A Bound for the Order of Derivatives in the Rosenfeld-Gröbner Algorithm

Oleg Golubitsky

oleg.golubitsky@gmail.com

NSERC Postdoctoral Fellow

Queen's University

Kingston, Ontario

In collaboration with

Marc Moreno Maza, Marina V. Kondratieva, and Alexey Ovchinnikov

Outline

- Introduction:
  - Jacobi bound for ODE systems
  - Ritt's proof of the Jacobi bound for linear systems
  - Motivation for our bound on the order of derivatives

Outline

- Introduction:
  - Jacobi bound for ODE systems
  - Ritt's proof of the Jacobi bound for linear systems
  - Motivation for our bound on the order of derivatives
- Bound for the special case of 2 variables

Outline

- Introduction:
  - Jacobi bound for ODE systems
  - Ritt's proof of the Jacobi bound for linear systems
  - Motivation for our bound on the order of derivatives
- Bound for the special case of 2 variables
- General case: n variables
  - What if the set of leading variables is fixed?
  - How can the set of leading variables change?
  - Weak d-triangular sets (E. Hubert's modification of the Rosenfeld-Gröbner algorithm)
  - Algebraic reduction w.r.t. a weak d-triangular set preserving the bound
  - Final algorithm and proof of the bound

Notation

- $\mathbb{K}$ is an ordinary differential field of characteristic zero with derivation $\delta : \mathbb{K} \to \mathbb{K}$:

$$\delta(a+b) = \delta(a) + \delta(b), \quad \delta(ab) = \delta(a)b + a\delta(b).$$

- $Y = \{y_1, \ldots, y_n\}$ is a set of *differential indeterminates*.
- $\delta^\infty Y = \{\delta^m y \mid y \in Y,\ m = 0, 1, 2, \ldots\}$ is the set of *derivatives*.
- $\mathbb{K}\{Y\} = \mathbb{K}[\delta^\infty Y]$ endowed with $\delta : \mathbb{K}\{Y\} \to \mathbb{K}\{Y\}$ is the *differential ring* of differential polynomials.

Jacobi bound for linear systems

- Given a system of n linear differential polynomials

$$L_1, \ldots, L_n \in \mathbb{K}\{Y\}$$

which for every $y \in Y$ implies an equation in y alone.

- Let $a_{ij} = \operatorname{ord}_{y_j} L_i$, $1 \le i, j \le n$
(here we assume that $\operatorname{ord}_y f = -\infty$ if f does not involve any derivatives of y)

- For a permutation $\pi \in S_n$, let

$$d_\pi = a_{1\pi(1)} + \ldots + a_{n\pi(n)}$$

be called a *diagonal sum*.

- Let $h = \max_{\pi \in S_n} d_\pi$.

Jacobi bound for linear systems

Theorem *[Ritt, 1935] There exists a triangular system of differential polynomials $R_1, \ldots, R_n$ equivalent to $L_1, \ldots, L_n$ and satisfying* $\sum_{i=1}^{n} \operatorname{ord}_{y_i} R_i \leq h$.

Proof...

- Show that there exists a finite diagonal sum.
- Consider elimination ranking $y_1 > \ldots > y_n$.
- If a_{i1} participates in a maximum diagonal sum, then reduction w.r.t. L_i, if it is possible, does not increase h.

Jacobi bound for linear systems

Theorem *[Ritt, 1935] There exists a triangular system of differential polynomials* $R_1, \ldots, R_n$ *equivalent to* $L_1, \ldots, L_n$ *and satisfying* $\sum_{i=1}^{n} \operatorname{ord}_{y_i} R_i \leq h$.

Proof...

If such reductions are not possible, this is because

- Only one L_i involves $y_1 \Rightarrow$ proceed similarly with the elimination of $y_2, \ldots, y_{n-1}$.
- There exists i such that a_{i1} is maximal among $a_{11}, \ldots, a_{n1}$ and participates in a finite diagonal sum. Without loss of generality, assume that $i = n$.

