Enumeration of Rota-Baxter and differential Rota-Baxter Words

Li Guo, Rutgers University at Newark
William Sit, City College of New York

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Overview

- Let **k** be a (differential) field of characteristic zero.
- The set of **Rota-Baxter words** forms a vector space basis over **k** for a **free Rota-Baxter k-algebra** over a given set of symbols.
- Likewise, the set of **differential Rota-Baxter words** forms a vector space basis over **k** for a **free differential Rota-Baxter k-algebra** over a given set of symbols.
- These free **k**-algebras may be considered as a (differential) Rota-Baxter analog of a polynomial ring, suitable to express **integral and differential-integral equations**.

Overview (continued)

- Enumeration of a basis is often a first step to choosing a **data representation** in implementation of algorithms involving free algebras, and in particular, free (differential) Rota-Baxter algebras and several related algebraic structures.
- This talk describes a method to **enumerate the set of (differential) Rota-Baxter words** and **outlines an algorithm** for their generation according to a quad-graded structure.
- In addition, the generating functions obtained provide counting for combinatorial structures and give rise to new integer sequences.

Rota-Baxter algebra

- A **Rota-Baxter algebra** is an associative algebra R (not necessarily unitary or commutative) together with a $\mathbf{k}$-linear operator $P : R \to R$ (called a **Rota-Baxter operator**) and a fixed constant $\lambda \in \mathbf{k}$ (called the **weight** of P), satisfying the identity

$$P(r_1)P(r_2) = P(r_1 P(r_2) + P(r_1)r_2 + \lambda r_1 r_2) \tag{1}$$

for all elements r_1, r_2 in R.

Free Rota-Baxter algebras

- Let A be a non-unitary $\mathbf{k}$-algebra, and let B be a given $\mathbf{k}$-basis of A.
- **Rota-Baxter words** (RBWs) are **certain** (finite) strings formed by concatenating elements of $b \in B$ and their images under P, iteratively. For example: $b_1 P(b_2 P(b_1))$.
- We can explicitly construct a **free Rota-Baxter algebra** $Ш^{NC,0}(A)$ over A using Rota-Baxter words on B. (Ebrahimi-Fard and Guo, 2005, 2008)
- The set of RBWs is denoted by $\mathfrak{M}^0(B)$ and if the empty word is included, by $\mathfrak{M}^1(B)$.

Free Rota-Baxter algebras on X

- Let X be an arbitrary set.
- A **free Rota-Baxter algebra on X** is defined by the usual universal property.
- It can be shown to be isomorphic to $Ш^{NC,0}(\mathbf{k}\langle X\rangle)$, the free (non-commutative) Rota-Baxter algebra on the non-commutative polynomial algebra $\mathbf{k}\langle X\rangle$, using the canonical $\mathbf{k}$-basis $B = B(X)$ consisting of non-commuting monomials in X.
- A **Rota-Baxter word on X** is an element of $\mathfrak{M}^1(B(X))$.

Strings on a $\mathbf{k}$-basis with brackets

- We now review the construction of Rota-Baxter words from a $\mathbf{k}$-basis B (in particular, when $B = B(X)$).
- the product of $b_1, b_2 \in B$ in the algebra A is denoted by $b_1 b_2$ or by $b_1 \cdot b_2$.
- Let $\lfloor$ and $\rfloor$ be symbols, called brackets, and let $B' = B \cup \{\lfloor, \rfloor\}$.
- Let $S(B')$ be the free (non-commutative) semigroup generated by B', the multiplication of which is denoted by the concatenation operator $\sqcup$ (often omitted and not part of the string).

Rota-Baxter words ($\mathtt{RBWs}$)

- A **Rota-Baxter word (**$\mathtt{RBW}$**)** is an element w of $S(B')$ that satisfies the following conditions.
- The number of $\lfloor$ in w equals the number of $\rfloor$ in w;
- Counting from the left to the right, the cumulative number of $\lfloor$ at each location is always greater than or equal to that of $\rfloor$;
- There is no occurrence of $b_1 \sqcup b_2$ in w, for any $b_1, b_2 \in B$;
- There is no occurrence of $\rfloor\lfloor$ or $\lfloor\rfloor$ in w.

