Q}[x]\{\overline{y}\} \approx \operatorname{Diffspec} \mathbb{Q}[x] \approx \operatorname{Diffspec} \mathbb{Q}(x).$$

Warning. The following theorem may be false.

Franck Benoist sent me an email pointing out a gap in the "proof" I gave of this theorem. I had taken it from Carra Ferro, who has a similar gap. At this time we do not know if it is true or false.

Theorem

$\iota_{\widehat{R}} \colon \widehat{R} \longrightarrow \widehat{\widehat{R}}$ *is a* Δ*-isomorphism.*

Thus "taking hat" is a kind of closure operation.

The mapping $\iota\colon R \to \widehat{R}$ and induces

$$^{a}\iota\colon \widehat{X} = \operatorname{Diffspec} \widehat{R} \to X = \operatorname{Diffspec} R$$

Theorem

$^{a}\iota$ is a homeomorphism of topological spaces.

This is due to Franck Benoist.
However we do not know whether or not it must be an isomorphism of schemes.

Differential zeros

In a ring, the condition $1 \in \operatorname{Ann}(z)$ is equivalent to the condition that $z = 0$.

Definition

$z \in R$ is a Δ-*zero* if $1 \in [\operatorname{Ann}(z)]$. The set of Δ-zeros of R is denoted by $\mathcal{Z} = \mathcal{Z}(R)$.

Theorem

$\mathcal{Z}$ is a Δ-ideal of R.

Proof.

We show that $\mathcal{Z}$ is closed under addition. If $z, w \in \mathcal{Z}$ then

$$\begin{aligned} 1 &\in \sqrt{[\operatorname{Ann}(z)]}\sqrt{[\operatorname{Ann}(w)]} = \sqrt{[\operatorname{Ann}(z)\operatorname{Ann}(w)]} \\ &\subseteqq \sqrt{[\operatorname{Ann}(z+w)]}, \end{aligned}$$

and therefore $1 \in [\operatorname{Ann}(z+w)]$. □

Example

Example

Let

$$R = \mathbb{Q}[x]\{y\}/[xy] = \mathbb{Q}[x]\{\overline{y}\}.$$

Then $x\overline{y} = 0$ so $x \in \mathrm{Ann}(\overline{y})$ and therefore

$$1 = x' \in [\mathrm{Ann}(\overline{y})]$$

so $\overline{y}$ is a Δ-zero.

In fact $\mathcal{Z} = [\overline{y}]$.

Theorem

$a \in R$ is a Δ-zero if and only if it goes to 0 in $R_{\mathfrak{p}}$ for every Δ-prime $\mathfrak{p}$.

Proof.

Suppose that $a/1 = 0 \in R_{\mathfrak{p}}$ for every prime Δ-ideal. Then there exist $b_{\mathfrak{p}} \notin \mathfrak{p}$ with with $b_{\mathfrak{p}} \in \mathrm{Ann}(a)$. This implies that $\mathrm{Ann}(a)$ is not contained in any prime Δ-ideal. By a previous lecture, this means that $1 \in [\mathrm{Ann}(a)]$, i.e. that a is a Δ-zero.
Now suppose that $a \in \mathcal{Z}$. Then $1 \in [\mathrm{Ann}(a)]$ so no prime Δ-ideal $\mathfrak{p}$ can contain $\mathrm{Ann}(a)$. Thus there exists $b_{\mathfrak{p}} \in \mathrm{Ann}(a)$ with $b_{\mathfrak{p}} \notin \mathfrak{p}$. But this means that

$$\frac{a}{1} = 0 \in R_{\mathfrak{p}}.$$

□

Kernel of ι

Corollary

The kernel of

$$\iota\colon R \longrightarrow \widehat{R}$$

is $\mathbb{Z}$.

Theorem

$\mathcal{Z}$ is contained in the nil radical of R, so reduced rings have no non-zero Δ-zeros.

Proof.

Let $a \in \mathcal{Z}$. Then

$$a = a \cdot 1 \in \sqrt{[a]}\sqrt{[\mathrm{Ann}(a)]} = \sqrt{[a\,\mathrm{Ann}(a)]} = \sqrt{[0]} = \text{nil radical}.$$

□

Corollary

If R is reduced then

$$\iota\colon R \to \widehat{R}$$

is injective.

A weaker condition is that R have no non-zero Δ-zero. However that condition is not preserved by taking rings of fractions. A better condition is the following.

Definition

R is *AAD* (Annihilators Are Differential) if

$$\mathrm{Ann}(a)$$

is a Δ-ideal for every $a \in R$.

We will not discuss this condition here. The analogous condition for difference rings is that the ring be “well-mixed”.

