Free Commutative Integro-differential Algebras

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(Joint work with G. Regensburger and M. Rosenkranz, and X. Gao and S. Zheng)

Free differential algebras

- Differential algebra

$$d(xy) = d(x)y + xd(y) + \lambda d(x)d(y).$$

$$d(uv) \mapsto_\phi d(u)v + ud(v) + \lambda d(u)d(v), \forall u, v \in R.$$

This leads to normal forms w with no products in d.

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- Thus free commutative differential algebra (of weight λ) on a set X is of the form

$$\mathbf{k}\{X\} := \mathbf{k}[\Delta X], \quad \Delta X := \{x^{(n)} \mid x \in X, n \geq 0\}$$

with concatenation product. Define $d_X : \mathbf{k}\{X\} \to \mathbf{k}\{X\}$ as follows. Let $w = u_1 \cdots u_k, u_i \in \Delta(X)$, $1 \leq i \leq k$, be a commutative word from the alphabet set $\Delta(X)$. If $k = 1$, so that $w = x^{(n)} \in \Delta(X)$, define $d_X(w) = x^{(n+1)}$. If $k > 1$, recursively define

$$d_X(w) = d_X(u_1)u_2 \cdots u_k + u_1 d_X(u_2 \cdots u_k) + \lambda d_X(u_1)d_X(u_2 \cdots u_k).$$

Further define $d_X(1) = 0$. Then $(\mathbf{k}\{X\}, d_X)$ is the free commutative differential algebra of weight λ on the set X.

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- Let $\mathbf{k}$ be a commutative ring. Let $\lambda \in \mathbf{k}$ be fixed. A Rota-Baxter operator of weight λ on a $\mathbf{k}$-algebra R is a linear map $P : R \to R$ such that

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- References:

1. L. Guo, WHAT IS a Rota-Baxter Algebra, *Notice of Amer. Math. Soc.* **56** (2009), 1436-1437.
2. L. Guo, An Introduction to Rota-Baxter Algebra, International Press, 2012.

The integration operator I

- For continuous functions $f(x)$ and $g(x)$, define

$$F(x) := I[f](x) := \int_0^x f(s)ds, G(x) := I[g](x) := \int_0^x g(s)ds. \quad (1)$$

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- Using Eq. (1), get $I[f\,I[g]](x) = I[f]I[g](x) - I[I[f]\,g](x)$.

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$$I[f\,I[g]] = I[f]I[g] - I[I[f]\,g], \quad I[f]I[g] = I[f\,I[g]] + I[I[f]\,g].$$

- An integral algebra is an algebra R together with a linear operator $I : R \to R$ that satisfies

$$I[f]I[g] = I[f\,I[g]] + I[g\,I[f]], \quad \forall f, g \in R.$$

Free commutative Rota-Baxter algebras

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- In a commutative Rota-Baxter algebra, this means

$$\mathfrak{a} = a_0 P(a_1 P(a_2 P(\cdots P(a_n)\cdots))), a_i \in A.$$

$$\mathfrak{a} = a_0 \otimes a_1 \otimes \cdots \otimes a_n \in A^{\otimes(n+1)}.$$

The product is given by

$$\mathfrak{a}\mathfrak{b} = (a_0 b_0) \otimes \big((a_1 \otimes \cdots a_n) \text{ш}_\lambda (b_1 \otimes \cdots \otimes b_m)\big).$$

ш_λ is a shuffle like product, called mixable shuffle product.

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- A mixable shuffle is a shuffle in which some pairs $a_i \otimes b_j$ are merged into $a_i b_j$.
  Define $(a_1 \otimes \ldots \otimes a_m) \text{Ш}_\lambda (b_1 \otimes \ldots \otimes b_n)$ to be the sum of mixable shuffles of $a_1 \otimes \ldots \otimes a_m$ and $b_1 \otimes \ldots \otimes b_n$.