Intuitive view of RBWs

- Intuitively, $P(w) = \lfloor w \rfloor$ and since $P(w_1)P(w_2)$ can be reduced by the Rota-Baxter identity, $\rfloor\lfloor$ does not occur in a RBW.
- A Rota-Baxter word w can be represented uniquely by a finite string composed of one or more elements of B, separated (if more than one b) by a left brackets $\lfloor$ or by a right bracket $\rfloor$, where the set of brackets formed balanced pairs, but neither the string $\rfloor\lfloor$ nor the string $\lfloor\,\rfloor$ appears as a substring.
- For example, when $B = \{ b \}$, the word $w = \lfloor\lfloor b \rfloor b \lfloor b \rfloor\rfloor b \lfloor b \rfloor$ is an RBW, but $\lfloor b \sqcup b \rfloor$, $\lfloor b^2 \rfloor$, $\lfloor b \rfloor\lfloor b \rfloor$, $b \rfloor b \lfloor b$, and $\lfloor b \lfloor\,\rfloor b \rfloor$ are not.

The diamond product for Rota-Baxter words

- Let B be a **k**-basis of a k-algebra A.
- Let $Ш^{NC,0}(A)$ be the free **k**-module with basis $\mathfrak{M}^0(B)$ (set of `RBWs` on B without the empty word).
- Consider the following properties where $\diamond$ is the intended multiplication operation in $Ш^{NC,0}(A)$:

$$\begin{aligned} b \diamond b' &= b \cdot b' \\ b \diamond \lfloor w \rfloor &= b\lfloor w \rfloor \\ \lfloor w \rfloor \diamond b &= \lfloor w \rfloor b \\ \lfloor w \rfloor \diamond \lfloor w' \rfloor &= \lfloor \lfloor w \rfloor \diamond w' \rfloor + \lfloor w \diamond \lfloor w' \rfloor \rfloor + \lambda \lfloor w \diamond w' \rfloor \end{aligned}$$

for all $b, b' \in B$ and all $w, w' \in \mathfrak{M}^0(B)$.

Construction of free Rota-Baxter algebra

- **Theorem.** (Guo). These properties uniquely define an associative bilinear product $\diamond$ on $Ш^{NC,0}(A)$. This product, together with the linear operator

$$P_A : Ш^{NC,0}(A) \to Ш^{NC,0}(A), \quad P_A(w) = \lfloor w \rfloor \text{ if } w \in \mathfrak{M}^0(B),$$

and the natural embedding

$$j_A : A \to Ш^{NC,0}(A), \qquad j_A(b) = b \text{ if } b \in B,$$

makes $Ш^{NC,0}(A)$ the free (non-unitary, non-commutative) Rota-Baxter algebra over A.

Leaf-decorated rooted trees and forests

- Let $\mathfrak{T}(X)$ be the set of (planar) **rooted trees with leaves decorated by X**.
- Let $\mathfrak{F}(X)$ be the set of **forests of (planar) rooted trees with leaves decorated by X** (that is, $\mathfrak{F}(X)$ is the set of finite tuples with entries in $\mathfrak{T}(X)$). Forests can be concatenated to form larger forests.
- If $F = (T_1, \ldots, T_b)$ is a forest, we can define its **grafting** $\lfloor F \rfloor$ to be the $T \in \mathfrak{T}(X)$ formed by adding a root and connecting this root to the roots of $T_1, \ldots, T_b$.
- If $T \in \mathfrak{T}(X)$ is a tree, we can define a forest F by removing its root. We denote F by $\overline{T}$.

Leaf-spaced forests

- Let $\mathfrak{F}_\ell(X)$ be the subset of $\mathfrak{F}(X)$ consisting of forests that do not have a vertex with adjacent non-leaf branches. These are called **leaf-spaced forests**.
- A product $\diamond$ is defined on the $\mathbf{k}$-vector space with basis $\mathfrak{F}_\ell(X)$.
- **Theorem.**(Guo) $(\mathbf{k}\mathfrak{F}_\ell(X), \Diamond, \lfloor\ \rfloor)$ is a free (non-unitary, non-commutative) Rota-Baxter algebra.
- $w = \lfloor a\lfloor bc\rfloor d\rfloor e\lfloor\lfloor f\rfloor g\lfloor h\rfloor\rfloor$ is a RBW over $B(X)$ if $a, b, c, d, e, f, g, h \in X$. It corresponds to a forest in $\mathfrak{F}_\ell(X)$ and $\lfloor w\rfloor$ is a leaf-spaced tree (with e and g as separators).
- The **free non-unitary, non-commutative Rota-Baxter algebra on X is denoted by $Ш^{NC,0}(X)$**. We see that $Ш^{NC,0}(\mathbf{k}\langle X\rangle)$ is isomorphic to $\mathbf{k}\mathfrak{F}_\ell(X)$ as $\mathbf{k}$-vector spaces.