Reduced schemes

Definition

A Δ-scheme X is *reduced* if $\mathcal{O}_X(U)$ is reduced for each open set U.

This is equivalent to saying that each stalk is reduced.

Theorem

If R is a reduced ring then $X = \operatorname{Diffspec} R$ is a reduced Δ-scheme.

But the converse is false.

Counterexample

Example

We have seen that

$$X = \text{Diffspec}\,\mathbb{Q}[x] \approx \text{Diffspec}\,\mathbb{Q}[x]\{\overline{y}\}$$

where $x\overline{y} = 0$. X is reduced since $\mathbb{Q}[x]$ is a reduced ring. However $\mathbb{Q}[x]\{\overline{y}\}$ is not reduced, indeed

$$(x\overline{y}')^2 = x(x^2\overline{y}') = 0$$

but $x\overline{y}' \neq 0$.

Reduced schemes

Theorem

If X is reduced then $\widehat{R}$ is reduced and $X \approx \widehat{X} = \operatorname{Diffspec} \widehat{R}$.

Thus we can always choose a reduced ring S with $X \approx \operatorname{Diffspec} S$.

Differential units

In a ring, the condition $1 \in (u)$ is equivalent to the condition that u is a unit (is invertible).

Definition

$u \in R$ is a Δ-*unit* if $1 \in [u]$. The set of Δ-units R is denoted by $\mathcal{U} = \mathcal{U}(R)$.

Theorem

$\mathcal{U}$ is a multiplicative set of R.

Example

Example

Let $R = \mathbb{Q}[x]$.
If $f \in R$ is not zero then the differential ideal $[f]$ is not zero and therefore is R itself. (Recall that R is Δ-simple.) Hence

$$1 \in [f]$$

and f is a Δ-unit.
Therefore

$$\mathcal{U} = R^*$$

The phenomenon of Δ-units was first noted by Cassidy.

Example

Let
$$R = \mathbb{Q}\{y\}/[y' - y] = \mathbb{Q}[e^x].$$
Then, for every $c \in \mathbb{Q}$, $c \neq 0$,
$$e^x - c$$
is a Δ-unit.
Indeed
$$c = (e^x - c)' - (e^x - c) \in [e^x - c]$$

In Cassidy's language
$$\frac{1}{y - c}$$
is an everywhere defined function on the variety defined by $y' = y$.

Differential units and zero divisors

Because $\mathcal{U}$ is a multiplicative set, we may form the ring of fractions $R\mathcal{U}^{-1}$, and we have a canonical homomorphism

$$j\colon R \to R\mathcal{U}^{-1}.$$

However j need not be injective.

Example

Let

$$R = \mathbb{Q}[x]\{y\}/[xy] = \mathbb{Q}[x]\{\overline{y}\}.$$

Then $1 \in [x]$ so x is a Δ-unit.
But it is also a zero divisor:

$$x\overline{y} = 0.$$

So

$$\overline{y} \to \frac{\overline{y}}{1} = 0 \in R\mathcal{U}^{-1}.$$

Kernel of $R \to R\mathfrak{U}^{-1}$

Theorem

The kernel of

$$R \to R\mathfrak{U}^{-1}$$

is contained in $\mathcal{Z}$*, the* Δ*-ideal of* Δ*-zeros.*

Hence $R \to R\mathfrak{U}^{-1}$ is injective if R is reduced.

I do not know, in general, if the kernel always equals $\mathcal{Z}$ or not.

Theorem

$a \in R$ is a Δ-unit if and only if it goes to a unit in $R_{\mathfrak{p}}$ for every Δ-prime $\mathfrak{p}$.

Proof.

Suppose that $a/1 \in R_{\mathfrak{p}}$ is a unit for every prime Δ-ideal. Then a cannot be in any prime Δ-ideal so $1 \in [a]$, i.e. a is a Δ-unit. Conversely if a is a Δ-unit, then it is not contained in any prime Δ-ideal (since $1 \in [a]$) and so it is a unit in every localization $R_{\mathfrak{p}}$. □

Global sections

Theorem

Every Δ-unit in $\widehat{R}$ is a unit.

Therefore $\iota\colon R \to \widehat{R}$ may be extended to a homomorphism

$$R\mathcal{U}^{-1} \to \widehat{R}.$$

Is this surjective? No.