Jacobi bound for linear systems

Theorem *[Ritt, 1935] There exists a triangular system of differential polynomials* $R_1, \ldots, R_n$ *equivalent to* $L_1, \ldots, L_n$ *and satisfying* $\sum_{i=1}^{n} \operatorname{ord}_{y_i} R_i \leq h$.

Proof...

- Change the indices of $L_1, \ldots, L_{n-1}$ and $y_2, \ldots, y_n$ so that
$$a_{11} + \ldots + a_{n-1,n-1}$$
is maximal. This sum is finite.
- Then one can reduce L_n w.r.t. L_1 without increasing h.

□

Notation

- Fix a ranking $<$: a total order on derivatives such that for all $u, v \in \delta^\infty Y$

$$[u < \delta u] \text{ and } [u < v \Rightarrow \delta u < \delta v].$$

Notation

- Fix a ranking $<$: a total order on derivatives such that for all $u, v \in \delta^\infty Y$

$$[u < \delta u] \text{ and } [u < v \Rightarrow \delta u < \delta v].$$

- For a polynomial f, let $u_f = \delta^k y_i$ be the derivative of the highest rank w.r.t. $\leq$ occurring in f. Then

$$f = \mathbf{i}_f u_f^d + g(u_f), \ \deg g < d.$$

Notation

- Fix a ranking $<$: a total order on derivatives such that for all $u, v \in \delta^\infty Y$

$$[u < \delta u] \ \text{and} \ [u < v \ \Rightarrow \ \delta u < \delta v].$$

- For a polynomial f, let $u_f = \delta^k y_i$ be the derivative of the highest rank w.r.t. $\leq$ occurring in f. Then

$$f = \mathbf{i}_f u_f^d + g(u_f), \ \ \deg g < d.$$

- $\mathrm{lv}\, f = y_i, \ \ \mathrm{ld}\, f = u_f, \ \ \mathrm{rk}\, f = u_f^d, \ \ \mathbf{s}_f = \mathbf{i}_{\delta f}.$

Notation

- Fix a ranking $<$: a total order on derivatives such that for all $u, v \in \delta^\infty Y$

$$[u < \delta u] \text{ and } [u < v \Rightarrow \delta u < \delta v].$$

- For a polynomial f, let $u_f = \delta^k y_i$ be the derivative of the highest rank w.r.t. $\leq$ occurring in f. Then

$$f = \mathbf{i}_f u_f^d + g(u_f), \ \deg g < d.$$

- $\operatorname{lv} f = y_i, \ \operatorname{ld} f = u_f, \ \operatorname{rk} f = u_f^d, \ \mathbf{s}_f = \mathbf{i}_{\delta f}$.
- Ranks $u_1^{d_1}$ and $u_2^{d_2}$ can be compared w.r.t. $<$:

$$u_1^{d_1} < u_2^{d_2} \iff [u_1 < u_2] \text{ or } [u_1 = u_2 \text{ and } d_1 < d_2].$$

Some basics of differential algebra

- Polynomial f is *algebraically reduced* w.r.t. g, if $\deg_{u_g} f < \deg_{u_g} g$.
- f is *partially reduced* w.r.t. g, if f is free of $\delta^k u_g$, $k > 0$.
- f is *(fully) reduced* w.r.t. g, if f is algebraically and partially reduced w.r.t. g.
- Set A is *autoreduced*, if every element of A is reduced w.r.t. every other element of A.
- For an autoreduced set A, let $\min A$ denote the polynomial in A of the least rank.
- For autoreduced sets A and B, $\operatorname{rk} A < \operatorname{rk} B$ iff

$$[\operatorname{rk} B \subset \operatorname{rk} A] \text{ or } [\min(\operatorname{rk} A \setminus \operatorname{rk} B) < \min(\operatorname{rk} B \setminus \operatorname{rk} A)].$$

Regular ideals

- For any finite polynomial sets A, H, ideal

$$[A] : H^\infty = \{f \mid \exists\, h \in H^\infty \;\; hf \in [A]\}$$

  is differential.