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- **Example:**

$$
\begin{aligned}
& a_1 \text{Ш}_\lambda (b_1 \otimes b_2) \\
& = a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 \quad (\text{shuffles}) \\
& + a_1 b_1 \otimes b_2 + b_1 \otimes a_1 b_2 \quad (\text{merged shuffles}).
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Quasi-shuffle product

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- Let $\text{Ш}^+(A) = \bigoplus_{n\geq 0} A^{\otimes n} (= T(A))$.
- Let $\mathbf{1_k} \in \mathbf{k}$ denote the unit. Let $\mathfrak{a} = a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$. Write $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$, $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$. We have

$$(a_1 \otimes \mathfrak{a}') \text{Ш} (b_1 \otimes \mathfrak{b}') = a_1 \otimes (\mathfrak{a}' \text{Ш} (b_1 \otimes \mathfrak{b}'))) + b_1 \otimes ((a_1 \otimes \mathfrak{a}') \text{Ш} \mathfrak{b}') + a_1 b_1 \otimes (\mathfrak{a}' \text{Ш} \mathfrak{b}'),$$

with the convention that if $\mathfrak{a} = a_1$, then $\mathfrak{a}'$ multiples as the identity.

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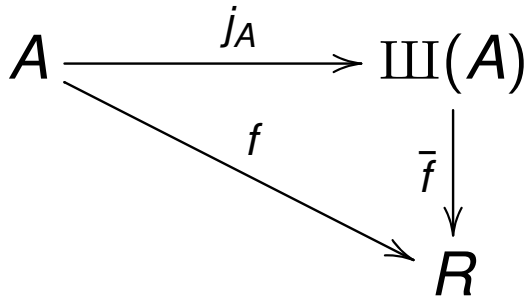
  with the convention that if $\mathfrak{a} = a_1$, then $\mathfrak{a}'$ multiples as the identity.
- **Example.**
  $a_1 \text{Ш} (b_1 \otimes b_2) = a_1 \otimes (\mathfrak{a}' \text{Ш} (b_1 \otimes b_2)) + b_1 \otimes (a_1 \text{Ш} b_2) + (a_1 b_1) \otimes (\mathfrak{a}' \text{Ш} b_2)$
  $= a_1 \otimes (b_1 \otimes b_2) + b_1 \otimes (a_1 \text{Ш} b_2) + (a_1 b_1) \otimes b_2.$
  $= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 + b_1 \otimes a_1 b_2 + a_1 b_1 \otimes b_2.$

Free commutative Rota-Baxter algebras

- A **free Rota-Baxter algebra over another algebra** A is a Rota-Baxter algebra $\text{Ш}(A)$ with an algebra homomorphism $j_A : A \to \text{Ш}(A)$ such that for any Rota-Baxter algebra R and algebra homomorphism $f : A \to R$, there is a unique Rota-Baxter algebra homomorphism making the diagram commute



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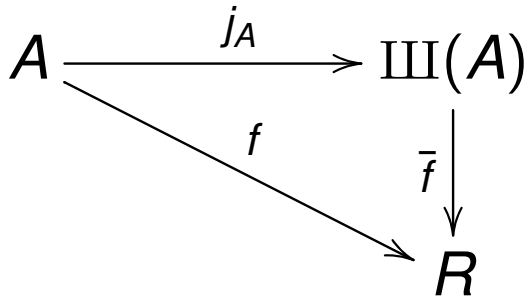

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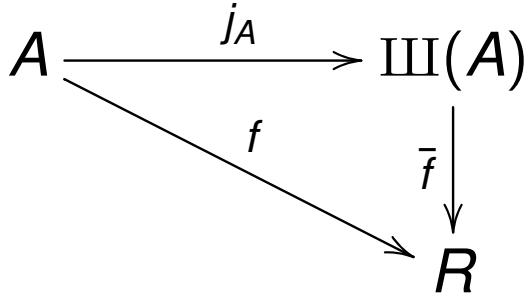

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- Recall $(\text{Ш}^+(A), \text{ш}_\lambda)$ is an associative algebra. Then the tensor product algebra (scalar extension) $\text{Ш}(A) := A \otimes \text{Ш}^+(A)$ is an A-algebra.
  Theorem (Guo-Keigher) $\text{Ш}(A)$ with the shift operator $P(\mathfrak{a}) := 1 \otimes \mathfrak{a}$ is the free commutative RBA over A.