P-degree, P-run

- **Example.** Let $b_1, b_2 \in B$.

$$w = \lfloor \lfloor \lfloor \mathbf{b_1} \rfloor b_2 \lfloor \mathbf{b_1} \rfloor \rfloor \rfloor b_1 \lfloor \mathbf{b_2} \rfloor = \mathbf{P^{(2)}}\big(\mathbf{P}(\mathbf{b_1}) b_2 \mathbf{P}(\mathbf{b_1})\big) b_1 \mathbf{P}(\mathbf{b_2})$$

  is an RBW in B.

- The number of balanced pairs of brackets in an RBW is called its ***P*-degree**. The P-degree of w in the above example is 5.
- For any RBW w, a ***P*-run** is any occurrence in w of consecutive compositions of $\lfloor\ \rfloor$ of maximal length (that is, of immediately nested $\lfloor\ \rfloor$, where the **length** is the number of consecutive applications of P).
- We denote a P-run by $P^{(\mu)}$ or $\lfloor\ \rfloor^{(\mu)}$ if its run length μ is > 1.
- The RBW w has one P-run of length 2 and three P-runs of length 1.

X-arity, x-arity, X-run and x-run

- **Example.** Let $\mathbf{x_1}, \mathbf{x_2} \in X$ and $B = B(X)$. Then

$$w = \lfloor\lfloor\lfloor \mathbf{x_1} \rfloor \mathbf{x_2}^2 \lfloor \mathbf{x_1} \rfloor\rfloor\rfloor \mathbf{x_1} \mathbf{x_2} \lfloor \mathbf{x_2} \rfloor$$

  is an RBW in X.

- When $B = B(X)$, the **X-arity** of a RBW w is the number of $x \in X$ appearing in w, counted with multiplicities. If $X = \{ \mathbf{x_1}, \mathbf{x_2} \}$ and $B = B(X)$, the X-arity of above w is 7.

- If we only count appearances of x for a particular $x \in X$, we will call this the **x-arity**. The $\mathbf{x_1}$-arity of above w is 3 and its $\mathbf{x_2}$-arity is 4.

- For any fixed generator $x \in X$, an **x-run** is any occurrence in w of consecutive products (in $B(X)$) of x of maximal length.

X-arity, x-arity, X-run and x-run

- We denote an x-run by x^ν if its run length ν is > 1.
- We define an **X-run** to be any occurrence in w of consecutive products (in $B(X)$) of the x's (with whatever subscripts) of maximal length.
- As an example, let $x_1, x_2 \in X$ and let

$$w = \mathbf{x_1^2 x_2} P^{(2)}(P(\mathbf{x_1 x_2} P(\mathbf{x_1}))\mathbf{x_1}) = \mathbf{x_1^2 x_2} \lfloor\lfloor\lfloor \mathbf{x_1 x_2} \lfloor \mathbf{x_1} \rfloor\rfloor \mathbf{x_1} \rfloor\rfloor. \quad (2)$$

Then w has P-degree 4, with three P-runs of lengths 2, 1, and 1, and X-arity 7, with four X-runs of lengths 3, 2, 1, and 1.

Quadgrading of general RBWs

- We denote the set of RBWs by R.
- If there is only one P and one x, let $R_{u,v}$ be the set of Rota-Baxter words such that the maximum length of any P-runs is $\leqslant u$ and the maximum length of any x-runs is $\leqslant v$.
- $R_{u,v}(n, m; k, \ell)$ is the subset of $R_{u,v}$ consisting of RBWs with P-degree n, X-arity m, and having k P-runs and ℓ X-runs.
- If there are p Rota-Baxter operators and q generators, let $R_{\vec{u},\vec{v}}$ be the set of Rota-Baxter words such that the maximum length of any P_i-runs is $\leqslant u_i$ and the maximum length of any x_j-runs is $\leqslant v_j$.
- $R_{\vec{u},\vec{v}}(n, m; k, \ell)$ is defined similarly.

