Wild Behavior of Truncation in Hahn Fields

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Valued Fields

A **valued field** is a field K equipped with a surjective map $v : K \to \Gamma \cup \{\infty\}$, such that for all $f, g \in K$ we have

(V0) $v(f) = \infty \iff f = 0$

(V1) $v(fg) = v(f) + v(g)$,

(V2) $v(f + g) \geq \min\{v(f), v(g)\}$

1. The valuation ring is $\mathcal{O} := \{f \in K : v(f) \geq 0\}$,
2. the maximal ideal of $\mathcal{O}$, $\wp := \{f \in K : v(f) > 0\}$, and
3. the residue field of K is defined to be $\mathbf{k} := \mathcal{O}/\wp$.

A **monomial group of** K is a subgroup $\mathfrak{M} \subseteq K^{\times}$ such that $v : \mathfrak{M} \to \Gamma$ is a group isomorphism.

An **additive complement to** $\mathcal{O}$ is an additive subgroup V of K such that $K = V \oplus \mathcal{O}$.

Hahn Series

Let $\mathbf{k}$ be a field, and Γ be an ordered abelian group. We consider the Hahn field $K = \mathbf{k}((t^\Gamma))$ consisting of elements $f = \sum_{\gamma\in\Gamma} f_\gamma t^\gamma$ with well-ordered support, where $\mathrm{supp}(f) = \{\gamma : f_\gamma \neq 0\}$.

$$f + g = \sum_{\gamma\in\Gamma} (f_\gamma + g_\gamma) t^\gamma; \quad fg = \sum_{\gamma\in\Gamma} \left(\sum_{\delta\in\Gamma} f_\delta g_{\gamma-\delta} \right) t^\gamma$$

For Hahn fields we have the following

- $\mathcal{O} := \{f \in K : \mathrm{supp}(f) \geq 0\}$,
- a (canonical) additive complement of $\mathcal{O}$, namely

$$V = \{f \in K : \mathrm{supp}(f) < 0\},$$

- and a (canonical) choice of monomial group $\mathfrak{M} = t^\Gamma$.

Examples of Hahn Fields

1. For $\mathbf{k} = \mathbb{R}$ and $\Gamma = \mathbb{Z}$, we obtain the Laurent series field in t

$$\mathbb{R}((t^{\mathbb{Z}})) = \left\{ \sum_{i>k} f_i t^i : k \in \mathbb{Z},\ f_i \in \mathbb{R} \right\}$$

2. For $\mathbf{k} = \mathbb{R}$ and $\Gamma = \mathbb{Q}$ an example of an element in $\mathbf{k}((t^{\Gamma}))$ is

$$f = \sum_{n>0} t^{1-\frac{1}{n}} + \sum_{n>0} t^n$$

Note that the order type of the support of S is $\omega + \omega$

Truncation In Hahn Fields

Given $f \in \mathbf{k}((t^\Gamma))$ and $\delta \in \Gamma$ we define the truncation of f at δ as

$$f|_\delta = \sum_{\gamma<\delta} f_\gamma t^\gamma.$$

1. For any Laurent series a truncation is a Laurent polynomial
2. For $f = \sum_{n>0} t^{1-\frac{1}{n}} + \sum_{n>0} t^n \in \mathbb{R}((t^\mathbb{Q}))$ as above

$$f|_2 = \sum_{n>0} t^{1-\frac{1}{n}} + t$$

We call $S \subseteq \mathbf{k}((t^\Gamma))$ **truncation closed** if for all $f \in S$ all truncations of f lie inside S (i.e. for all $\gamma \in \Gamma$ we have $f|_\gamma \in S$).

Some derivations in Hahn fields

A **derivation** ∂ on a field K is an additive map that satisfies the Liebniz rule.

- $\partial(f+g) = \partial(f) + \partial(g)$
- $\partial(fg) = \partial(f)g + f\partial(g)$

A **strong** derivation in a Hahn field opens sums

- $\partial(\sum_\gamma f_\gamma t^\gamma) = \sum_\gamma \partial(f_\gamma t^\gamma)$

Any strong derivation induces an additive map $c : \Gamma \to \mathbf{k}((t^\Gamma))$ given by $c(\gamma) = (t^\gamma)^\dagger = (t^\gamma)'/t^\gamma$.