- Ideal $[A] : H^\infty$ is called *regular*, if
  - A is autoreduced
  - $H \supseteq H_A = \{\mathbf{i}_f, \mathbf{s}_f \mid f \in A\}$
  - H is partially reduced w.r.t. A.

- **Theorem.** [Boulier et al, 1995] *Regular ideals are radical.*

- **Rosnefeld's Lemma.** *If differential ideal $[A] : H^\infty$ is regular and polynomial f is partially reduced w.r.t. A, then*

$$f \in [A] : H^\infty \iff f \in (A) : H^\infty$$

Regular decomposition

- The Rosenfeld-Gröbner algorithm yields a regular decomposition of a radical differential ideal:

$$\{F\} = \bigcap_{i=1}^{k} R_i, \quad R_i = [A_i] : H_i^\infty.$$

- There exist efficient *algebraic* methods (plus parallel and modular Monte-Carlo algorithms currently under development by M. Moreno Maza et al) for computing a regular decomposition of a radical ideal:

$$\sqrt{G} = \bigcap_{i=1}^{l} J_i, \quad J_i = (A_i) : H_i^\infty.$$

Motivation for our bound

- Given a system of differential polynomials F, find a number d, so that every algebraic regular decomposition of the radical algebraic ideal

$$\sqrt{F^{(d)}}, \quad F^{(d)} = \{f^{(i)} \mid f \in F, \ \ 0 \le i \le d\}$$

  "yields" a regular decomposition of $\{F\}$.

- **First step:** estimate the order of differential polynomials in a regular decomposition

$$\{F\} = \bigcap_{i=1}^{k} [A_i] : H_i^{\infty}.$$

Rosenfeld-Gröbner algorithm

Algorithm Rosenfeld-Gröbner(F_0)

Input: A finite set of differential polynomials F_0

Output: A finite set T of regular systems such that $\{F_0\} = \bigcap_{(A,H)\in T} [A] : H^\infty$

$T := \varnothing$

$U := \{(F_0, \varnothing)\}$

while $U \neq \varnothing$ **do**

  Take and remove any $(F, H) \in U$

  Let C be an autoreduced subset of F of the least rank

  $R :=$ d-rem$(F \setminus C, C) \setminus \{0\}$

  if $R = \varnothing$ **then**

    if $1 \notin (C) : (\text{d-rem}(H, C) \cup H_C)^\infty$ **then** $T := T \cup \{(C, \text{d-rem}(H, C) \cup H_C)\}$

  else

    $U := U \cup \{(C \cup R, H \cup H_C)\}$

  end if

  $U := U \cup \{(F \cup \{h\}, H) \mid h \in H_C,\ h \in \mathbb{K}\}$

end while

return T

Special case: $n = 2$

- Let $F \subset \mathbb{K}\{y, z\}$.
- Let $m_y(F)$ and $m_z(F)$ be the the maximal orders of derivatives of y and z occurring in F.
- Let $M(F) = m_y(F) + m_z(F)$.

Lemma. *For all* $(F, H) \in U$ *in the Rosenfeld-Gröbner algorithm,*

$$M(F) \leq M(F_0).$$

Proof...
Show that $M(F)$ cannot increase in the Rosenfeld-Gröbner algorithm:

- Let $(F, H) \in U$.
- Let C be an autoreduced subset of F of the least rank.

Special case: $n = 2$

- Let $F \subset \mathbb{K}\{y, z\}$.
- Let $m_y(F)$ and $m_z(F)$ be the the maximal orders of derivatives of y and z occurring in F.
- Let $M(F) = m_y(F) + m_z(F)$.

Lemma. *For all $(F, H) \in U$ in the Rosenfeld-Gröbner algorithm,*

$$M(F) \leq M(F_0).$$

Proof...

- $|C| \leq 2$.
- Let $R = \text{d-rem}(F \setminus C, C)$.