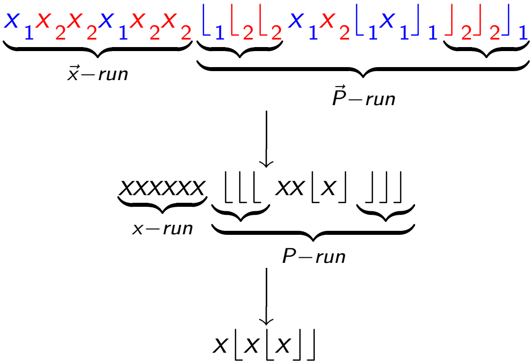
Bigrading of Idempotent RBWs

- Let $R_{1,1}$ be the set of Rota-Baxter words in one operator P and on one generator x, such that the lengths of all P-runs and x-runs are 1. These RBWs are called **idempotent**.
- For an idempotent Rota-Baxter word w, the **P-degree** is the number of times P is applied in forming w and the **x-arity** is the number of x used in forming w.
- Four subclasses of idempotent RBWs:

$\langle \mathbf{a} \rangle$	(associate):	$x\lfloor x \rfloor$, $x\lfloor x \rfloor x$, $\lfloor x \rfloor x$
$\langle \mathbf{b} \rangle$	(bracketed):	$\lfloor \lfloor x \rfloor x \rfloor$, $\lfloor x \rfloor x \lfloor x \rfloor$
$\langle \mathbf{i} \rangle$	(indecomposable):	$\lfloor \lfloor x \rfloor x \rfloor$
$\langle \mathbf{d} \rangle$	(decomposable):	$\lfloor x \rfloor x \lfloor x \rfloor$

Reduction to Idempotent RBWs

- Our enumeration approach simplifies the structure of RBWs by means of a forgetful map and a collapsing map.



Generating Function for Idempotent RBWs

- **Generating functions for the general RBWs** are obtained by understanding the structure of RBWs via compositions of integers and coloring. They can be expressed in terms of generating functions of the idempotent case.
- Let $R_{1,1}(n, m)$ be the set of RBWs in this case with P-degree n and x-arity m. Let $r_{1,1}(n, m)$ be its cardinality.
- **Theorem.** (Guo-Sit) The generating function for $r_{1,1}(n, m)$ is given by:

$$\mathbf{R}_{1,1}(z, t) := \sum_{n,m \geqslant 0} r_{1,1}(n, m) z^n t^m = \frac{1 - \sqrt{1 - 4zt - 4zt^2}}{2tz}. \tag{3}$$

Grading and counting Idempotent RBWs

- A **formal grammar** to describe idempotent RBWs.
- The grammar provides a **system of recurrence equations** for counts.
- Solving the recurrence system gives **generation functions**.
- **Bivariate generating function suggests an algorithm to generate RBWs recursively and irredundantly.**

Formal Grammar for $R_{1,1}$

- To define a **formal language**, we start with an alphabet Σ of **terminals** consisting of a special symbol $\emptyset$ and the three symbols $\lfloor$, x, and $\rfloor$, a set of **non-terminals** consisting of $\langle \mathbf{b} \rangle$, $\langle \mathbf{i} \rangle$, $\langle \mathbf{d} \rangle$, $\langle \mathbf{a} \rangle$ and the sentence symbol $\langle \mathtt{RBW} \rangle$.
- Let the **production rules** be:

$$
\begin{aligned}
\langle \mathtt{RBW} \rangle &\longrightarrow \emptyset \mid \langle \mathbf{b} \rangle \mid \langle \mathbf{a} \rangle \\
\langle \mathbf{a} \rangle &\longrightarrow x \mid x\langle \mathbf{b} \rangle \mid \langle \mathbf{b} \rangle x \mid x\langle \mathbf{b} \rangle x \\
\langle \mathbf{b} \rangle &\longrightarrow \langle \mathbf{i} \rangle \mid \langle \mathbf{d} \rangle \\
\langle \mathbf{i} \rangle &\longrightarrow \lfloor \langle \mathbf{d} \rangle \rfloor \mid \lfloor \langle \mathbf{a} \rangle \rfloor \\
\langle \mathbf{d} \rangle &\longrightarrow \langle \mathbf{b} \rangle x \langle \mathbf{b} \rangle
\end{aligned}
$$