Examples

- The free commutative Rota-Baxter algebra on $\mathbf{k}$ (i.e., on the empty set) is

$$Ш(\emptyset) = \oplus_{k\geq 1}\mathbf{k}a_k,$$

$$a_m a_n = \sum_{r=0}^{\min(m,n)} \binom{m+n-r}{m}\binom{m}{r}\lambda^r \mathbf{1}^{\otimes(m+n-r)}.$$

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- The free commutative integral algebra (Rota-Baxter algebra of weight 0) on $\mathbf{k}[x]$ (i.e., on one generator x):
  Let $\mathcal{I} := \coprod_{k\geq 1}\mathbb{N}_{\geq 0}^k$. For $I = (i_0, \cdots, i_k) \in \mathcal{I}$, denote

$$x^{\otimes I} := x^{i_0} \otimes \cdots \otimes x^{i_k}.$$

  Then $\text{Ш}(\mathbf{k}[x]) = \oplus_{I\in\mathcal{I}}\mathbf{k}x^{\otimes I}$.
  For $x^{\otimes I} = x^{i_0} \otimes x^{\bar{I}}$ and $x^{\otimes J} = x^{j_0} \otimes x^{\bar{J}}$, we have

$$x^{\otimes I}x^{\otimes J} = x^{i_0+j_0} \otimes \left(x^{\bar{I}} \text{Ш} x^{\bar{J}}\right),$$

  where Ш is the shuffle product (partitions and multiple zeta values).

Differential Rota-Baxter algebra

- A differential Rota-Baxter algebra (DRB) is a triple (R, d, P) where (R, d) is a differential algebra (of weight λ), (R, P) is a Rota-Baxter algebra (of weight λ) such that $d \circ P = \mathrm{id}_R$.

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- These give three rewriting rules that imply that a normal form for the DRB algebra is of the form $\mathfrak{x} := x_0 \otimes x_1 \otimes \cdots \otimes x_n, x_i \in \Delta X$.
- More generally, let (A, d) be a differential algebra of weight λ. On the free commutative Rota-Baxter algebra $(\mathrm{Ш}(A), P_A)$, define

$$d_A : \mathrm{Ш}(A) \to \mathrm{Ш}(A),$$

$$\begin{aligned} d_A(x_0 \otimes x_1 \otimes \cdots \otimes x_n) &= d_0(x_0) \otimes x_1 \otimes \ldots \otimes x_n \\ &\quad + x_0 x_1 \otimes x_2 \otimes \ldots \otimes x_n + \lambda d_0(x_0) x_1 \otimes x_2 \otimes \ldots \otimes x_n. \end{aligned}$$

Then $(\mathrm{Ш}(A), d_A, P_A)$ is the free commutative differential Rota-Baxter algebra on A.

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  Then $(\text{Ш}(A), d_A, P_A)$ is the free commutative differential Rota-Baxter algebra on A.
- Theorem (Guo-Keigher) Let X be a set. The differential Rota-Baxter algebra $(\text{Ш}(\mathbf{k}\{X\}), d_{\mathbf{k}\{X\}}, P_{\mathbf{k}\{X\}})$ is the free differential Rota-Baxter algebra on X.

Integro-differential algebras

- Note that the "integral by parts" formula in Rota-Baxter algebra

$$I(f)I(g) = I(fI(g)) + I(I(f)g)$$

is a "purified" version of the original formula

$$FG = I(F'G) + I(FG')$$

by taking the differentiation out of the picture. This needs to be put back in order to understand fully the algebraic structure in differential equations.