Special case: $n = 2$

- Let $F \subset \mathbb{K}\{y, z\}$.
- Let $m_y(F)$ and $m_z(F)$ be the the maximal orders of derivatives of y and z occurring in F.
- Let $M(F) = m_y(F) + m_z(F)$.

Lemma. *For all $(F, H) \in U$ in the Rosenfeld-Gröbner algorithm,*

$$M(F) \leq M(F_0).$$

Proof...

- Let $|C| = 1$. Without loss of generality, $\operatorname{ld} C = \{y^{(d_y)}\}$.
- $m_y(C \cup R) = d_y$, $m_z(C \cup R) \leq m_z(F) + (m_y(F) - d_y)$.
- Therefore $M(C \cup R) \leq M(F)$.

Special case: $n = 2$

- Let $F \subset \mathbb{K}\{y, z\}$.
- Let $m_y(F)$ and $m_z(F)$ be the the maximal orders of derivatives of y and z occurring in F.
- Let $M(F) = m_y(F) + m_z(F)$.

Lemma. *For all $(F, H) \in U$ in the Rosenfeld-Gröbner algorithm,*

$$M(F) \leq M(F_0).$$

Proof...

- Let $|C| = 2$. Then $\operatorname{ld} C = \{y^{(d_y)}, z^{(d_z)}\}$ and

$$M(C \cup R) = d_y + d_z \leq M(F).$$

Special case: $n = 2$

- Let $F \subset \mathbb{K}\{y, z\}$.
- Let $m_y(F)$ and $m_z(F)$ be the the maximal orders of derivatives of y and z occurring in F.
- Let $M(F) = m_y(F) + m_z(F)$.

Lemma. *For all* $(F, H) \in U$ *in the Rosenfeld-Gröbner algorithm,*

$$M(F) \leq M(F_0).$$

Proof...

- Finally, if $G \subset F \cup H_F$, then $M(G) \leq M(F)$.

□

General case; fixed leading variables

- Let $F \subset \mathbb{K}\{y_1, \ldots, y_n\}$.
- Let C be an autoreduced subset of F of the least rank with

$$\operatorname{ld} C = \{y_1^{(d_1)}, \ldots, y_k^{(d_k)}\}.$$

- Then

$$m_i(C \cup R) \leq \begin{cases} d_i, & i = 1, \ldots, k \\ m_i(F) + \max\limits_{1 \leq j \leq k} (m_j(F) - d_j), & i = k+1, \ldots, n \end{cases}$$

General case; fixed leading variables

Define

$$M_{\mathrm{lv}\,C}(F) = M_{y_1,\dots,y_k}(F) = (n-k)\sum_{i=1}^{k} m_i(F) + \sum_{i=k+1}^{n} m_i(F) \quad (1 \le |C| < n).$$

Then inequality

$$m_i(C \cup R) \le \begin{cases} d_i, & i = 1,\dots,k \\ m_i(F) + \max\limits_{1\le j\le k}(m_j(F) - d_j), & i = k+1,\dots,n \end{cases}$$

implies:

$$\begin{aligned} &M_{\mathrm{lv}\,C}(C \cup R) = M_{y_1,\dots,y_k}(C \cup R) = \\ &\quad (n-k)\sum_{i=1}^{k} m_i(C \cup R) + \sum_{i=k+1}^{n} m_i(C \cup R) \le \\ &\quad (n-k)\sum_{i=1}^{k} d_i + \sum_{i=k+1}^{n} m_i(F) + (n-k)\max_{1\le j\le k}(m_j(F) - d_j) \le \\ &\quad (n-k)\sum_{i=1}^{k} m_i(F) + \sum_{i=k+1}^{n} m_i(F) - \\ &\quad -(n-k)\sum_{i=1}^{k}(m_i(F) - d_i) + (n-k)\max_{1\le j\le k}(m_j(F) - d_j) \le M_{\mathrm{lv}\,C}(F). \end{aligned}$$

Changing leading variables

- A non-leading variable y_{k+1} becomes leading:

$$
\begin{aligned}
&M_{y_1,\dots,y_{k+1}}(F) = \\
&\quad (n-k-1)\sum_{i=1}^{k+1} m_i(F) + \sum_{i=k+2}^{n} m_i(F) \le \\
&\quad M_{y_1,\dots,y_k}(F) + (n-k-2)m_{k+1}(F) \le \\
&\quad (n-k-1)M_{y_1,\dots,y_k}(F).
\end{aligned}
$$

- A leading variable becomes non-leading: make sure this does not happen!

