The Logarithmic Derivative Homomorphism on an Elliptic Curve Defined over Constants: Addendum

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1 Introductory remarks

At the Friday meeting, the logarithmic derivative map came into play, since it is the analogue for elliptic curves defined over constants of the Manin homomorphism. Let $\mathcal{G}$ be a differentially closed differential field, with derivation operator $\delta = \frac{d}{dt}$, and with (algebraically closed) constant field $\mathcal{C}$ of characteristic 0. Note that $\mathcal{G}$ is also algebraically closed. Let x, y, z be *differential* indeterminates. For any subset S of $\mathbb{P}^2(\mathcal{G})$, and subfield $\mathcal{L}$ of $\mathcal{G}$, set $S(\mathcal{L})$ equal to the set of points in $\mathcal{L}$ that are rational over $\mathcal{L}$. Let $E = E(\mathcal{G})$ be the elliptic curve in $\mathbb{P}^2(\mathcal{G})$ defined by the affine Legendre equation

$$y^2 = (x - e_1)(x - e_2)(x - e_3), \quad e_1, e_2, e_3 \text{ distinct elements of } \mathcal{C}.$$

We equip E with the Kolchin topology, making it a differential algebraic group. Let E_{tors} be the torsion group of E, and denote by $E^{\#}$ the Kolchin closure of E_{tors} . It is clear that $E^{\#} = E(\mathcal{C})$. Let $\ell\delta : E \to \mathbb{G}_a(\mathcal{G})$ be the logarithmic derivative map. Dating back to Kolchin's 1953 paper, the fact that $\ell\delta$ is a surjective differential rational homomorphism with kernel $E^{\#}$ has been crucial in the development of the Galois theory of differential fields. An essential hypothesis is that E be defined over constants. The Manin homomorphism is its replacement in the case where E does not descend to constants.

In this addendum to Friday's talk, I will show that $\ell\delta$ is everywhere defined and surjective, and that its kernel is $E^{\#}$. For the fact that it is a group homomorphism, I refer you to Kovacic, "On the inverse problem in the Galois theory of differential fields: II", *Annals of Mathematics*, 93, 269-284. The logarithmic derivative map is the dual of the invariant differential on the elliptic curve. However, I would like to discuss it *ab initio* without reference to this – and in the language of differential algebraic groups. mainly because it is a good exercise in defining a global section of the *differential* algebraic variety E by patching differential rational functions on an affine open cover. I hope to give a proof in a later note that $\ell\delta$ is an invariant derivation on E, thus proving that it is a homomorphism of differential algebraic groups. The proof of surjectivity in this language is very easy.

2 The kernel of $\ell\delta$

In this section, we will define $\ell\delta$ on the affine open $yz \neq 0$, and assume that is everywhere defined on E. We will show that $\ker \ell\delta = E^{\#}$.

The defining equation of $E = E(\mathcal{G})$ is

$$y^2 = (x - e_1)(x - e_2)(x - e_3), \quad e_1, e_2, e_3 \text{ distinct elements of } \mathcal{C}.$$

The associated homogeneous equation is

$$y^2 z = (x - e_1 z)(x - e_2 z)(x - e_3 z).$$

Let U be the affine open subset $yz \neq 0$ of $\mathbb{P}^2(\mathcal{G})$. We identify the affine open subset $z \neq 0$ of the projective plane with the affine (x, y)-plane. So, U is the complement of the x-axis. We define the logarithmic derivative map

$$\ell\delta : E(U) \longrightarrow \mathbb{G}_a(\mathcal{G})$$

by the formula

$$\ell\delta(x, y) = \frac{x'}{y}.$$

The constant trace $E(\mathcal{C})$ of E is a Zariski dense connected differential algebraic subgroup of $E(\mathcal{G})$. As we remarked in the introduction, it is the Kolchin closure $E^{\#}$of E_{tors} . We will show that $\ker(\ell\delta \mid U) = U(\mathcal{C})$.

$$\begin{aligned} y^2 &= (x - e_1)(x - e_2)(x - e_3). \\ &= x^3 - (e_1 + e_2 + e_3)x^2 + (e_1 e_2 + e_1 e_3 + e_2 e_3)x - e_1 e_2 e_3. \end{aligned}$$

Let

$$\begin{aligned} P(x) &= x^3 - (e_1 + e_2 + e_3)x^2 + (e_1 e_2 + e_1 e_3 + e_2 e_3)x - e_1 e_2 e_3. \\ y^2 &= P(x). \end{aligned}$$