Definition of Integro-differential Algebras

- An integro-differential **k**-algebra of weight λ (also called a λ-integro-differential **k**-algebra) is a differential **k**-algebra (R, D) of weight λ with a linear operator $\Pi\colon R \to R$ such that

$$D \circ \Pi = \mathrm{id}_R$$

and the initialization

$$J := \Pi \circ D$$

satisfies

$$J(x)J(y) = J(x)y + xJ(y) - J(xy) \quad \text{for all } x, y \in R.$$

Equivalent conditions

- Let (R, D) be a differential algebra of weight λ with a linear operator Π on R such that $D \circ \Pi = \mathrm{id}_R$. Denote $J = \Pi \circ D$, called the initialization, and $E = \mathrm{id}_R - J$, called the evaluation. Then the following statements are equivalent:
  1. (R, D, Π) is an integro-differential algebra;
  2. $E(xy) = E(x)E(y)$ for all $x, y \in R$;
  3. $\ker E = \mathrm{im} J$ is an ideal;
  4. $J(xJ(y)) = xJ(y)$ and $J(J(x)y) = J(x)y$ for all $x, y \in R$;
  5. $J(x\Pi(y)) = x\Pi(y)$ and $J(\Pi(x)y) = \Pi(x)y$ for all $x, y \in R$;
  6. $x\Pi(y) = \Pi(D(x)\Pi(y)) + \Pi(xy) + \lambda\Pi(D(x)y)$ and $\Pi(x)y = \Pi(\Pi(x)D(y)) + \Pi(xy) + \lambda\Pi(xD(y))$ for all $x, y \in R$;
  7. (R, D, Π) is a differential Rota-Baxter algebra and $\Pi(E(x)y) = E(x)\Pi(y)$ and $\Pi(xE(y)) = \Pi(x)E(y)$ for all $x, y \in R$;
  8. (R, D, Π) is a differential Rota-Baxter algebra and $J(E(x)J(y)) = E(x)J(y)$ and $J(J(x)E(y)) = J(x)E(y)$ for all $x, y \in R$.

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  6. $x\Pi(y) = \Pi(D(x)\Pi(y)) + \Pi(xy) + \lambda\Pi(D(x)y)$ and $\Pi(x)y = \Pi(\Pi(x)D(y)) + \Pi(xy) + \lambda\Pi(xD(y))$ for all $x, y \in R$;
  7. (R, D, Π) is a differential Rota-Baxter algebra and $\Pi(E(x)y) = E(x)\Pi(y)$ and $\Pi(xE(y)) = \Pi(x)E(y)$ for all $x, y \in R$;
  8. (R, D, Π) is a differential Rota-Baxter algebra and $J(E(x)J(y)) = E(x)J(y)$ and $J(J(x)E(y)) = J(x)E(y)$ for all $x, y \in R$.
- We will focus on 6: $\Pi(D(x)\Pi(y)) = x\Pi(y) - \Pi(xy) - \lambda\Pi(D(x)y)$.

Integral by parts revisited

- (R, D, Π) is an integro-differential algebra if and only if (R, D) is a differential algebra, $D \circ \Pi = \mathrm{id}_R$ and

$$\Pi(D(x)\Pi(y)) - x\Pi(y) + \Pi(xy) + \lambda\Pi(D(x)y) = 0, \quad \forall x, y \in R.$$

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- Theorem Let (A, D) be a differential algebra. Let I_{ID} be the differential Rota-Baxter ideal of $\amalg(A)$ generated by elements in the above equations. Then the quotient differential Rota-Baxter algebra $\amalg(A)/I_{ID}$ is the free integro-differential algebra on (A, D).

Integral by parts revisited

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- Theorem Let (A, D) be a differential algebra. Let I_{ID} be the differential Rota-Baxter ideal of $\text{Ш}(A)$ generated by elements in the above equations. Then the quotient differential Rota-Baxter algebra $\text{Ш}(A)/I_{ID}$ is the free integro-differential algebra on (A, D).
- The last equation suggests the rewriting rule

$$\Pi(D(x)\Pi(y)) \mapsto_{ID} x\Pi(y) - \Pi(xy) - \lambda\Pi(D(x)y).$$

Working in the free differential Rota-Baxter algebra $\text{Ш}(A)$ where (A, d) is a differential algebra, this means that $d(x)$ should not appear in Π. More precisely, in $\mathfrak{a} = a_0 \otimes a_1 \otimes \cdots \otimes a_n$, $a_1, \cdots, a_{n-1} \in A$ should be "in complement of" $d(A)$, i.e., in A_T such that $A = \mathrm{im}d \oplus A_T$. Such an A is called regular.