Leading variables become non-leading

Example 1:

- $F = \{x, x^2 + z, y^2 + z\}$, $x > y > z$
- $C = \{x, y^2 + z\}$, $\mathrm{lv}\, C = \{x, y\}$
- $R = \text{d-rem}(F \setminus C, C) = \{z\}$
- $F_1 = C \cup R = \{x, y^2 + z, z\}$
- $C_1 = \{x, z\}$, $\mathrm{lv}\, C_1 = \{x, z\}$
- $y \in \mathrm{lv}\, C$ but $y \notin \mathrm{lv}\, C_1$
- y disappeared from leading variables only temporarily: reduce $y^2 + z$ w.r.t. z, and y becomes a leading variable again.
- $\Rightarrow$ one can try to replace autoreduced sets by weak d-triangular sets in the Rosenfeld-Gröbner algorithm

Leading variables become non-leading

Example 2:

- $F = \{x, x^2 + z, zy^2\}$, $x > y > z$
- $C = \{x, zy^2\}$, $\mathrm{lv}\, C = \{x, y\}$
- $R = \text{d-rem}(F \setminus C, C) = \{z\}$
- $F_1 = C \cup R = \{x, zy^2, z\}$
- $C_1 = \{x, z\}$, $\mathrm{lv}\, C_1 = \{x, z\}$
- y disappeared from leading variables permanently:

$$zy^2 \to_z 0.$$

- **Observation:** In the component (F_1, H_1), where $H_1 = H \cup H_C$, we have $z \in F_1 \cap H_1$, hence

$$\{F_1\} : H_1^\infty = (1).$$

Differentially triangular sets

- A set of polynomials A is a *weak differentially triangular* set, if $\operatorname{ld} A$ is autoreduced.
- A weak differentially triangular set A is *differentially triangular*, if every element of A is partially reduced w.r.t. the other elements of A.
- One can expand the definition of regular ideals [Hubert]: Ideal $[A] : H^\infty$ is called *regular*, if
  - A is differentially triangular
  - $H \supseteq \mathbf{s}_A$
  - H is partially reduced w.r.t. A.

Modified Rosenfeld-Gröbner algorithm

Algorithm Rosenfeld-Gröbner(F_0) (based on [Hubert, 2001])

Input: A finite set of differential polynomials F_0

Output: A finite set T of regular systems such that $\{F_0\} = \bigcap_{(A,H)\in T} [A] : H^\infty$

$T := \varnothing$

$U := \{(F_0 \setminus \{\min F_0\}, \{\min F_0\}, \varnothing)\}$

while $U \neq \varnothing$ **do**

  Take and remove any $(F, C, H) \in U$

  $R :=$ d-rem$(F, C) \setminus \{0\}$

  if $R = \varnothing$ **then** $T := T \cup$ Autoreduce&Check(C,$H \cup H_C$)

  else $C^> := \{p \in C \mid \operatorname{lv} p = \operatorname{lv}(\min R)\}$

    $\bar{C} := C \setminus C^> \cup \{\min R\}$ # Note: $\bar{C}$ is a weak d-triangular set s.t.

    $\bar{F} := C^> \cup R \setminus \{\min R\}$ # $\operatorname{rk} \bar{C} < \operatorname{rk} C$ and $\operatorname{lv} C \subseteq \operatorname{lv} \bar{C}$.