Enumeration Experiments and Observations

- Brute-force method using the production rules inductively
- **A decomposable word like $\lfloor x \rfloor x \lfloor x \rfloor x \lfloor x \rfloor$ may be derived in more than one way.**
- Many duplicates need to be removed.
- The number r_n of RBWs with n balanced pairs of brackets $n = 0, 1, 2, \ldots$ are:

$$2, 4, 16, 80, 448, 2688, \ldots$$

- These matched the sequence to **A025225 in the Sloane database**: $2^{n+1}C_n$ where (C_n is n-th **Catalan number**).
- **Proof?**

Relationship with Catalan Numbers

- We strip the RBWs of all the x's and obtain a **skeleton** of brackets alone.
- $\lfloor x \lfloor x \rfloor x \lfloor x \lfloor x \rfloor x \rfloor x \rfloor$ has skeleton $\lfloor \lfloor \rfloor \lfloor \lfloor \rfloor \rfloor \rfloor$.
- These correspond bijectively with **planar rooted trees on n vertices**.
- Skeleton $\lfloor \lfloor \rfloor \lfloor \lfloor \rfloor \rfloor \rfloor$, corresponds to .
- **Their counts are the Catalan numbers!**
- But how many ways are there to form RBWs using this skeleton? The total seems to be 2^{n+1}. How many are there with a fixed number m of x's?

Recurrence System

- Let r_n, a_n, b_n, i_n, d_n respectively be the number of RBWs in the classes $\langle \mathtt{RBW} \rangle$, $\langle \mathbf{a} \rangle$, $\langle \mathbf{b} \rangle$, $\langle \mathbf{i} \rangle$, $\langle \mathbf{d} \rangle$ using exactly n times the operator P.
- From the production rules, they satisfy, for $n > 0$:

$$
\begin{array}{llll}
\langle \mathtt{RBW} \rangle & \to \emptyset \mid \langle \mathbf{b} \rangle \mid \langle \mathbf{a} \rangle & \Longrightarrow & r_n = b_n + a_n \\
\langle \mathbf{a} \rangle & \to x \mid x\langle \mathbf{b} \rangle \mid \langle \mathbf{b} \rangle x \mid x\langle \mathbf{b} \rangle x & \Longrightarrow & a_n = 3b_n \\
\langle \mathbf{b} \rangle & \to \langle \mathbf{i} \rangle \mid \langle \mathbf{d} \rangle & \Longrightarrow & b_n = i_n + d_n \\
\langle \mathbf{i} \rangle & \to \lfloor \langle \mathbf{d} \rangle \rfloor \mid \lfloor \langle \mathbf{a} \rangle \rfloor & \Longrightarrow & i_n = a_{n-1} + d_{n-1} \\
\langle \mathbf{d} \rangle & \to \langle \mathbf{b} \rangle x \langle \mathbf{b} \rangle & \Longrightarrow & d_n = \sum_{\substack{(n_1,\dots,n_p;n) \\ p>1}} i_{n_1} \cdots i_{n_p}
\end{array}
$$

Compositions and Decomposable RBWs

- A **Composition of an integer** n into p positive parts is an **ordered partition** of n into p parts: $n = n_1 + n_2 + \cdots + n_p$.
- A decomposable RBW eventually becomes (or comes from) a product of indecomposables (with x separating two adjacent ones).
- If there are p indecomposables each using n_i P's, then we have a composition of n into p parts.
- Let $(n_1, \ldots, n_p; n)$ denotes the set of all compositions of n into p positive integers.
- $d_n = \sum_{\substack{(n_1,\ldots,n_p;n) \\ p>1}} i_{n_1} \cdots i_{n_p}$

Solving for the Generating Functions

$$\mathbf{R}(z) = \sum_{n=0}^{\infty} r_n z^n = \frac{1-\sqrt{1-8z}}{2z},$$

$$\mathbf{B}(z) = \sum_{n=0}^{\infty} b_n z^n = \frac{1-4z-\sqrt{1-8z}}{8z},$$

$$\mathbf{I}(z) = \sum_{n=0}^{\infty} i_n z^n = \frac{1-2z-\sqrt{1-8z}}{2(z+1)},$$

$$\mathbf{D}(z) = \sum_{n=0}^{\infty} d_n z^n = \frac{1-7z+4z^2+(3z-1)\sqrt{1-8z}}{8z(z+1)},$$

$$\mathbf{A}(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{3-4z-3\sqrt{1-8z}}{8z}.$$