Kolchin gave an example and said that “it can be shown that ...”. The example is

$$R = \mathbb{Q}\{y, z\}/[yz'' - z'(y' + 1)] = \mathbb{Q}\{\overline{y}, \overline{z}\}.$$

So

$$\overline{yz}'' = \overline{z}'(\overline{y}' + 1).$$

This is an integral domain. Consider the global section:

$$s(\mathfrak{p}) = \begin{cases} \dfrac{z'}{y} & \mathfrak{p} \in D(y) \\ \dfrac{z''}{y' + 1} & \mathfrak{p} \in D(y' + 1) \end{cases}$$

(I've dropped the bars to simplify the notation.)

s is not in $R\mathcal{U}^{-1}$. This means that there do not exist $a \in \mathcal{R}$, $b \in \mathcal{U}$ with

$$s(\mathfrak{p}) = \frac{a}{b} \qquad \text{for all } \mathfrak{p}.$$

Almost surjective

For reduced rings $\iota\colon R \to \widehat{R}$ is injective. But it is not necessarily surjective.

Definition

A mapping of Δ-rings $\phi\colon R \to S$ is *almost surjective* if for every $s \in S$ there exist $a_1, b_1, \ldots, a_n, b_n \in R$ such that

$$\phi(b_i)s = \phi(a_i)$$

and

$$1 \in [b_1, \ldots, b_n].$$

Thus ϕ is like a map of R into a ring of fractions, except that we allow several denominators.
The name "almost surjective" is due to Franck Benoist. A better name is solicited.

Global sections

Recall that a global section $s \in \widehat{R}$ satisfies

$$s(\mathfrak{p}) = \begin{cases} \dfrac{a_1}{b_1} & \text{for } \mathfrak{p} \in D(b_1) \\ \vdots & \vdots \\ \dfrac{a_n}{b_n} & \text{for } \mathfrak{p} \in D(b_n) \end{cases}$$

and $1 \in [b_1, \ldots, b_n]$.

This says that $\iota(b_i)s_i(\mathfrak{p}) = \iota(a_i)$ for all $\mathfrak{p} \in D(b_i)$.
But it does *not* say that $\iota(b_i)s(\mathfrak{q}) = \iota(a_i)$ for $q \notin D(b_i)$.
So it does not imply that ι is almost surjective.

Theorem

If R is reduced then $\iota \colon R \to \widehat{R}$ is almost surjective.

We do not know whether this is true or false in general.

Let $X = \operatorname{Diffspec} R$ and $\widehat{X} = \operatorname{Diffspec} \widehat{R}$.
We mentioned earlier that

$$^{a}\iota \colon \widehat{X} \to X$$

is always a homeomorphism.

Theorem

If ι is almost surjective, then $^{a}\iota$ is an isomorphism of schemes.

I don't know if the converse is true or false.

Corollary

If R is reduced then $X \approx \widehat{X}$.

Δ-S-schemes

Let R and S be Δ-rings and set $X = \operatorname{Diffspec} R$ and $Y = \operatorname{Diffspec} S$.

Definition

X is a Δ-scheme over Y if there is a morphism $X \to Y$.

However, this does not imply that R is a S-algebra.

Example

Let $R = \mathbb{Q}[x]$ and $S = \mathbb{Q}(x)$. Then X is a Δ-scheme over Y (in fact, we saw earlier that they are isomorphic). However R is not a S-algebra.

Theorem

If X is a Δ-scheme over Y and R is reduced then $X \approx \operatorname{Diffspec} \widehat{R}$ and $\widehat{R}$ is a Δ-S algebra.

Thus we can replace R with a ring that is indeed a Δ-S algebra.

Products

This affects the existence of products. If $X = \operatorname{Diffspec} R$ and $Z = \operatorname{Diffspec} T$ are Δ-schemes over $Y = \operatorname{Diffspec} S$ then

$$X \times_Y Z \stackrel{?}{=} \operatorname{Diffspec} R \otimes_S T.$$

But the right-hand side need not make sense!
If R and T are reduced then we can replace them with S-algebras and we can define the product.
However, in general, we do not know if products exist in the category of Δ-schemes.

Subschemes

Suppose that $Y \subset X$ is a closed subscheme. Does there exist an ideal $\mathfrak{a} \subset R$ such that $Y \approx \mathrm{Diffspec}(R/\mathfrak{a})$?
I don't know.

Theorem

If $Y \subset X$ are both reduced then there is a radical Δ-ideal $\mathfrak{a} \subset R$ such that $Y \approx \mathrm{Diffspec}(R/\mathfrak{a})$.

Adjunction

In algebraic geometry there is an adjunction between Spec and the global sections functor. Here too.