    $\bar{H} :=$ d-rem$(H \cup H_{\bar{C}}, \bar{C})$

    if $0 \notin \bar{H}$ **then** $U := U \cup \{(\bar{F}, \bar{C}, \bar{H})\}$

  end if

  $U := U \cup \{(F \cup \{h\}, C, H) \mid h \in H_C,\ h \notin \mathbb{K}\}$

end while

return T

Reduction w.r.t. a weak d-$\triangle$ set

Algorithm Rosenfeld-Gröbner(F_0)

Input: A finite set of differential polynomials F_0

Output: A finite set T of regular systems such that $\{F_0\} = \bigcap_{(A,H)\in T} [A] : H^\infty$

...

while $U \neq \varnothing$ **do**

Take and remove any $(F, C, H) \in U$

Let $m_i = \max\{\mathrm{ord}_{y_i} f \mid f \in F \cup C\}$, $i = 1, \ldots, n$

B :=Differentiate&Autoreduce$(C, \{m_i\}_{i=1}^n)$

if $B \neq \varnothing$ **then**

$R :=$ alg-rem$(F, B) \setminus \{0\}$

if $R = \varnothing$ **then** $T := T\cup$Autoreduce&Check$(C, H \cup H_C)$

else

...

end if

$U := U \cup \{(F \cup \{h\}, C, H) \mid h \in H_C,\ h \notin \mathbb{K}\}$

end if

end while

return T

Algorithm Differentiate&Autoreduce

Algorithm Differentiate&Autoreduce$(C, \{m_i\})$

Input: a weak d-triangular set $C = C_1, \ldots, C_k$ with $\operatorname{ld} C = y_1^{(d_1)}, \ldots, y_k^{(d_k)}$, and a set of non-negative integers $\{m_i\}_{i=1}^n$, $m_i \geq m_i(C)$

Output: set $B = \{B_i^j \mid 1 \leq i \leq k,\ 0 \leq j \leq m_i - d_i\}$ satisfying

$B \subset [C]$, $\operatorname{rk} B_i^0 = \operatorname{rk} C_i$, $\operatorname{rk} B_i^j = y_i^{(d_i+j)}$ $(j > 0)$

$\mathbf{i}_{B_i^j} \in H_C^\infty + [C]$ $(j \geq 0)$

B_i^j is partially reduced w.r.t. $C \setminus \{C_i\}$

$m_i(B) \leq m_i + \sum_{j=1}^k (m_j - d_j)$, $i = k+1, \ldots, n$

or $\varnothing$, if it is detected that $[C] : H_C^\infty = (1)$

for $i := 1$ **to** k **do**
  $B_i^0 :=$ alg-rem$(C_i, \{B_l^r \mid 1 \leq l < i,\ 0 < r \leq m_l - d_l\})$
  if $\operatorname{rk} B_i^0 \neq \operatorname{rk} C_i$ **then return** $\varnothing$
  for $j := 1$ **to** $m_i - d_i$ **do**
    $B_i^j :=$ alg-rem$(\delta B_i^{j-1}, \delta(C \setminus \{C_i\}))$
    if $\operatorname{ld} B_i^j \neq y_i^{(d_i+j)}$ **then return** $\varnothing$
  end for
end for
return B

Differentiate&Autoreduce is correct

Lemma. *Let C be a weak d-triangular set, and let f be a polynomial such that* $\operatorname{lv} f \notin \operatorname{lv} C$ *and* $\mathbf{i}_f \in H_C^\infty + [C]$*. Let* $f \rightarrow_C g$*. Then*

- $\operatorname{rk} g \neq \operatorname{rk} f \;\Rightarrow\; [C] : H_C^\infty = (1)$
- $\operatorname{rk} g = \operatorname{rk} f \;\Rightarrow\; \mathbf{i}_g \in H_C^\infty + [C]$