More Experiments

- We have solve for the generating functions using the difference equations with suitable initial conditions.
- **In particular,**

$$b_n = 2^{n-1} C_n, \qquad n = 0, 1, 2, \ldots \qquad \textbf{A003645}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ **is the n-th Catalan number.**

- $r_n = 2^{n+1} C_n, \qquad n = 0, 1, 2, \ldots$ **A025225**.
- Knowing the sequence does not enable us to enumerate RBWs of degree n, but the counts enable us to **verify the program**, which is inefficient.

Recurrence System, Again

- The key is to do a finer analysis of the recurrence equations.
- Let $r_{n,m}$, $a_{n,m}$, $b_{n,m}$, $i_{n,m}$, $d_{n,m}$ be the number of RBWs in the classes $\langle \mathtt{RBW} \rangle$, $\langle \mathbf{a} \rangle$, $\langle \mathbf{b} \rangle$, $\langle \mathbf{i} \rangle$, $\langle \mathbf{d} \rangle$ using exactly n applications of P and exactly m x's.
- Then for $n \geqslant 2, m \geqslant 2$,

$$
\begin{aligned}
r_{n,m} &= b_{n,m} + a_{n,m} \\
a_{n,m} &= 2b_{n,m-1} + b_{n,m-2} \\
b_{n,m} &= i_{n,m} + d_{n,m} \\
i_{n,m} &= d_{n-1,m} + a_{n-1,m} \\
d_{n,m} &= \sum_{p=2}^{\min(n,m)} \sum_{(m_1,\ldots,m_p;\, m-p+1)} \sum_{(n_1,\ldots,n_p;\, n)} (i_{n_1,m_1}) \cdots (i_{n_p,m_p})
\end{aligned}
$$

Algebraic Relations among Generating Functions

- These recurrence equations with suitable initial conditions translates easily to algebraic relations of the generating functions in two variables.
- Let $R(z,t)$, $A(z,t)$, $B(z,t)$, $I(z,t)$, $D(z,t)$ be respectively the generating functions for $r_{n,m}$, $a_{n,m}$, $b_{n,m}$, $i_{n,m}$, $d_{n,m}$.
- We get "immediately" (indeed from the **grammar**)

$$\begin{aligned} R &= 1 + B + A \\ A &= t + 2tB + t^2 B \\ B &= I + D \\ I &= zD + zA \end{aligned}$$

- Eliminate A and D: $(1+z)I(z,t) - zt = z(1+t)^2 B(z,t)$

Elimination

- We also get $D = \sum_{p \geqslant 2} I^p t^{p-1}$.
- Since $B = I + D$, this yields

$$B(z,t) = \frac{I(z,t)}{1 - tI(z,t)}$$

- From these two identities involving B and I, we can solve all the generating functions.
- In particular, for $n \leqslant m \leqslant 2n-1, n \geqslant 0$,

$$r_{n,m} = \binom{n+1}{m-n} C_n, \qquad b_{n,m} = \binom{n-1}{m-n} C_n$$

$$r_{n,m} = b_{n,m} = 0 \textbf{ otherwise.}$$

- Now we know why summing over m gives a power of 2.

New Algorithm Suggested by Generating Functions

- B satisfies a quadratic equation after eliminating I from

$$(1+z)I - zt = z(1+t)^2 B, \qquad B(1-tI) = I$$

- $B - zt = 2zt(1+t)\,B + zt(1+t)^2\,B^2.$
- In explicit form, for $(n,m) \neq (1,1)$:

$$b_{n,m} = 2\,b_{n-1,m-1} + 2\,b_{n-1,m-2} + \sum_{k=1}^{n-2}\sum_{\ell=1}^{m-2} b_{k,\ell} b_{n-1-k,m-1-\ell} + 2\sum_{k=1}^{n-2}\sum_{\ell=1}^{m-3} b_{k,\ell} b_{n-1-k,m-2-\ell} + \sum_{k=1}^{n-2}\sum_{\ell=1}^{m-4} b_{k,\ell} b_{n-1-k,m-3-\ell}$$

- This last equation provides a very efficient and non-redundant algorithm to generate all bracketed RBWs.