Regular differential algebras

- Let (A, d) be a differential algebra. A linear map $Q : A \to A$ is called a quasi-antiderivative if $d \circ Q \circ d = d$ and $Q \circ d \circ Q = Q$ (and $\ker Q \leq A$ if $\lambda \neq 0$). Then (A, d) is called regular. Take $T := \mathrm{Id} - d \circ Q$.

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- (A, d) is regular if and only if $A = A_T \oplus \mathrm{im} d = A_J \oplus \ker d$.
- Let X be a well ordered set. For $x_1^{(i_1)}, x_2^{(i_2)} \in \Delta X$ with $x_1, x_2 \in X$ and $i_1, i_2 \geq 0$, define

$$x_1^{(i_1)} \leq x_2^{(i_2)} \Leftrightarrow (x_1, -i_1) \leq (x_2, -i_2) \text{ lexicographically.}$$

For example $x^{(2)} < x^{(1)} < x$. Also, $x_1 < x_2$ implies $x_1^{(i_1)} < x_2^{(i_2)}$ for all $i_1, i_2 \geq 0$.

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- Let $u \in C(\Delta X)$ (free commutative monoid) be in the form

  $u = u_0^{j_0} \cdots u_k^{j_k}$, where $u_0, \cdots, u_k \in \Delta X, u_0 > \cdots > u_k$ and $j_0, \cdots, j_k \geq 1$.

  Call u functional if either $u \in C(X)$ or $j_k > 1$. Let $A = \mathbf{k}[\Delta X]$ and A_T be the linear span of the functional monomials. Then $\mathbf{k}[\Delta X] = A_T \oplus \mathrm{im} d$ and $\mathbf{k}\{X\}$ is a regular differential algebra.

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For $\mathfrak{a} := a_0 \otimes \cdots \otimes a_n \in A \otimes (A_\psi)^{\otimes n}$, write $\mathfrak{a} = a_0 \otimes \overline{\mathfrak{a}}, \overline{\mathfrak{a}} \in A_\psi^{\otimes n}$. Define
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- Theorem (Guo-Regensburger-Rosenkranz) The triple $(ID(A), d_u, P_u)$ is the free commutative integro-differential algebra on (A, d).

Normal forms of integro-differential algebras

- Back to the rewriting rule

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- Then in the free differential Rota-Baxter algebra $\text{Ш}(\mathbf{k}\{X\})$. Elements of the form $d(x)$ should not appear in $\Pi(-\Pi(v))$. So for $\mathfrak{a} = a_0 \otimes a_1 \otimes \cdots \otimes a_n$ to be normal, we should have $a_i \in A_T, 1 \leq i \leq n-1$. This is quite hard to verify directly.

Free integro-differential algebras by normal forms

- By the method of Gröbner-Shirshov basis, we obtain.
  Theorem(Gao-Guo-Zheng) Let X be a nonempty well-ordered set and $A := \mathbf{k}\{X\}$. Let $\text{Ш}(\mathbf{k}\{X\}) = \text{Ш}(\mathbf{k}[\Delta X])$, with the derivation d and Rota-Baxter operator P, be the free commutative differential Rota-Baxter algebra of weight λ on X. Let I_{ID} be the differential Rota-Baxter ideal of $\text{Ш}(\mathbf{k}\{X\})$ generated by

  $$S := \{P(d(u)P(v)) - uP(v) + P(uv) + \lambda P(d(u)v) \mid u, v \in \text{Ш}(\mathbf{k}\{X\})\}.$$

  Let A_T be the submodule of $A = \mathbf{k}\{X\}$ spanned by functional monomials. Then the composition

  $$\text{Ш}(A)_T := A \oplus \left(\bigoplus_{k \geq 0} A \otimes A_T^{\otimes k} \otimes A \right) \hookrightarrow \text{Ш}(A) \to \text{Ш}(A)/I_{ID}$$

  of the inclusion and the quotient map is a linear bijection. Thus $\text{Ш}(A)_T$ gives an explicit construction of the free integro-differential algebra $\text{Ш}(A)/I_{ID}$.

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- **Thank You!**