- $B_1^0 = C_1$ is partially reduced w.r.t. $C_2, \ldots, C_k$.
- δB_1^0 is reduced w.r.t. $\delta^l C_i$, $l > 1$, $i = 2, \ldots, k$
- $\operatorname{rk} \delta B_1^0 = y_1^{(d_1+1)}$
- $B_1^1 = \text{alg-rem}(\delta B_1^0, \delta(C \setminus \{C\})$
- Lemma $\Rightarrow [C] : H_C^\infty = (1)$ **or**

$$\operatorname{rk} B_1^1 = y_1^{(d_1+1)} \text{ and } \mathbf{i}_{B_1^1} \in H_C^\infty + [C].$$

Differentiate&Autoreduce is correct

Lemma. *Let C be a weak d-triangular set, and let f be a polynomial such that $\operatorname{lv} f \notin \operatorname{lv} C$ and $\mathbf{i}_f \in H_C^\infty + [C]$. Let $f \to_C g$. Then*

- $\operatorname{rk} g \neq \operatorname{rk} f \;\Rightarrow\; [C] : H_C^\infty = (1)$
- $\operatorname{rk} g = \operatorname{rk} f \;\Rightarrow\; \mathbf{i}_g \in H_C^\infty + [C]$

- B_1^1 is partially reduced w.r.t. $C_2, \ldots, C_k$.
- $\Rightarrow$ similarly for all B_1^r, $1 < r < m_1 - d_1$.
- For $B_1 = B_1^0, \ldots, B_1^{m_1-d_1}$, we have:

$$B_1 \subset [C], \quad H_{B_1} \subset H_C^\infty + [C]$$

Differentiate&Autoreduce is correct

Lemma. *Let C be a weak d-triangular set, and let f be a polynomial such that* $\operatorname{lv} f \notin \operatorname{lv} C$ *and* $\mathbf{i}_f \in H_C^\infty + [C]$*. Let* $f \to_C g$*. Then*

- $\operatorname{rk} g \neq \operatorname{rk} f \;\Rightarrow\; [C] : H_C^\infty = (1)$
- $\operatorname{rk} g = \operatorname{rk} f \;\Rightarrow\; \mathbf{i}_g \in H_C^\infty + [C]$

- $B_2^0 = \text{alg-rem}(C_2, \{B_1^0, \ldots, B_1^{(m_1-d_1)}\})$
- C_2 is partially reduced w.r.t. $C_3, \ldots, C_k$ and $y_1^{(d_1+l)}$, $l > m_1 - d_1$.
- By Lemma, two cases are possible:
  - $\operatorname{rk} B_2^0 = \operatorname{rk} C_2$, $\mathbf{i}_{B_2^0} \in H_{B_1}^\infty + [B] \subset H_C^\infty + [C]$
  - $[B] : H_B^\infty = (1) \;\Rightarrow\; [C] : H_C^\infty = (1)$
- Similarly for $B_2^1, \ldots, B_2^{m_2-d_2}$ and B_i^r, $i > 2$.

Differentiate&Autoreduce is correct

Inequality

$$m_i(B) \leq m_i + \sum_{j=1}^{k}(m_j - d_j), \quad i = k+1, \ldots, n$$

follows from the fact that the two nested loops

for $i := 1$ **to** k **do**

$\ldots$

for $j := 1$ **to** $m_i - d_i$ **do**

$\ldots$

end for

end for

have $\sum_{j=1}^{k}(m_j - d_j)$ iterations, and at each iteration each polynomial is differentiated at most once.

Final algorithm and bound

Algorithm Rosenfeld-Gröbner(F_0)

Output: $\{F_0\} = \bigcap_{(A,H)\in T} [A] : H^\infty$ satisfying $M(A) \le (n-1)!M(F_0), \ \ (A,H) \in T$