Algorithm for RBWs in $R_{1,1}(n, m)$

- $2\,b_{n-1,m-1}$: For each RBW $w \in B(n-1, m-1)$, form two RBWs $f_{1,1}(w) = \lfloor x\,w \rfloor$ and $f_{1,2}(w) = \lfloor w\,x \rfloor$.
- $2\,b_{n-1,m-2}$: For each RBW $u \in B(n-1, m-2)$, form the RBWs $f_2(u) = \lfloor x\,u\,x \rfloor$
- $b_{k,\ell} b_{n-1-k,m-1-\ell}$: For each pair of RBWs $(v, y) \in I(k, \ell) \times B(n-1-k, m-1-\ell)$, form the RBW $f_3(v, y) = \lfloor v\,x\,y \rfloor$.
- $b_{k,\ell} b_{n-1-k,m-2-\ell}$: For each pair of RBWs $(v, y) \in D(k, \ell) \times B(n-1-k, m-1-\ell)$ form the RBW $f_4(v, y) = \lfloor v \rfloor\,x\,y$
- $\vdots$

Number Sequences and Combinatorial Objects

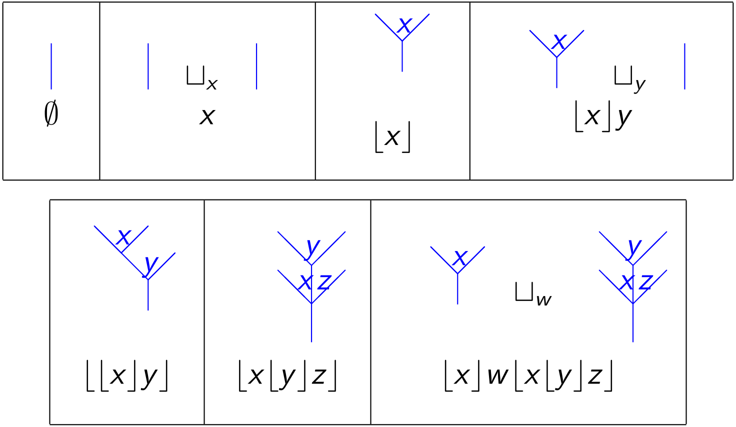
- We obtained parameterized generating functions.
- Same generating functions beg for natural bijections.
- Some new sequences for various z values from one generating function:

$$\mathbf{R}_{1,1}(z,t) = \frac{1-\sqrt{1-4zt-4zt^2}}{2tz}.$$

$$\begin{array}{ll} z=2: & 1,3,12,66,408,2712,18912,136488,\dots, \\ z=3: & 1,4,24,192,1728,16704,169344,\dots, \\ z=4: & 1,5,40,420,4960,62880,835840,\dots,, \\ z=5: & 1,6,60,780,1140,178800,2940000,\dots,. \end{array}$$

- An example is between lattice paths and RBWs. Another example we is the correspondence between angularly decorated forests and bracketed RBWs.

Bracketed RBWs and Angularly Decorated Forests



Differential Rota-Baxter algebra

- A **differential operator** δ (of weight λ) on a $\mathbf{k}$-algebra $\mathcal{R}$ is a $\mathbf{k}$-linear operator $\delta : \mathcal{R} \to \mathcal{R}$ satisfying the identities

$$\delta(r_1 r_2) = \delta(r_1) r_2 + r_1 \delta(r_2) + \lambda \delta(r_1) \delta(r_2), \tag{4}$$

for r_1, r_2 in $\mathcal{R}$,

$$\delta(1) = 0, \tag{5}$$

- A **differential Rota-Baxter algebra** is an associative algebra $\mathcal{R}$ together with a Rota-Baxter operator P and a differential operator δ, each of weight λ such that

$$\delta \circ P = \mathrm{id}_{\mathcal{R}}. \tag{6}$$

Free differential Rota-Baxter algebras

- Let Y be a set.
- (Guo-Keigher) A non-unitary **free differential Rota-Baxter algebra over Y** can be constructed using the free Rota-Baxter algebra $Ш^{NC,0}(\mathbf{k}\langle X\rangle)$ over X, where $X = \{x_i \mid i \geqslant 1\}$ and $B(X)$ is the set of non-commutative differential monomials in y after identifying x_i with $\delta^{(i-1)}y$ for $i \geqslant 1$.
- A **differential Rota-Baxter word** $(\mathtt{DRBW})$ is any element from the set $\mathfrak{M}^1(B(X))$ of Rota-Baxter words on the symbol set X. The set $\mathfrak{M}^1(B(X))$ is denoted by E.

