November 4, 18, and, December 2, 2005 (tentative dates)
Phyllis J. Cassidy, Smith College, and, CCNY

Three talks on affine differential algebraic groups.

We have heard talks on Picard-Vessiot theory, and, the Ritt-Kolchin elimination theory of the differential polynomial ring. We now move to differential algebraic geometry, which Alexandru Buium, in his book, *Differential algebra and Diophantine Geometry*, calls a "new geometry,"and, to differential algebraic groups, the group objects of the new geometry. In the affine case, the objects of differential algebraic geometry are the sets of zeros of differential polynomial ideals. The symmetries present in the defining differential polynomial ideals of groups and their homogeneous spaces, makes these ideals much more tractable. In particular, characteristic sets are much easier to compute.Let ∂ be a set of m commuting derivation operators on a differentially closed differential field of characteristic 0. My talks will be a narrative of the present theory of affine differential algebraic groups, with emphasis on linear groups, the Galois groups of parametrized Picard-Vessiot theory. Our point of view will be classical. In particular, linear differential algebraic groups will be subgroups of $GL\left(n\right)$.

November 4, 2005 Introduction to affine differential algebraic groups.

We introduce the Kolchin topology on affine n-space $\mathbb{A}^n\left(\mathcal{U}\right)$, $\mathcal{U}$ a differentially closed differential field of characteristic 0, and, discuss the basic notions of affine differential algebraic group theory. Since affine algebraic groups are defined by order 0 differential polynomial equations, every affine algebraic group is a differential algebraic group. In contrast to classical algebraic group theory, however, not all affine differential algebraic groups are linear. We hope to close the first talk with the classification of the differential algebraic subgroups of $\mathbb{G}_m\left(\mathcal{U}\right) = GL_1\left(\mathcal{U}\right)$, the multiplicative group of the field $\mathcal{U}$.

November 18, 2005 (tentative date). The classification of the infinite simple differential algebraic groups.

Anand Pillay proved that all simple differential algebraic groups are linear. This means that simple differential algebraic groups are easily described.

Every simple differential algebraic group is isomorphic to a Zariski dense subgroup of a simple algebraic group H defined over $\mathbb{Q}$ (a Chevalley group).

From this point on, for ease of exposition, we assume that $\mathcal{U}$ is an ordinary differential field, with distinguished derivation operator ∂ and, field $\mathcal{C}$ of constants. We define the Lie algebra of matrices of a linear differential algebraic group, and, the logarithmic derivative map $\ell\partial$, so important in Picard-Vessiot theory, from an *algebraic group* H, defined over $\mathcal{C}$, onto its Lie algebra $\ell(H)$of matrices. $\ell\partial(h) = \partial h h^{-1}$, where ∂h is the matrix whose entries are the derivatives of the entries of h. Associated with the logarithmic derivative map is the *gauge action* of H on $\ell(H)$, defined by the formula $a \longmapsto \ell\partial(h) + hah^{-1}$. Note that the gauge action maps H into the group of affine transformations of the finite dimensional vector space $\ell(H)$ over $\mathcal{U}$.

December 2, 2005 The action of the proper Zariski dense differential algebraic subgroups of $SL_2(\mathcal{U})$ on Riccati varieties.

Let G be a Zariski dense differential algebraic subgroup of a simple Chevalley group H. Then, one can show that either $G = H$ (the generic case), or, G is the isotropy group, under the gauge action, of an $n \times n$ matrix a in the Lie algebra $\ell(H)$. $G = \{g \in H : \partial g = [a, g]\}$, where $[a, g] = ag - ga$. Thus, if $G \neq H$, it is conjugate (by a matrix $h \in H$ such that $\ell\partial(h) = a$) to the group $H(\mathcal{C})$, of matrices in H with entries in $\mathcal{C}$. So, every simple differential algebraic group can be realized as a simple *algebraic* group, either over $\mathcal{U}$, or, over the field $\mathcal{C}$ of constants of $\mathcal{U}$.

The lectures this semester on Picard-Vessiot theory introduced us to Riccati varieties. A Riccati equation $\partial y = y^2 + b_1 y + b_0$ defines a *Riccati variety* V in $\mathbb{A}^1(\mathcal{U})$. V is an interesting object of differential algebraic geometry, since it is an affine variety isomorphic under a linear fractional transformation to the projective line $\mathbb{P}^1(\mathcal{C})$ We now set $H = SL_2(\mathcal{U})$. Let G be a proper Zariski dense differential algebraic subgroup of H. Let a be the trace zero matrix defining G as its isotropy group under the gauge action. Associated with a is a Riccati equation. The Riccati variety V defined by this equation is a homogeneous space for G, acting on V by linear fractional transformations. Let $\partial = \frac{d}{dt}$.Since the Riccati variety $\partial y = y^2 + \frac{1}{2}t$ is a subvariety of the second *Painlevé variety* $\partial^2 y = 2y^3 + ty + \frac{1}{2}$, this gives us an interesting action of a differential algebraic group on a Painlevé variety, in the context of differential algebraic geometry, opening up an intriguing topic to explore.