Robustness of Truncation

Theorem (van den Dries, Fornasiero, Mourgues, Ressayre, Marker, Macintyre)

Let S be a truncation closed subset of $\mathbf{k}((t^{\Gamma}))$.

- *The ring and the field generated by S are truncation closed.*

If F is a truncation closed subfield of $\mathbf{k}((t^{\Gamma}))$ and $\operatorname{char}(\mathbf{k}) = 0$

- *The henselization of F in $\mathbf{k}((t^{\Gamma}))$ is truncation closed.*
- *If F is henselian, then any algebraic extension is truncation closed.*
- *Any extensions of F given by $\exp(f)$ or $\log(1+f)$, for f ranging over element of positive support in F, are also truncation closed.*

Robustness with derivation

Theorem

Let ∂ be a derivation on $K = \mathbf{k}((t^{\Gamma}))$. Let F be a truncation closed subfield of K If the induced map is such that $c(\Gamma) \subseteq \mathbf{k}$, then the differential ring generated by F is truncation closed.

Definition

An ordered valued differential field is an H-field if

- $\mathcal{O} = \mathrm{conv}(C) = C \oplus \mathcal{o}$
- If $f > C$, then $f' > 0$

Definition

An H-field K is Liouville closed if K is real closed, $(K)' = K$ and $K^{\dagger} = K$.

The field of Logarithmic-Exponential series $\mathbb{T}$ is (informaly) a series field in which the monomials are

- Real powers of x
- e^f for f a series with negative support
- products of the above
- precomposition of the above by iterations of the logarithm

Theorem

Let F be a truncation closed differential subfield of $\mathbb{T}$ closed under logarithms, then the Liouville closure of F is truncation closed.

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- it is closed under projections ($A \in \mathcal{S}_{m+n} \to \pi_m(A) \in \mathcal{S}_m$), and
- it is closed under Cartesian products ($A \in \mathcal{S}_m, B \in \mathcal{S}_n \to A \times B \in \mathcal{S}_{m+n}$)

We call the elements of $\mathcal{S}_n$ **definable sets**.

Concrete example of $\mathbb{R}$

Consider $(\mathbb{R}; +, \times)$
polynomials can be defined (since composition is defined)

$$Graph(g \circ f) = \pi_{x,z}(\mathbb{R} \times Graph(g) \cap Graph(f) \times \mathbb{R})$$
$$\{(x,z) : f(x) = y \,\&\, g(y) = z\}$$

An ordering can be defined

$$x \leq y \iff \exists z(x + z^2 = y)$$

Theorem (Tarski)

All definable sets are of the form

$$\{x \in \mathbb{R}^n : f(x) = 0 \,\&\, g_i(x) > 0\}$$

for some $f, \{g_i\}_{i<m} \in \mathbb{R}[X_1, \ldots, X_n]$

Examples of Model Theoretic Structures

Tame (Stable)

- (S) Any set with no structure
- $(\mathbb{C}; \times, +)$, in fact any algebraically closed field
- Differentially closed fields of characteristic 0

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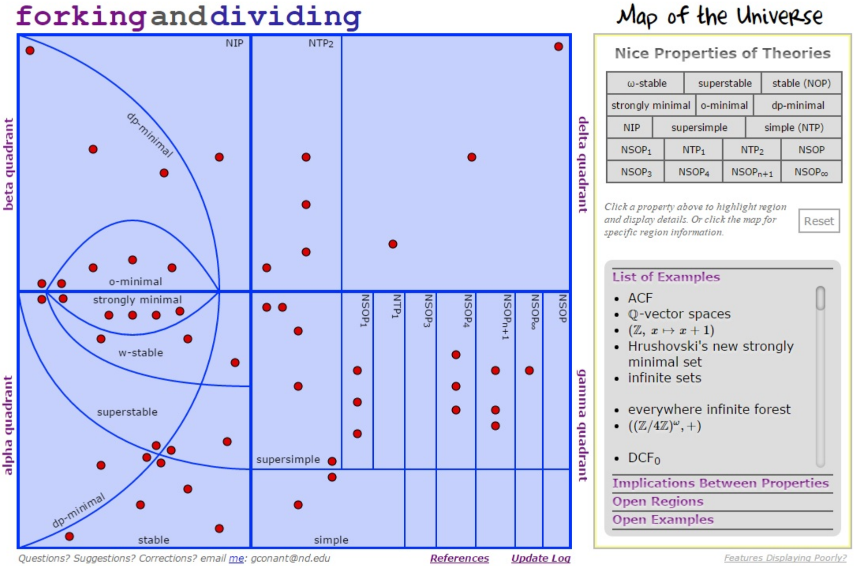
Completely Wild