Runs in DRBWs

- For any (non-commutative) differential monomial in y whose corresponding monomial is $x_{i_1}\cdots x_{i_m}$ in $B(X)$, where $i_1,\ldots,i_m$ are not necessarily distinct integers $\geqslant 1$, we define its **δ-arity** to be the sum $i_1+\cdots+i_m$.
- A **P-run** is a run of consecutive P applications and an **X-run** is a run of consecutive x_i (where i may vary and repeat).
- An example of a `DRBW` and its corresponding `RBW` is

$$w = \lfloor (y^{(2)})^3 \lfloor (y^{(1)})^4 (y^{(2)})^2 \rfloor \rfloor = \lfloor x_3^3 \lfloor x_2^4 \cdot x_3^2 \rfloor \rfloor,$$

where $P(w)$ is denoted by $\lfloor w \rfloor$. Here w has P-degree 2, with 2 P-runs of run length 1 each, and δ-arity 23, with 2 X-runs of run length 3 and 6.

Finer grading using runs

- Let n, d, k, ℓ be natural numbers.
- Let $E(n, d) \subset E$ denote the set of all RBWs with P-degree n and δ-arity d.
- Let $e(n, d)$ denote the cardinality of $E(n, d)$.
- Let $E(n, d; k, \ell)$ denote the set of all RBWs of E with P-degree n distributed into exactly k P-runs, and δ-arity d distributed into exactly ℓ X-runs.
- Let $e(n, d; k, \ell)$ denote the cardinality of $E(n, d; k, \ell)$.

Partition of integer

- Let b and m be natural numbers.
- Let $G(b, m)$ denotes the set of compositions of b into m parts.
- Let $G(b)$ the set of compositions of b.
- **Theorem.** For any natural numbers n, k, d, ℓ, we have a bijection between $E(n, d; k, \ell)$ and the set

$$R_{1,1}(k, \ell) \times G(n, k) \times \coprod_{\vec{d} \in G(d, \ell)} G(d_1) \times \cdots \times G(d_\ell) \qquad (7)$$

Generating functions

- We have the disjoint union:

$$E(n,d) = \coprod_{k=0}^{n} \coprod_{\ell=0}^{d} E(n,d;k,\ell). \tag{8}$$

- **Theorem.** The generating function $\mathbf{E}(z,s)$ is given by

$$\mathbf{R}_{1,1}\left(\frac{z}{1-z}, \frac{s}{1-2s}\right) + \left(\frac{1}{1-z}\right)\left(\frac{1-s-s^2}{(1-s)(1-2s)}\right) - \left(\frac{1-s}{1-2s}\right)$$

where $\mathbf{R}_{1,1}(z,t)$ is given by Eq. (3).

Enumeration algorithm for compositions

- The set of compositions of any positive integer b without restriction on the number of parts can be enumerated by readily available, efficient, and well-known algorithms (see COMP_NEXT of SUBSET library in Nijenhuis and Wilf for example).
- The set of compositions of b into exactly m parts can also be enumerated by the same algorithm using a slight modification.
- We generate all compositions $\vec{n}$ of n into exactly k parts, and all compositions $\vec{d} = (d_1, \ldots, d_\ell)$ of d in $G(d, \ell)$.
- Then for each $\vec{d}$, we generate the compositions of $d_1, \ldots, d_\ell$ in $G(d_1), \ldots, G(d_\ell)$ respectively.

Enumeration of DRBWs

- We enumerate the entire set E of `DRBWs` by enumerating $E(n, d)$ for each n and d.
- Using the disjoint union in Eq. (8), we can enumerate $E(n, d)$ by enumerating $E(n, d; k, \ell)$ for given k and ℓ.
- We base our algorithm to enumerate the sets $E(n, d; k, \ell)$ of `DRBWs` using our Theorem.
- (Guo-Sit) We already have an efficient algorithm for the enumeration of $R_{1,1}(k, \ell)$ for any positive k, ℓ.

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