Theorem

Suppose that $X = \operatorname{Diffspec} R$ *and* $Y = \operatorname{Diffspec} S$. *Then there is a bijection*

$$\operatorname{Mor}(Y, X) \approx \operatorname{Hom}(R, \widehat{S})$$

This theorem is due to William Keigher.
This is not symmetric in R and S!
It is a problem for compositions:

$$\operatorname{Mor}(Z, Y) \times \operatorname{Mor}(Y, X) \to \operatorname{Mor}(Z, X)$$

gives

$$\operatorname{Hom}(R, \widehat{S}) \times \operatorname{Hom}(S, \widehat{T}) \to \operatorname{Hom}(R, \widehat{T}).$$

They don't match up!

Adjunction for reduced Δ-schemes

Theorem

If R and S are reduced then

$$\mathrm{Mor}(Y, X) \approx \mathrm{Hom}(R, \widehat{S}) \approx \mathrm{Hom}(\widehat{R}, \widehat{S}).$$

It's not pretty but at least it is symmetric.

I do not know if it is true or false in general.

Rittian Δ-rings

Definition

A Δ-ring R is *Rittian* if every strictly increasing chain of radical Δ-ideals is finite.

Theorem

If R is finitely Δ-generated over a Δ-field K then R is Rittian.

Theorem

R is Rittian if and only if X is a Noetherian topological space.

This is actually better than the algebraic version!

Finite type

Definition

$X = \operatorname{Diffspec} R$ is a Δ-scheme of finite type over $Y = \operatorname{Diffspec} S$ if R is a Δ-finitely generated S-algebra.

Suppose that X is also reduced. We can replace R by $\widehat{R}$, which is reduced, and $X \approx \operatorname{Diffspec} \widehat{R}$, and $\widehat{R}$ will be a reduced Δ-ring. But it is usually *not* finitely Δ-generated over S.

Theorem

If X is reduced and of finite type over Y then there exists a reduced Δ-ring R_1 which is finitely Δ-generated over S such that $X \approx \operatorname{Diffspec} R_1$.

So, again, we can replace R by a more amenable Δ-ring.

Reduced Rittian Ritt algebras

Suppose that R is both reduced and Rittian.

Theorem

R has a finite number of minimal prime ideals and they are Δ-ideals.

Theorem

The complete ring of quotients of R is a finite product of Δ-fields.

Theorem

R has a finite number of minimal idempotents.

Using these we get the usual theorems about connected and irreducible components of X.

Rational functions

If R is reduced then the canonical mapping $\iota\colon R \to \widehat{R}$ is injective and we identify R with a subring of $\widehat{R}$.

Theorem

If R is a domain then so is $\widehat{R}$ and $\mathrm{qf}(R) = \mathrm{qf}(\widehat{R})$.

Hence the field of rational functions of an irreducible variety is what you expect.

If R is not a domain, the ring of rational functions classically is $Q(R)$, the complete of fractions of R, i.e. $R\Sigma^{-1}$ where Σ is the multiplicative set of elements that are not zero divisors in R.

Rational functions of a reducible Δ-scheme

Recall that a global section has the form

$$s(\mathfrak{p}) = \begin{cases} \dfrac{a_1}{b_1} & \text{for } \mathfrak{p} \in D(b_1) \\ \vdots & \vdots \\ \dfrac{a_n}{b_n} & \text{for } \mathfrak{p} \in D(b_n) \end{cases}$$

We would like to have $s \in Q(R)$, however it could happen that every b_i is a zero divisor. Nonetheless

Theorem

If R is a reduced Rittian Ritt algebra then there is an injective homomorphism

$$\widehat{R} \longrightarrow Q(R).$$

If we identify $\widehat{R}$ with its image then $\widehat{R}$ is the subring of $Q(R)$ consisting all everywhere defined functions (in the sense of Cassidy)

Work to be done

Almost all of the known results are for reduced Δ-schemes (or the slight generalization AAD). Any results for schemes without this hypothesis would be welcome.
Nothing has been done about tangent space, sheaves of modules, cohomology, Δ-group schemes, singularities, etc.

In fact, if you don't see it here, it probably doesn't exist.

Differential Galois theory

My reason for interest in differential schemes is the following.

Definition

Let X^Δ be the local ringed space where

1. the topological spaces X^Δ and X are the same,
2. $\mathcal{O}_{X^\Delta}(U)$ is the ring of constants of $\mathcal{O}_X(U)$, i.e.

$$\mathcal{O}_{X^\Delta}(U) = \mathcal{O}_X(U)^\Delta.$$

In general X^Δ is a local ringed space but not a scheme.
Let K be a strongly normal (differential Galois) extension of k and

$$R = K \otimes_k K, \qquad X = \operatorname{Diffspec} R.$$

Theorem

X^Δ is a group scheme of finite type over k^Δ whose closed points are canonically identified with the Galois group of K over k.

Thank you

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