- $(\mathbb{Z}; +, \times)$
- $(V, \in)$ any model of ZFC(axiomatic set theory)
- $(\mathcal{B}; \wedge, \neg)$ any atomless boolean algebra

The Model Theoretic Universe

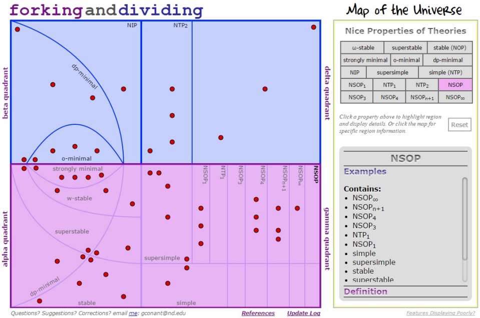


The Strict Order Property

Definition

A relation $\varphi(x;y)$ on $M^x \times M^y$ has the **Strict Order Property**, SOP for short, if there are $\{b_i\}_{i\in\mathbb{N}} \subseteq M^y$ such that

$$\varphi(M^x;b_i) \subseteq \varphi(M^x,b_j) \iff i<j$$



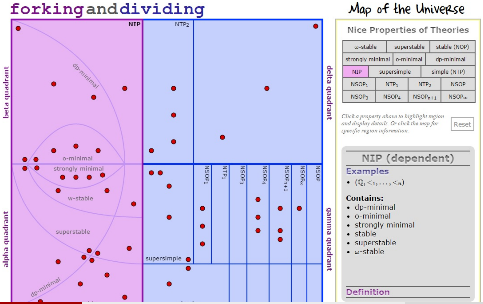
The Independence Property

Definition

A relation $\varphi(x;y)$ on M has the **independence property**, IP for short, if there are $A = \{a_i\}_{i\in\mathbb{N}} \subseteq M^x$, and $\{b_I\}_{I\in\mathcal{P}(\mathbb{N})} \subseteq M^y$ such that

$$\models \varphi(a_i;b_I) \iff i \in I$$

In such case we say that the relation $\varphi(x;y)$ **shatters** A.



Some definable properties

Let $K = \mathbf{k}((t^\Gamma))$ be a Hahn field.
Let $\mathcal{K} = (K; 0, 1, +, \times, \mathcal{O}, \mathfrak{M}, V)$.

Note that for $x \in f$

$$x|_0 = v \iff \exists\, y \in \mathcal{O} : x = v + y$$

and for any $\mathfrak{m} \in \mathfrak{M}$

$$x|_{v(\mathfrak{m})} = \mathfrak{m}((\mathfrak{m}^{-1}x)|_0).$$

Moreover we can identify when a monomial belongs to the "support" of an element

$$\mathfrak{m} \in \operatorname{supp}(f) \iff f - f|_{v(\mathfrak{m})} \asymp \mathfrak{m}$$

IP in Truncation

Definition

A relation $\varphi(x; y)$ on M has the **IP**, if there are $A = \{a_i\}_{i\in\mathbb{N}} \subseteq M^x$, and $\{b_I\}_{I\in\mathcal{P}(\mathbb{N})} \subseteq M^y$ such that

$$\models \varphi(a_i; b_I) \iff i \in I$$

In such case we say that the relation $\varphi(x; y)$ **shatters** A.

Proposition

"x is in the support of y" has IP

Proof.