Set

$$\begin{aligned} F(x, y) &= y^2 - P(x). \\ \frac{\partial F}{\partial y} &= 2y. \quad \frac{\partial F}{\partial x} = -\frac{dP}{dx}. \end{aligned}$$

$$\frac{\partial F}{\partial y} y' = \frac{dP}{dx} x'$$

Let $(x, y) \in U$. Since E is a smooth algebraic curve, either $\frac{\partial F}{\partial y}$ or $\frac{dP}{dx}$ fails to vanish at (x, y).

$$\begin{aligned} \frac{\partial F}{\partial y} &= 2y = 0 \Longleftrightarrow y = 0 \\ &\Longleftrightarrow P(x) = 0 \\ &\Longleftrightarrow x = e_1, e_2, e_3. \end{aligned}$$

So, the fact that the roots of $P(x)$ are distinct guarantees the smoothness of the elliptic curve! We have eliminated them from U.

$$\forall\,(x, y) \in U, \quad \frac{\partial F}{\partial y} \neq 0 \text{ at } (x, y).$$

$$\forall\,(x, y) \in U, \quad x' = 0 \Longleftrightarrow y' = 0.$$

$$\begin{aligned} \ell\delta\,(x, y) &= \frac{x'}{y}. \\ \ker\,(\ell\delta \mid U) &= U\,(\mathcal{C})\,. \end{aligned}$$

$U\,(\mathcal{C})$ is an open subset of the Kolchin closed subgroup $E\,(C)$. Suppose $\ell\delta$ is everywhere defined on E, hence on $E\,(\mathcal{C})$. It follows that since it vanishes on a dense open subset of $E(C)$ it will vanish on all of $E(\mathcal{C})$. Can it vanish on a point (x, y, z) not in $E(C)$? What are the points in the complement of U? If $z = 0$, then $(x, y, z) = (0, 1, 0)$, the unique point at ∞. If $y = 0$, then $x = e_1$ or e_2 or e_3. These 4 points are all in $E(C)$. Thus, $\ker \ell\delta = E(\mathcal{C})$.

3 $\ell\delta$ is an everywhere defined differential rational function on E.

To show this, we must extend the definition to an open cover of E. So, we cover E by a finite number of affine open subsets on which the logarithmic derivative map is defined by everywhere differential rational functions that agree on the intersections.

I decided to transform the Legendre form of the defining equation of E to the Weierstrass form, which is easier, since it is missing the second degree term in the cubic $P(x)$. So, we choose $a \in \mathcal{C}$ so that the translation $x \mapsto x - a$, which fixes y, removes the degree 2 term. This transformation does not change the formula for the logarithmic derivative in (x, y) coordinates. It suffices to show that $\ell\delta$ is a global section of the differential structure sheaf on E, now defined by the Weierstrass equation

$$\begin{aligned} y^2 &= x^3 - g_2 x - g_3, \quad g_2, g_3 \in \mathcal{C}, \quad 4g_2^2 - 27g_3^3 \neq 0. \\ y^2 z &= x^3 - g_2 x z^2 - g_3 z^3. \end{aligned}$$

In Weierstrass form,the inequation $4g_2^2 - 27g_3^3 \neq 0$, which tells us that the discriminant of the cubic vanishes nowhere on the curve, guarantees that its roots are distinct, and the elliptic curve is smooth, as we saw in the first section.

3.1 The affine open $yz \neq 0$.

As usual, we define $\ell\delta : E \longrightarrow E(\mathcal{G})$, by the formulae

$$\begin{aligned} \ell\delta(x,y) &= \frac{x'}{y}, \\ \ell\delta(x,y,z) &= \frac{zx' - xz'}{yz} \end{aligned}$$

first in affine coordinates, then in homogeneous coordinates, on the affine open $U_1 : z \neq 0, y \neq 0$. $\ell\delta$ is everywhere defined on U_1. $\ell\delta$ is not everywhere defined on the entire open subset$z \neq 0$ of E. We have not covered the points (x,y), $y \neq 0$. We need an open neighborhood of these 3 points on E.

