THE INFINITE QUATERNION GROUP $\mathbf{Q}_\infty$ AND THE AIRY EQUATION

The following material was presented by Lourdes Juan in the KSDA on Sunday, May 07, 2006 at Hunter College. It is part of a joint paper with Arne Ledet.

Example. Let Q_∞ denote the group generated by $\mathbb{G}_m$ and j, subject to the conditions $j^2 = -1 \in \mathbb{G}_m$ and $jxj^{-1} = x^{-1}$ for $x \in \mathbb{G}_m$.

Then Q_∞ is generated by the matrices

$$a = \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}, \quad j = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where $a \in \mathbb{G}_m$. This group contains all the quaternion groups $Q_{4n} = \langle \zeta, j \rangle$, where ζ is a primitive $2n^{\text{th}}$ root of unity, hence the name Q_∞.

Since the connected component $\mathbb{G}_m$ has trivial cohomology, we can take the crossed homomorphism to be of the form $f\colon \mathrm{Gal}(M/K) \to Q_\infty(M)$, where M/K is a quadratic extension. Let τ be a generator for $\mathrm{Gal}(M/K)$, and let $f_\tau = cj$, $c \in M^*$. Since $1 = f_1 = f_\tau \tau(f_\tau) = cj\tau(c)j = -c/\tau(c)$, we have $\tau(c) = -c$, and can write $M = K(\sqrt{b})$, where $b = c^2$ and $c = \sqrt{b}$.

The coordinate ring for the torsor is $R = M[x, 1/x]$, where $\left(\begin{smallmatrix} x & 0 \\ 0 & 1/x \end{smallmatrix}\right)$ is the generic element in $\mathbb{G}_m$. The $Q_\infty(\mathfrak{C})$-action on R is then

$$\begin{aligned} a\colon \sqrt{b} &\mapsto \sqrt{b}, \quad && x \mapsto ax, \\ j\colon \sqrt{b} &\mapsto -\sqrt{b}, && x \mapsto \sqrt{b}/x. \end{aligned}$$

If we want a derivation on R, given by $x' = ax$, $a \in M$, such that $Q_\infty(\mathfrak{C})$ provides differential automorphisms, we must have

$$\begin{aligned} \tau(a)\frac{\sqrt{b}}{x} &= j(ax) = j(x') = (j(x))' = \Big(\frac{\sqrt{b}}{x}\Big)' \\ &= \frac{\frac{b'}{2\sqrt{b}}x - \sqrt{b}ax}{x^2} = \Big(\frac{b'}{2b} - a\Big)\frac{\sqrt{b}}{x}, \end{aligned}$$

i.e.,

$$a + \tau(a) = \frac{b'}{2b},$$

or

$$a = \frac{b'}{4b} + \alpha\sqrt{b}, \quad \alpha \in K$$

(note that $a + \tau(a)$ is the trace of a).

Of course, we only get a Picard-Vessiot extension if there are no new constants, i.e., if a is not a rational multiple of a logarithmic derivative in $K(\sqrt{b})$. In that case, the extension $K(\sqrt{b}, x)/K$ is the Picard-Vessiot extension for the differential equation

$$y'' - (\frac{\alpha'}{\alpha} + \frac{b'}{b})y' - \left((\frac{b'}{4b})' - (\frac{\alpha'}{\alpha} + \frac{3b'}{4b})\frac{b'}{4b} + \alpha^2 b\right)y = 0,$$

which has x and $\sqrt{b}/x$ as linearly independent solutions.

For instance: Let $K = \mathfrak{C}(t)$, with $t' = 1$, and take $b = t$. Since all logarithmic derivatives in $\mathfrak{C}(\sqrt{t})$ are rational functions in $\sqrt{t}$ of negative degree, we can then take $\alpha = 1$, and get a $Q_\infty(\mathfrak{C})$-extension

$$\mathfrak{C}(\sqrt{t}, x)/\mathfrak{C}(t),$$

where $t' = 1$ and $x' = (1/4t + \sqrt{t})x$.

More 'generically', we can let $K = \mathfrak{C}\langle \alpha, b\rangle$, where α and b are differential indeterminates.

Remark. Let $K = \mathfrak{C}((1/t))$ be the Laurent series field in $1/t$, with the usual derivation. Then the Ricatti equation $v' + v^2 = t$ has exactly two solutions in $M = K(\sqrt{t}) = \mathfrak{C}((1/\sqrt{t}))$, and these are conjugate under the Galois action. We let

$$a = \sqrt{t} + \sum_{n=0}^{\infty} a_n t^{-n/2}$$

be one of them.

The differential equation $w' - 2aw = 1$ has a (unique) solution in M, which we will simply call w. Then $w'/w - a$ is a solution to the Ricatti equation, i.e., $w'/w - a = \tau(a)$, when τ is the generator for $\mathrm{Gal}(M/K)$. In particular, $w'/w = a + \tau(a) \in K$, which means that w and $\tau(w)$ have the same logarithmic derivative, and therefore that they differ by a constant: $\tau(w) = cw$ for some $c \in \mathfrak{C}^*$. Since $w \notin K$ and $\tau^2 = 1$, we get $c = -1$ and $\tau(w) = -w$.

Consequently, $b = w^2 \in K$, and $M = K(\sqrt{b})$, with $\sqrt{b} = w$.

Now, with $\alpha = -\frac{1}{2}w^{-1}$, we get $a = b'/4b + \alpha\sqrt{b}$. A logarithmic derivative in M has no terms in degree ≥ -1, so a is not a rational multiple of a logarithmic derivative, and if we let $x' = ax$, we get a Picard-Vessiot extension with differential Galois group $Q_\infty(\mathfrak{C})$. The corresponding differential equation is the Airy equation

$$y'' - ty = 0.$$

Note. The Airy equation is also discussed in M. van der Put & M. F. Singer, *Galois Theory of Linear Differential Equations*, Grundlehren der mathematischen Wissenschaften **328**, Springer-Verlag, 2003.