Let $\Theta = \{\theta_i : i \in \mathbb{N}\}$ be a well ordered subset of Γ.
For each $I \subseteq \mathcal{P}(\mathbb{N})$ let $f_I = \sum_{i\in I} t^{\theta_i}$.
Then the formula "x is in the support of y" shatters the set t^Θ $\square$

We say that a formula $\varphi(x; y)$ has the **tree property of the second kind**, or TP2 for short, if there are tuples $\{b^i_j : i, j \in \mathbb{N}\} \subseteq M^y$ such that for any $\sigma : \mathbb{N} \to \mathbb{N}$ the set $\{\varphi(x; b^i_{\sigma(i)}) : i \in \mathbb{N}\}$ is consistent and for any i and $j \neq k$ we have $\{\varphi(x; b^i_j), \varphi(x; b^i_k)\}$ is inconsistent.

The Tree Property of the Second Kind

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$$
\begin{array}{cccccc}
\varphi(x; b^0_0) & \varphi(x; b^0_1) & \varphi(x; b^0_2) & \dots & \varphi(x; b^0_j) & \dots \\
\\
\varphi(x; b^1_0) & \varphi(x; b^1_1) & \varphi(x; b^1_2) & \dots & \varphi(x; b^1_j) & \dots \\
\\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
\\
\varphi(x; b^i_0) & \varphi(x; b^i_1) & \varphi(x; b^i_2) & \dots & \varphi(x; b^i_j) & \dots
\end{array}
$$

The Tree Property of the Second Kind

Consistent

$$
\begin{array}{cccccc}
\varphi(x; b_0^0) & \varphi(x; b_1^0) & \varphi(x; b_2^0) & \dots & \varphi(x; b_j^0) & \dots \\
\varphi(x; b_0^1) & \varphi(x; b_1^1) & \varphi(x; b_2^1) & \dots & \varphi(x; b_j^1) & \dots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
\varphi(x; b_0^i) & \varphi(x; b_1^i) & \varphi(x; b_2^i) & \dots & \varphi(x; b_j^i) & \dots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots
\end{array}
$$

The Tree Property of the Second Kind

Inconsistent

$$
\begin{array}{cccccc}
\varphi(x; b^0_0) & \varphi(x; b^0_1) & \varphi(x; b^0_2) & \dots & \varphi(x; b^0_j) & \dots \\[2em]
\varphi(x; b^1_0) & \boxed{\color{red}{\varphi(x; b^1_1)}} & \varphi(x; b^1_2) & \dots & \boxed{\color{red}{\varphi(x; b^1_j)}} & \dots \\[2em]
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\vdots & \vdots & \vdots & \ddots & \vdots & \ddots
\end{array}
$$

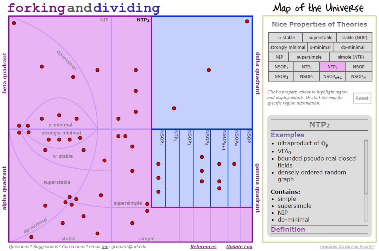
Checking TP2

Lemma

Let T be a complete theory and $M \models T$. Let $A = \{a_i : i \in \mathbb{N}\} \subseteq M^x$ and $B = \{b_I : I \in \mathcal{P}(\mathbb{N})\} \subseteq M^y$. Assume that there is $\varphi(x; y)$ such that for any fixed $b_I \in B$

$$\models \varphi(a; b_I) \iff \text{there is } i \in I \text{ such that } a = a_i.$$

Then φ has TP2*. In this case we say φ* ***exclusively shatters*** *A.*



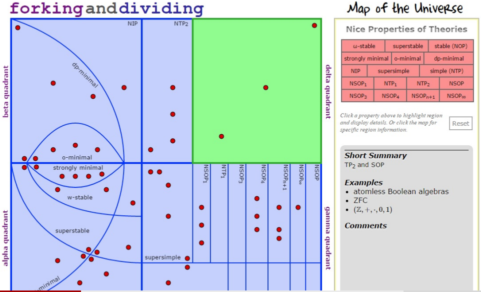
The Place of Truncation in the Model Theoretic Universe

Corollary

"x is in the support of y" has TP2

Proof.

Let Θ be a well ordered subset of Γ. For each $I \subseteq \mathcal{P}(\mathbb{N})$ let $f_I = \sum_{i \in I} t^{\theta_i}$. Then the formula "$x$ is in the support of y" exclusively shatters the set t^{Θ} $\square$



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