$y = 0 \Longleftrightarrow x$ is a root of the cubic

$$Q(x) = x^3 - g_2 x - g_3.$$

Since all the roots are simple roots, $\frac{dQ}{dx} = 3x^2 - g_2$ does not vanish at any of these points in E. Let U_2 be the affine open subset of the open affine subset $z \neq 0$, defined by the inequation $3x^2 - g_2 \neq 0$. Define

$$\begin{aligned} \ell\delta(x,y) &= \frac{2y'}{3x^2 - g_2}, \\ \ell\delta(x,y,z) &= \frac{2(zy' - yz')}{3x^2 - g_2 z^2}, \end{aligned}$$

on U_2

What's missing? We havent shown that the log derivative is defined on an open neighborhood of the unique point at infinity on E $\infty = (0,1,0)$, which we get by setting $z = 0$ in the homogenized equation. Now, set $y = 1$ in the homogeneous equation for E. We now have affine coordinates (x,z), and we get the defining affine equation and its homogenization

$$\begin{aligned} z &= x^3 - g_2 x z^2 - g_3 z^3, \\ y^2 z &= x^3 - g_2 x z^2 - g_3 z^3. \end{aligned}$$

satisfied by $\infty = (0,1)$. The defining polynomial of E in the (x,z) plane is

$$P(x,z) = z - \left(x^3 - g_2 x z^2 - g_3 z^3\right).$$

We follow the technique used to capture the roots of the cubic: We use the fact that E is smooth. Therefore, one of the partial derivatives of $P(x,z)$ must be nonzero at ∞. Indeed, although $\frac{\partial P}{\partial x}$ vanishes at $\infty = (0,0)$,

$$\frac{\partial P}{\partial z} = 1 + 2g_2 xz + 3g_3 z^2$$

does not vanish. So, to capture ∞, we use the same trick we used to define $\ell\delta$ at the 3 roots of the cubic. Let U_3 be the affine open defined by the inequation $y \neq 0$, $1 + 2g_2xz + 3g_3z^2 \neq 0$. Define

$$\begin{aligned}\ell\delta\left(x,z\right) &= \frac{2x'}{1+2g_2xz+3g_3z^2}, \\ \ell\delta\left(x,y,z\right) &= \frac{2(yx'-xy')}{y^2+2g_2xz+3g_3z^2}.\end{aligned}$$

The union of the three affine open sets equals $E = E\left(\mathcal{G}\right)$. The differential rational functions defining the logarithmic derivative on these open subsets are everywhere defined. However, we must verify agreement on the intersections $U_1 \cap U_2$, $U_1 \cap U_3$, $U_2 \cap U_3$. We will then have a global differential rational map on E. Note that although for ease of exposition I have avoided it, the logarithmic derivative function is actually a differential polynomial function on E.

3.1.1 $U_1 \cap U_2$

We must show that if $z \neq 0, y \neq 0, 3x^2 - {g_2}^2 \neq 0$, the formulae for $\ell\delta$ agree. On the one hand,

$$\ell\delta\left(x,y\right) = \frac{x'}{y}.$$

On the other hand,

$$\ell\delta\left(x,y\right) = \frac{2y'}{3x^2-g_2}$$

But, recall

$$y^2 = Q\left(x\right), \quad \text{and } \frac{dQ}{dx} = 3x^2 - g_2.$$

Differentiate $y^2 = Q(x)$, where $Q(x) = x^3 - g_2x - g_3$

$$2yy' = \frac{dQ}{dx}x'.$$

So,

$$\left(3x^2-g_2\right)x' = 2yy',$$

and the two differential rational functions agree on the intersection.

3.1.2 $U_1 \cap U_3$

We must now turn to the homogeneous expressions since, on the one hand we are working in the (x, y) plane and on the other hand, we are in the (x, z) plane. So, spose $z \neq 0, y \neq 0, y^2 + 2g_2xz + 3g_3z^2 \neq 0$. On the one hand,

$$\ell\delta\left(x,y,z\right) = \frac{zx'-xz'}{yz}.$$

On the other hand,

$$\ell\delta\left(x,y,z\right)=\frac{2(yx'-xy')}{y^2+2g_2xz+3g_3z^2}.$$

Show

$$\left(y^2+2g_2xz+3g_3z^2\right)\left(zx'-xz'\right)=2yz\left(yx'-xy'\right).$$

Differentiate the affine (x,z) equation

$$z+g_2xz^2-g_3z^3=x^3.$$

We get

$$\left(1+2g_2xz+3g_3z^2\right)z'=3x^2x'.$$

Homogenize:

$$\left(y^2+2g_2xz+3g_3z^2\right)\left(zy'-yz'\right)=\left(3x^2-g_2z^2\right)\left(yx'-xy'\right).$$