$T := \varnothing$, $U := \{(F_0 \setminus \{\min F_0\}, \{\min F_0\}, \varnothing)\}$

while $U \ne \varnothing$ **do**

Take and remove any $(F, C, H) \in U$

Let $m_i = \max\{\mathrm{ord}_{y_i} f \mid f \in F \cup C\}$, $i = 1, \ldots, n$

B :=Differentiate&Autoreduce($C, \{m_i\}_{i=1}^n$)

if $B \ne \varnothing$ **then**

$R :=$ alg-rem$(F, B) \setminus \{0\}$

if $R = \varnothing$ **then** $T := T \cup$ Autoreduce&Check(C,$H \cup H_C$)

else $C^> := \{p \in C \mid \ \mathrm{lv}\, p = \mathrm{lv}(\min R)\}$

$\bar{C} := C \setminus C^> \cup \{\min R\}$

$\bar{F} := C^> \cup R \setminus \{\min R\}$

$\bar{H} :=$ d-rem$(H \cup H_{\bar{C}}, \bar{C})$

if $0 \notin \bar{H}$ **then** $U := U \cup \{(\bar{F}, \bar{C}, \bar{H})\}$

end if

$U := U \cup \{(F \cup \{h\}, C, H) \mid h \in H_C, \ h \notin \mathbb{K}\}$

end if

end while

return T

Final proof of the bound

- $m_i(B) \le m_i + \sum_{j=1}^{k}(m_j - d_j),\ k < i \le n$
-
$$m_i(R) \le \begin{cases} d_i, & 1 \le i \le k \\ m_i(B), & k < i \le n \end{cases}$$
- $M_{\operatorname{lv} C}(R) \le (n-k)\sum_{i=1}^{k} d_i + \sum_{i=k+1}^{n} m_i + \sum_{i=k+1}^{n}(m_i - d_i) \le M_{\operatorname{lv} C}(F \cup C)$
- Two cases are possible:
  - $|C| < n$: Again two cases:
    - $|\bar{C}| < n$:
$$M_{\operatorname{lv} \bar{C}}(\bar{F} \cup \bar{C}) \le \begin{cases} M_{\operatorname{lv} C}(F \cup C), & \operatorname{lv} \bar{C} = \operatorname{lv} C \\ (n - |\operatorname{lv} C| - 1) M_{\operatorname{lv} C}(F \cup C), & \operatorname{lv} \bar{C} = \operatorname{lv} C \cup \{y\}, \\ & y \notin \operatorname{lv} C \end{cases}$$
    - $|\bar{C}| = n$: $M(\bar{F} \cup \bar{C}) = M_{\operatorname{lv} C}(\bar{F} \cup \bar{C}) \le M(F \cup C)$.
  - $|C| = n$. Then also $|\bar{C}| = n$ and $M(\bar{F} \cup \bar{C}) \le \sum_{i=1}^{n} d_i \le M(F \cup C)$.
- Therefore $M(A) \le (n-1)! M(F_0)$.

References

1. F. Boulier, D. Lazard, F. Ollivier, M. Petitot: Representation for the radical of a finitely generated differential ideal. *Proc. ISSAC 1995*, ACM Press, 158–166.

2. R.M. Cohn: The Greenspan bound for the order of differential systems. *Proc. AMS*, Vol. 79, No. 4, 1980, 523–526.

3. X. Dahan, M. Moreno Maza, É. Schost, W. Wu, Y. Xie: Lifting techniques for triangular decompositions. *Proc. ISSAC 2005*, 108–115.

4. O.G., M. Kondratieva, M. Moreno Maza, A. Ovchinnikov: Algebraic transformation of differential characteristic sets. *Preprint, 2006.*

5. E. Hubert: Notes on triangular sets and triangulation-decomposition algorithms II: differential systems. *In: Lecture Notes in Computer Science*, Vol. 2630, Springer Verlag, 2003, 40–87.

6. M. Moreno Maza: On triangular decompositions of algebraic varieties, Tech. Rep. 4/99, NAG Ltd, 1999.

7. J.F. Ritt, Jacobi's problem on the order of a system of differential equations. *The Annals of Mathematics*, 2nd Ser., Vol. 36, No. 2, 1935, 303–312.

8. J.F. Ritt: Systems of algebraic differential equations. *The Annals of Mathematics*, 2nd Ser., Vol. 36, No. 2, 1935, 293–302.