Suppose $3x^2-g_2z^2\neq 0$. In this case, (x,y,z) is in $U_1\cap U_2\cap U_3$. We saw that on $U_1\cap U_2$,

$$\left(3x^2-g_2\right)x'=2yy'.$$

Homogenize.by replacing x by $\frac{x}{z}$ and y by $\frac{y}{z}$. Then,

$$\left(3x^2-g_2z^2\right)\left(zx'-xz'\right)=2yz\left(zy'-yz'\right).$$

Thus,

$$\begin{aligned}\frac{zx'-xz'}{yz} &= 2\left(\frac{zy'-yz'}{3x^2-g_2z^2}\right).\\ &= \left(\frac{2(yx'-xy')}{y^2+2g_2xz+3g_3z^2}\right).\end{aligned}$$

So, the two expressions for $\ell\delta$ agree.

We established the equation

$$\left(y^2+2g_2xz+3g_3z^2\right)\left(zy'-yz'\right)=\left(3x^2-g_2z^2\right)\left(yx'-xy'\right).$$

If $3x^2-g_2z^2=0$, then $zy'-yz'=0$. Since $z\neq 0$, $x\neq 0$. So, we have $xyz\neq 0$. First, we divide the equation $3x^2-g_2z^2=0$ by z^2.Thus, $3\left(\frac{x}{z}\right)^2=g_2$. Therefore, $6x\left(zx'-xz'\right)=0$. So, the Wronskian $zx'-xz'=0$. Since $y\neq 0$, $3\left(\frac{x}{y}\right)^2=g_2\left(\frac{z}{y}\right)^2$. Therefore, $6x\left(yx'-xy'\right)=2g_2y\left(yz'-zy'\right)=0$. So, the Wronskian $yx'-xy'=0$. So, both numerators of $\ell\delta$ vanish at (x,y,z).

The two differential rational functions agree on $U_1\cap U_3$. As usual, the complication arose from the vanishing of a differential polynomial function. In this case, it was $3x^2-g_2z^2$.

3.1.3 $U_2 \cap U_3$.

We have $z \neq 0, y \neq 0, 3x^2 - g_2 z^2 \neq 0, y^2 + 2g_2 xz + 3g_3 z^2$. On U_2,

$$\ell\delta\,(x, y, z) = \frac{2\,(zy' - yz')}{3x^2 - g_2 z^2}.$$

On U_3,

$$\ell\delta\,(x, y, z) = \frac{2(yx' - xy')}{y^2 + 2g_2 xz + 3g_3 z^2}$$

But, (x, y, z) lies in $U_1 \cap U_2 \cap U_3$. We showed that the three expressions for $\ell\delta$ agree on $U_1 \cap U_2 \cap U_3$.

$$\frac{2(yx' - xy')}{y^2 + 2g_2 xz + 3g_3 z^2} = \frac{zx' - xz'}{yz} = \frac{2\,(zy' - yz')}{3x^2 - g_2 z^2}.$$

So, the two expressions agree on $U_2 \cap U_3 = U_1 \cap U_2 \cap U_3$. Using these formulae, we see again that $\ker \ell\delta = E\,(\mathcal{C}) = E^{\#}$.

4 The surjectivity of $\ell\delta$

Since $\ell\delta$ is an everywhere defined differential rational function on $E = E\,(\mathcal{G})$, its image is a constructible subset of $\mathbb{G}_{\mathrm{a}}\ (\mathcal{G})$. We assume that $\ell\delta$ is a homomorphism of groups. Then, it is a homomorphism of differential algebraic groups. Therefore, its image is a differential algebraic subgroup of $\mathbb{G}_{\mathrm{a}}\ (\mathcal{G})$. We know that the sum of the differential dimensions of the kernel and image equals the differential dimension of E. The differential dimension of E is 1 (x is differentially algebraic over $\mathcal{G}$ and y is an algebraic function of x). The differential dimension of $E(\mathcal{C})$ is 0. Therefore, the differential dimension of $\ell\delta\,(E)$ is 1. Since every proper differential algebraic subgroup of $\mathbb{G}_{\mathrm{a}}\ (\mathcal{G})$ has differential dimension 0, it follows that $\ell\delta\,(E) = \mathbb{G}_{\mathrm{a}}\ (\mathcal{G})$.